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ON A CONJECTURE OF MERGELYAN

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Аннотация. A solution is presented to the problem of uniform weighted polynomial approximation on bounded simply connected domains in the complex plane, that is analogous to Mergelyan's solution to the classical Bernstein problem on the real line.

To Sergei Nikitovich Mergelyan on his 80-th birthday

1. INTRODUCTION

Let Ω be a bounded simply connected domain in the complex plane $\mathbb{C}$, let Ω_∞ be the unbounded complementary component of its closure, and let w be a positive continuous function defined throughout Ω. Denote by dA the two-dimensional Lebesgue measure (area measure) and for each p, $1 \leq p < \infty$, let $H^p(\Omega, wdA)$ be the closed subspace of $L^p(\Omega, wdA)$ that is spanned by the complex analytic polynomials. Since w is bounded away from zero on each compact subset of Ω, it follows that

$$H^p(\Omega, w\, dA) \subseteq L^p_a(\Omega, w\, dA),$$

the apparently larger of the two spaces consisting of functions in $L^p(\Omega, wdA)$ analytic in Ω. It is an old problem to find necessary and sufficient conditions on either the region Ω or on the weight w in order that $H^p(\Omega, wdA) = L^p_a(\Omega, wdA)$. Whenever the two spaces coincide, the polynomials are said to be *complete* in $L^p_a(\Omega, wdA)$.

By analogy with uniform polynomial approximation on compact subsets of the plane, it would seem natural to assume from the outset that $\partial\Omega = \partial\Omega_\infty$, at least for $w \equiv 1$. Regions for which $\partial\Omega = \partial\Omega_\infty$ are known as *Carathéodory domains*: they evidently include all regions bounded by a single Jordan curve, as well as many other non-Jordan regions. By 1923 Carleman [11] was able to prove that $H^p(\Omega, dA) = L^p_a(\Omega, dA)$ for all p whenever Ω is a Jordan domain, and a decade later Markushevich [18] and Farrell [13] obtained independently the corresponding theorem for Carathéodory regions (cf. also [26], p. 112). That left unresolved the question of polynomial completeness on non-Carathéodory regions; regions for which the following are representative examples:

(a) the *crescent*, topologically a region bounded by two internally tangent circles;

(b) *domains with boundary cuts*, that is regions obtained from a Jordan domain by introducing cuts in the form of simple arcs extending outward to the boundary.

In 1939, Keldysh [16] made the initial and somewhat surprising discovery: For a crescent Ω the polynomials may or may not be dense in $L^p_a(\Omega, dA)$ depending on the *thickness*, or *metric properties*, of Ω in the vicinity of the multiple-boundary point. Not until 1947-48 was a condition found that is both necessary and sufficient for completeness of the polynomials in that context. That was due to the combined efforts of Djrbashyan [12], who established sufficiency, and Shaginyan [30], who established necessity. They showed that if Ω is a crescent lying between the two circles $|z-1| = 1$ and $|z - \frac{1}{2}| = \frac{1}{2}$, having its multiple-boundary point at the origin, and if $l(r)$ is the total length of $(|z| = r) \cap \Omega$ then $H^p(\Omega, dA) = L^p_a(\Omega, dA)$ if and only if

$$\int_0 \log\, l(r)\, dr = -\infty, \tag{1.1}$$

provided $l(r)$ is subject to certain additional and rather restrictive regularity conditions (cf. [26], p. 158). The condition that Ω lies between two tangent circles precludes the possibility of a cusp at the multiple-boundary point, and is essential to the theorem (cf. [4], p. 142). Between the years 1968 and 1977 more precise criteria were found for completeness in a much wider class of crescent domains by Maz'ja and Havin [15], [20], [21] and the author [3], [4]. In addition to vastly weaker regularity restrictions, the intersection of the exterior and the interior boundaries of the crescent was no longer assumed to be a singleton.

It had been noticed at a rather early stage that whenever the polynomials fail to be complete in a crescent domain that was due to, or at least accompanied by, the fact that every function in $H^p(\Omega, dA)$ then admits an analytic continuation across $\partial_i\Omega = \partial\Omega \setminus \partial\Omega_\infty$ into the bounded region complementary to $\bar{\Omega}$. If in a slit domain Ω the total planar measure of the cuts is zero, then the polynomials are evidently not complete in $L^p_a(\Omega, dA)$ for essentially the same reason.

However, for cuts sufficiently massive, Mergelyan and Tamadyan [29] have shown that even in a slit domain it can happen that $H^2(\Omega, dA) = L^2_a(\Omega, dA)$, and their argument extends to all $p \geq 1$. In an attempt to fully explain the completeness phenomenon when $w \equiv 1$ Mergelyan [27] subsequently conjectured, at least for $p = 2$, that in order to have $H^p(\Omega, dA) = L^p_a(\Omega, dA)$, it is necessary and sufficient that for each point $\xi \in \partial\Omega$, and each $\epsilon > 0$ a polynomial P should exist, such that

(1) $||P||_{L^p(\Omega,dA)} < \epsilon$;

(2) $\sup\limits_{|z-\xi|<\epsilon} |P(z)| > 1$.

In other words, completeness fails if and only if at least for one point $\xi \in \partial\Omega$ and a constant $C > 0$ the inequality

$$|P(z)| \le C||P||_{L^p(\Omega,dA)} \tag{1.2}$$

is satisfied for all polynomials P and all z in some (possibly small) neighborhood of ξ. In particular, every function in $H^p(\Omega, dA)$ must therefore admit an analytic continuation across $\partial\Omega$ to a fixed neighborhood of ξ. Mergelyan's conjecture has since been confirmed, not only in its original form, but in many instances for weighted polynomial approximation as well (cf. [5], p. 418 and [8]). Moreover, it has turned out, rather unexpectedly, that (1.2) is equivalent to demanding merely that the inequality

$$|P(\xi)| \le C||P||_{L^p(\Omega,dA)} \tag{1.3}$$

be satisfied for all polynomials P at a single point $\xi \in \partial\Omega$. Such points are generally referred to as *bounded point evaluations* (or BPE's for short), and they play a key role in connection with the dichotomy between completeness and analytic continuation as envisioned by Mergelyan (cf. [10]).

In order to study the completeness question for the most general regions where boundary cuts are present, we consider a weighted measure wdA. We might expect that $H^p(\Omega, wdA) = L^p_a(\Omega, wdA)$ if $w(z) \to 0$ sufficiently rapidly at $\partial_i\Omega$ so that the underlying measure respects any and all cuts.

But, to avoid certain technical difficulties we shall further assume that w depends only on Green's function. More specifically, $g(z, a)$ will denote Green's function with pole at some fixed point $a \in \Omega$. We put $g(z) = \min\,(g(z, a), 1)$ and require that $w(z) = w(g(z))$ be a function of $g(z)$ alone. This makes the problem conformally invariant, and every significant result concerning weighted polynomial approximation on open subsets of the plane, going back to Keldysh [17], is based on this or some roughly equivalent assumption.

If, in addition, we assume that $g\,\log w(g) \downarrow -\infty$ as $g \downarrow 0$ then $H^p(\Omega, wdA) = L^p_a(\Omega, wdA)$ for all p, $1 \le p < \infty$, whenever

$$\int_0 \log\,\log\,\frac{1}{w(g)}\,dg = +\infty. \tag{1.4}$$

Moreover, if $\partial_i\Omega$ contains an isolated *smooth* arc lying in the interior of $\bar{\Omega}$ then $H^p(\Omega, wdA) \neq L^p_a(\Omega, wdA)$ whenever the integral in (1.4) converges and furthermore every function in $H^p(\Omega, wdA)$ then extends analytically across the exposed arc, a fact consistent with Mergelyan's conjecture (cf. [8] and [6], p. 46).

Much of what has been said concerning L^p-completeness has a natural interpretation in the context of uniform weighted approximation. Beurling [2] has considered, for example, the following generalization of the classical Bernstein problem for weighted polynomial approximation on the real line to approximation on open subsets in the plane: With Ω and w as above, let $C_w(\Omega)$ be the Banach space of all complex-valued functions f for which the product $f(z)w(z)$ is continuous on $\bar{\Omega}$ and vanishes on $\partial\Omega$, the norm being defined by

$$||f||_w = \sup_{\Omega} |f|\, w.$$

Evidently, the collection of functions

$$A_w(\Omega) = \{f \in C_w(\Omega) : \ f \text{ is analytic in } \Omega\}$$

is a closed subspace of $C_w(\Omega)$. The problem is to determine whether or not the polynomials are dense in $A_w(\Omega)$. The present paper offers a solution in terms of suitably understood bounded point evaluations which is analogous to Mergelyan's solution [28] to the aforementioned Bernstein problem (cf. also [14], Chapter IV). With some minor restrictions on w we verify Mergelyan's conjecture in the context of uniform weighted approximation by establishing the general principle that either the polynomials are dense in $A_w(\Omega)$, or else, every function in $A_w(\Omega)$ admits an analytic extension to a fixed neighborhood of some point $\xi \in \partial\Omega$.

An extensive and in-depth discussion of the background and history of the completeness problem in its various aspects can be found in the survey articles of Mel'nikov and Sinanyan [25] Mergelyan [26], [28] and the author [9], as well as in the monograph of Walsh [35].

2. THE CAUCHY INTEGRAL AND ANALYTIC CAPACITY

In order to show that one collection of functions is dense in another we argue by duality, taking into account the fact that in this case every continuous linear functional on $A_w(\Omega)$ can be identified with a bounded complex-valued Borel measure μ on Ω. Hence we are immediately led to questions concerning the behavior of the

Cauchy integral

$$\hat{\mu}(z) = \int_\Omega \frac{d\mu(\zeta)}{\zeta - z}$$

at points belonging to $\partial_i\Omega$, and ultimately to the notion of analytic capacity.

The *analytic capacity* $\gamma(X)$ of a compact planar set X is defined as follows:

$$\gamma(X) = \sup\ |f'(\infty)|,$$

where the supremum is taken over all functions analytic in $\hat{\mathbb{C}} \setminus X$ and normalized so that

(a) $||f||_\infty = \sup\limits_{\hat{\mathbb{C}}\setminus X}\ |f| \leq 1$,

(b) $f(\infty) = 0$.

Here $\hat{\mathbb{C}}$ is the extended complex plane or Riemann sphere. For an arbitrary set E we let $\gamma(E) = \sup \gamma(X)$, the supremum now being taken over all compact sets $X \subseteq E$.

It is of utmost importance that γ is equivalent to a second auxiliary capacity γ^+ which is defined directly in terms of the Cauchy integral. For a compact set X we define

$$\gamma^+(X) = \sup_\nu \nu(X)$$

to be the supremum over all ***positive*** measures ν supported on X such that $\hat{\nu} \in L^\infty(\mathbb{C})$ and $||\hat{\nu}||_\infty \leq 1$. Since $\hat{\nu}$ is analytic in $\hat{\mathbb{C}} \setminus X$ and $|\hat{\nu}'(\infty)| = \nu(X)$, the function $\hat{\nu}$ is also admissible for γ and so $\gamma^+(X) \leq \gamma(X)$. As before, if E is an arbitrary planar set we let $\gamma^+(E) = \sup\ \gamma^+(X)$ where X is compact and $X \subseteq E$. Tolsa [32] has shown that there exists an absolute constant $C > 0$ such that

(i) $\gamma^+(E) \leq \gamma(E) \leq C\gamma^+(E)$ for all sets $E \subseteq \mathbb{C}$

(ii) $\gamma(\cup_n E_n) \leq C \sum_n \gamma(E_n)$ for any countable collection of Borel sets $E_n,\ n = 1, 2, 3, \cdots$

Since γ^+ is semiadditive in the sense that it enjoys property (ii), it follows that (i) implies (ii). These results of Tolsa have their roots in the work of Mattila, Mel'nikov and Verdera [19], [23] (cf. also [24] and [33]).

The capacity γ^+ can be used in order to establish a certain lower semicontinuity associated with the Cauchy integral, a property essential for our main theorem. Given a finite, complex, compactly supported measure μ, by $\hat{\mu}$ we denote the Cauchy transform as defined above, and

$$U^{|\mu|}(z) = \int \frac{d|\mu|(\zeta)}{|\zeta - z|}$$

will be the corresponding Newtonian potential.

Lemma 1. *Let x_0 be any point where $U^{|\mu|}(x_0) < \infty$. For each $r > 0$ let $B_r = B(x_0, r)$ be the disk with center at x_0 and radius r, and E be a set with the property that for every $r > 0$ there is a relatively large subset $E_r \subseteq (E \cap B_r)$ on which $U^{|\mu|}$ is bounded; that is,*

(i) $U^{|\mu|} \leq M_r < \infty$ *on* E_r,

(ii) $\gamma(E_r) \geq \epsilon\gamma(E \cap B_r)$ *for some absolute constant* ϵ.

If, moreover, E is thick at x_0 in the sense that

$$\limsup_{r \to 0} \frac{\gamma(E \cap B_r)}{r} > 0, \tag{2.1}$$

then necessarily

$$|\mu(x_0)| \leq \limsup_{z \to x_0, z \in E} |\hat{\mu}(z)|.$$

For a proof see [10], p. 224. Let us also emphasize that it is essential for our purpose that the conclusion of the lemma be valid at every point x_0 were $U^{|\mu|}(x_0) < \infty$, and not simply almost everywhere with respect to area.

3. BOUNDED POINT EVALUATIONS AND ANALYTIC CAPACITY

If Φ is a bounded linear functional on $A_w(\Omega)$, by Hahn-Banach theorem there exists a finite Borel measure μ on Ω such that

$$\Phi(f) = \int_\Omega fw\, d\mu$$

for all $f \in A_w(\Omega)$. Throughout this section we shall assume that μ is an annihilator for $A_w(\Omega)$; that is

$$\int_\Omega fw\, d\mu = 0,$$

whenever $f \in A_w(\Omega)$. By definition $\nu = w\mu$ and $\hat{\nu}$ is its Cauchy transform. Thus, $\hat{\nu} \equiv 0$ in Ω_∞. Our results are based on the following elementary fact:

Lemma 2. *If $H^1(|\hat{\nu}|dA)$ has a BPE at a point $x_0 \in \mathbb{C}$, then the polynomials also have a BPE at x_0 in the $A_w(\Omega)$-norm.*

Proof. By assumption there exists a function $h \in L^\infty(dA)$ with the property

$$P(x_0) = \int Ph|\hat{\nu}|\, dA$$

for every polynomial P. Setting $k = h\,\dfrac{|\hat{\nu}|}{\hat{\nu}}$ for $\hat{\nu} \neq 0$ and $k = 0$ otherwise, we have $||k||_\infty = ||h||_\infty$, and by an interchange in the order of integration

$$P(x_0) = \int Pk\hat{\nu}\, dA = -\int \widehat{Pk}\, d\nu.$$

On the other hand, by Weyl's lemma $\widehat{Pk} = P\hat{k} + F$ a.e.-dA where F is entire. But, $\widehat{Pk}$ and $P\hat{k}$ are continuous and so equality holds everywhere. Since, by assumption, ν annihilates all polynomials and so also F

$$P(x_0) = -\int P\hat{k}\, d\nu = -\int P\hat{k}w\, d\mu.$$

It follows that

$$|P(x_0)| \le C \sup_{\Omega} |P|w = C||P||_w$$

for all polynomials P and some absolute constant C. That is, the polynomials have a BPE at x_0 in the $A_w(\Omega)$-norm. □

The question therefore arises: Under what conditions might we expect $H^1(|\hat{\nu}|dA)$ to have a BPE at a given point? In order to provide a satisfactory answer we adopt a scheme due to Thomson [31], which has its origins in the work of Mel'nikov [22] and Vitushkin [34].

For each positive integer n we consider a grid in the plane consisting of lines parallel to the coordinate axes, and intersecting at those points with both coordinates integral multiples of 2^{-n}. The resulting collection of squares $\mathcal{G}_n = \{S_{nj}\}_{j=1}^{\infty}$ of side lengths 2^{-n} is an edge-to-edge tiling of the plane. Its members will be referred to as squares of the *n-th generation.* Let x_0 be any point in $\partial\Omega$ at which $U^{|\nu|}(x_0) < \infty$. Beginning with a fixed generation, the n-th say, pick a square $S^* \in \mathcal{G}_n$ with $x_0 \in S^*$. For each $\lambda > 0$ let $E_\lambda = \{z : |\hat{\nu}(z)| < \lambda\}$ and denote by $\mathcal{G}_n^\lambda$ the collection of all n-th generation squares S for which

$$|E_\lambda \cap S| > \frac{1}{100}|S|. \tag{3.1}$$

K_n will denote the union of all squares in $\mathcal{G}_n^\lambda$ that can be joined to S^* by a finite chain of squares also lying in $\mathcal{G}_n^\lambda$. If K_n is bounded or empty, then there exists a closed *corridor*, or *barrier*, $Q_n = \cup_j S_{nj}$ composed of squares from $\mathcal{G}_n$ abutting $S^* \cup K_n$, separating the latter from ∞, adjacent to one another along their sides, and such that for each j

$$|E_\lambda \cap S_{nj}| \le \frac{1}{100}|S_{nj}|. \tag{3.2}$$

The polynomial convex hull of Q_n is a polygon Π_n with its boundary Γ_n lying along the sides of squares for which (3.2) is satisfied. Thus, $|\hat{\nu}| \ge \lambda$ on a large portion of every square S_{nj} meeting Γ_n. By adjoining to Π_n additional n-th generation squares we obtain a polygon Π_n^* with boundary Γ_n^* in such a way that

(i) $\Pi_n^* \supseteq \Pi_n$,

(ii) $n^2 2^{-n} \leq \text{dist}\,(\Gamma_n^*, \Gamma_n) \leq 3n^2 2^{-n}$.

At this point let K_{n+1} denote the union of all squares in $\mathcal{G}_{n+1}^{\lambda}$ that can be joined to Π_n^* by a chain of squares in $\mathcal{G}_{n+1}^{\lambda}$ and continue as before. In this way we obtain a nested sequence of polygons

$$\Pi_n \subseteq \Pi_{n+1} \subseteq \cdots \subseteq \Pi_{n+l} \subseteq \cdots \tag{3.3}$$

and compact sets $K_j \subseteq (\Pi_j \setminus \Pi_{j-1})$, $j \geq n$, some of which may be empty, such that if $K_j \neq \emptyset$, then

(a) K_j is the union of squares in $\mathcal{G}_j$ connecting Γ_{j-1}^* to Q_j;
(b) $|E_\lambda \cap S| \geq |S|/100$ for each $S \subseteq K_j$;
(c) dist $(K_j, \Gamma_j^*) \leq$ dist $(K_j, \Gamma_j) +$ dist $(\Gamma_j, \Gamma_j^*) < 4j^2 2^{-j}$.

There are two mutually exclusive possibilities: either the sequence (3.3)

(A) terminates after l steps and $\infty \in \Pi_{n+l}$, or
(B) it continues indefinitely and $\infty \notin \Pi_j$ for any j.

In the second instance there exists an *infinite* sequence of barriers Q_j bounded by polygonal curves Γ_j extending outward from x_0, and accumulating in a finite portion of the plane. This implies

Lemma 3. *If there exists an infinite sequence of barriers Q_j, $j = n, n+1, n+2, \cdots$ surrounding a point x_0, then there is a BPE at x_0 for the polynomials in the $L^1(|\hat{\nu}|dA)$-norm. Hence, there is also a BPE at x_0 in the $A_w(\Omega)$-norm.*

A complete proof can be found in [10], pp. 230-232. Moreover, a closer examination of the argument shows that each point ξ in the region bounded by the initial barrier Q_n corresponds to a BPE for $A_w(\Omega)$ with norm depending only on dist$\,(\xi, Q_n)$. Thus, if Ω_0 is a neighborhood of x_0 with dist$\,(\Omega_0, Q_n) > 0$, then there exists a fixed constant $C > 0$ such that

$$|P(\xi)| \leq C \sup_{\Omega} |P|w = C||P||_w$$

for all $\xi \in \Omega_0$ and all polynomials P. It follows that every function f in $A_w(\Omega)$ must necessarily admit an analytic continuation to Ω_0. The role played here by the Cauchy integral of an annihilating measure is illustrative of a general principle associated with the analytic continuation of a given family of functions obtained in one or another completion process (cf. [7]).

4. THE WEIGHTED APPROXIMATION PROBLEM

We are now in a position to consider the weighted approximation problem in a general context. The notation is as before: Ω is a bounded simply connected domain, $g(z,a)$ is Green's function for Ω with pole at a fixed point $a \in \Omega$, $g(z) = \min\,(g(z,a),1)$, and $w(z) = w(g(z))$ is a weight depending only on $g(z)$. Moreover, $\phi:\ \Omega \to D$ will be a conformal map of Ω onto the open unit disk D with $\phi(a) = 0$, and $\psi = \phi^{-1}$. For each point $z \in \Omega$ let $\delta(z)$ be the Euclidean distance from z to $\partial\Omega$.

The following two Lemmas are corollaries of the Koebe distortion theorem (cf. [9], p. 15 for a reference).

Lemma 4. *There exist constants C_1 and C_2, depending only on $\delta(a) = \mathrm{dist}\,(a, \partial\Omega)$, such that for all $z \in \Omega$*

$$C_1 \frac{g(z)}{\delta(z)} \le |\phi'(z)| \le C_2 \frac{g(z)}{\delta(z)}.$$

Lemma 5. *There exists a constant C such that for all $z \in \Omega$*

$$g^2(z) \le C\,\delta(z).$$

Beurling [2] has studied the completeness problem for $A_w(\Omega)$ under the assumption that $\psi:\ D \to \Omega$ extends continuously to $\bar{D}$. This imposes a rather severe restriction on the region Ω, requiring that

(i) $\partial\Omega$ is arcwise connected and

(ii) $\partial\Omega_\infty$ is a Jordan curve.

The effect is to exclude from consideration any region for which either (i) or (ii) is violated; for example, the region Ω obtained by removing from D the spiral $z = re^{i\theta}$ defined by

$$r = e^{-1/\log\,\theta}, \quad \theta > 2\pi + 1.$$

The region Ω as described here was first examined by Keldysh in 1941 in order to exhibit an example where weighted L^p-completeness fails for weights having a slightly less than optimal rate of decay at $\partial\Omega$. The essential difficulty here lies in the fact that the rest of $\partial\Omega$ is effectively shielded from $\partial\Omega_\infty$. Our goal is to establish a general criterion sufficient for the completeness of the polynomials in the $A_w(\Omega)$-norm with no restrictions on Ω, save simple connectivity.

For an arbitrary weight w and any point $\zeta \in \partial\Omega$ let

$$M_w(\zeta) = \sup\,|P(\zeta)|,$$

the supremum being extended over all polynomials P for which $||P||_w \le 1$. Thus, the polynomials will have a BPE at ζ in the $A_w(\Omega)$-norm if and only if $M_w(\zeta) < +\infty$. In addition to demanding that $w \to 0$ at $\partial\Omega$ we adopt a standing assumption, that

$$\frac{w(g)}{g^2} \downarrow 0, \qquad \text{as } g \downarrow 0.$$

For convenience in notation let $w^*(g) = \dfrac{w(g)}{g^2}$.

Our contribution to Mergelyan's conjecture is this:

Theorem 1. *The polynomials are dense in $A_w(\Omega)$ whenever*

$$M_{w*}(\zeta) = +\infty \qquad \textit{for every } \zeta \in \partial\Omega.$$

If, conversely, the polynomials fail to be dense in $A_w(\Omega)$, then every $f \in A_w(\Omega)$ that can be approximated by polynomials admits an analytic continuation into a fixed neighborhood U intersecting $\partial\Omega$.

Proof. In order to establish the density of the polynomials in $A_w(\Omega)$ it is sufficient to verify that every function, bounded and analytic in Ω, can be so approximated by polynomials. To see why, we first transfer the problem to the open unit disk, setting $W = w(\psi)$ and thereby obtain a weight W on D which depends only on the radius. For any $f \in A_w(\Omega)$ the function $F = f(\psi)$ belongs to $A_W(D)$ and if $0 < r < 1$ the corresponding functions $f_r = F(r\phi)$ and $F_r = f_r(\psi)$ are bounded and analytic in Ω and D, respectively. Moreover, it is clear that

$$||f_r - f||_w = ||F_r - F||_W,$$

and it follows from the monotonicity of W that the right hand side approaches zero as $r \to 1$. Hence, $||f_r - f||_w \to 0$ as $r \to 1$. Since, by assumption, f_r lies in the closed span of the polynomials in $A_w(\Omega)$, the same must be true of f. The conclusion is that the polynomials are dense in $A_w(\Omega)$.

Suppose now that $M_{w*}(\zeta) = +\infty$ for every $\zeta \in \partial\Omega$. Let μ be an annihilating measure for the polynomials; that is

$$\int_\Omega Pw\, d\mu = 0$$

for all polynomials P. Consequently, $\hat{\nu} \equiv 0$ in Ω_∞ where $\nu = w\mu$. If $\zeta \in \partial\Omega$ it can then be inferred from Lemma 5 that

$$U^{|\nu|}(\zeta) = \int_\Omega \frac{w(z)|d\mu(z)|}{|z-\zeta|} \leq C \sup_\Omega \frac{w(g)}{g^2} < \infty.$$

It follows from the semi-continuity of the Cauchy integral as described in Lemma 1 that $\hat{\nu} \equiv 0$ on $\partial\Omega_\infty$ (cf. [10], p. 236). Our first task is to prove that $\hat{\nu} \equiv 0$ on the rest of $\partial\Omega$ as well. It is here that we will make use of the fact that $M_{w*}(\zeta) = +\infty$ at every point $\zeta \in \partial\Omega$. By assumption, for any polynomial P and any $\zeta \in \partial\Omega$ we have

$$\int_{\Omega} \frac{P(z) - P(\zeta)}{z - \zeta}\, d\nu(z) = 0.$$

By an argument due essentially to Cauchy,

$$P(\zeta) = \frac{1}{\hat{\nu}(\zeta)} \int_{\Omega} \frac{P(z)}{z - \zeta}\, d\nu(z)$$

provided $\hat{\nu}(\zeta) \neq 0$. Since $\zeta \in \partial\Omega$, Lemma 5 implies

$$|P(\zeta)| \leq \frac{1}{|\hat{\nu}(\zeta)|} \int_{\Omega} \frac{|P(z)|}{|z - \zeta|}\, |d\nu(z)| \leq C \sup_{\Omega} |P|\, \frac{w(g)}{g^2} = C||P||_{w*}.$$

Because $M_{w*}(\zeta) = +\infty$ this is a contradiction unless $\hat{\nu}(\zeta) = 0$. Therefore, $\hat{\nu} \equiv 0$ on $\partial\Omega$.

The next intermediate step is to establish the fact that the functions $\phi^n\phi'$, $n = 0, 1, 2, \cdots$ all lie in the closure of the polynomials in $A_w(\Omega)$. To that end we have to prove that

$$\int_{\Omega} \phi^n \phi' w\, d\mu = 0, \quad n = 0, 1, 2, \cdots$$

Given $\epsilon > 0$, let $\Omega_\epsilon = \{z \in \Omega :\ g(z) > \epsilon\}$, and ν_ϵ be the restriction of the measure ν to Ω_ϵ. Note that for any point $\zeta \in \partial\Omega$ by Lemma 5

$$|\hat{\nu}_\epsilon(\zeta)| = |\hat{\nu}(\zeta) - \hat{\nu}_\epsilon(\zeta)| = \left| \int_{g \leq \epsilon} \frac{d\nu(z)}{z - \zeta} \right| \leq C\, \frac{w(\epsilon)}{\epsilon^2}. \tag{4.1}$$

Let $\eta < \epsilon$. By an interchange in the order of integration

$$\frac{1}{2\pi i} \int_{g=\eta} \phi^n \phi' \hat{\nu}_\epsilon\, dz = \int_{\Omega_\epsilon} \left(\frac{1}{2\pi i} \int_{g=\eta} \frac{\phi^n \phi'}{\zeta - z}\, dz \right) d\nu(\zeta) = -\int_{\Omega_\epsilon} \phi^n \phi' w\, d\mu.$$

The contour integral on the left is independent of η when $0 < \eta < \epsilon$ and satisfies an estimate from above:

$$\left| \frac{1}{2\pi i} \int_{g=\eta} \phi^n \phi' \hat{\nu}_\epsilon\, dz \right| \leq \int_{g=\eta} |\hat{\nu}_\epsilon|\, |\phi'|\, \frac{|dz|}{2\pi}.$$

As $\eta \to 0$ the measures $|\phi'|\, \frac{|dz|}{2\pi}$ on $g = \eta$ converge $wk - *$ to harmonic measure $d\omega$ on $\partial\Omega$. By (4.1)

$$\left| \int_{\Omega_\epsilon} \phi^n \phi' w\, d\mu \right| \leq \int_{\partial\Omega} |\hat{\nu}_\epsilon|\, d\omega \leq C \frac{w(\epsilon)}{\epsilon^2}.$$

Finally, letting $\epsilon \to 0$ we conclude that

$$\int_{\Omega} \phi^n \phi' w\, d\mu = 0, \quad n = 0, 1, 2, \cdots$$

and therefore each of the functions $\phi^n\phi'$, $n = 0, 1, 2, \cdots$ lies in the closure of the polynomials in $A_w(\Omega)$ as claimed.

To complete the proof of the first half of the theorem let $f \in H^\infty(\Omega)$ and fix $\epsilon > 0$. If we can show that there exists a polynomial P such that

$$\sup_{\Omega} |f - P(\phi)\, \phi'|w = ||f - P(\phi)\phi'||_w < \epsilon,$$

the desired result will follow. To that end let us note that, by virtue of the Koebe distortion theorem (i.e. Lemmas 4 and 5),

$$|f - P(\phi)\phi'|w = \left|\frac{f}{\phi'} - P(\phi)\right| |\phi'|w \leq C \left|\frac{f}{\phi'} - P(\phi)\right| \frac{w(g)}{g}$$

for every polynomial P and some absolute constant C. In similar fashion

$$\left|\frac{f}{\phi'}\right| \frac{w(g)}{g} \leq C|f|\frac{w(g)}{g^2}.$$

Since f is assumed to be bounded, the right hand side $\to 0$ as $g \to 0$. Hence, the map $\phi : \Omega \to D$ carries the weight $\dfrac{w(g)}{g}$ from Ω into a corresponding weight W on D, while $\dfrac{f}{\phi'}$ goes over to a function $F \in A_W(D)$ and we have

$$|f - P(\phi)\phi'|w \leq C|F - P|W.$$

Since $W(r) \downarrow 0$ as $r \uparrow 1$ the same argument with which we began this discussion ensures that the right hand side can be made arbitrarily small by a suitable choice of P. Therefore, under the stated conditions the polynomials are dense in $A_w(\Omega)$.

Conversely, suppose now that the polynomials are not dense in $A_w(\Omega)$. It follows from the above argument that there exists an annihilating measure $\nu = w\mu$ for the polynomials and at least one point $x_0 \in \partial\Omega$ where $\hat{\nu}(x_0) \neq 0$. Since $U^{|\nu|}(x_0) < \infty$ we can further conclude that there exists an infinite sequence of barriers relative to a set where $|\hat{\nu}|$ is bounded away from zero, and surrounding the point x_0 as described above in Section 3.

Suppose for the sake of argument that this is not the case. For an arbitrary, but fixed, $\lambda > 0$ consider the set $E_\lambda = \{z : |\hat{\nu}(z)| < \lambda\}$. By assumption E_λ must in a sense escape from x_0 to ∞. More precisely, we can find a connected set X linking x_0 to ∞ such that X is the union of squares from some generation, the n-th say, and higher, and certain narrow rectangles R_j, $j > n$, where

(1) $|E_\lambda \cap S| > \dfrac{1}{100}\,|S|$ for each square $S \subset X$,
(2) diam $(R_j) \approx j^2 2^{-j}$.

Given $r > 0$, let $B_r = B(x_0, r)$. By discarding certain superfluous pieces we can assume that $X \cap B_r$ is connected and joins x_0 to ∂B_r. Thus,

$$\gamma(X \cap B_r) \geq \frac{1}{4} \operatorname{diam}\,(X \cap B_r) \geq \frac{r}{8}.$$

On the other hand, it follows from the countable semi-additivity of analytic capacity that

$$\frac{r}{16} \leq \gamma(X \cap B_{r/2}) \leq C\left[\gamma(K) + \sum_{j=n}^{\infty} j^2 2^{-j}\right],$$

where K is the union of squares in X for which (1) is satisfied, and C is an absolute constant. Since we are free to begin with an arbitrary generation, we can let $n \to \infty$. It follows that

$$\gamma(E_\lambda \cap B_r) \geq Cr$$

(cf. [10], p. 233 for details). The upshot is

$$\limsup_{r \to 0} \frac{\gamma(E_\lambda \cap B_r)}{r} > 0,$$

and so Lemma 1 implies that

$$|\hat{\nu}(x_0)| \leq \limsup_{z \to x_0, z \in E_\lambda} |\hat{\nu}(z)| \leq \lambda.$$

Since this is valid for all $\lambda > 0$, we must conclude, contrary to assumption, that $\hat{\nu}(x_0) = 0$. We come to the conclusion that for some $\lambda > 0$ there exists an infinite sequence of barriers surrounding x_0 that correspond to the set with $|\hat{\nu}| > \lambda$.

From the discussion following Lemma 3 it is now clear that there is a fixed neighborhood U of x_0 such that every function $f \in A_w(\Omega)$ lying in the closure of the polynomials extends analytically to U. $\square$

To the best of author's knowledge there is at present no available criterion for deciding whether a given point $\zeta \in \partial\Omega$ is, or is not, a BPE for the polynomials in the $A_w(\Omega)$-norm. In certain cases, however, it is possible to nearly quantify the rate of decay of w required in order for the polynomials to be dense in $A_w(\Omega)$. The following is a case in point and was obtained by Beurling [2], p. 413, under the slightly stronger assumption

$$\frac{w(g)}{g^4} \downarrow 0, \qquad \text{as } g \downarrow 0.$$

We shall continue to assume only that $\dfrac{w(g)}{g^2} \downarrow 0$ as $g \downarrow 0$.

Theorem 2. *Assume that the conformal map $\psi : D \to \Omega$ extends continuously to $\bar{D}$. Then, the polynomials are dense in $A_w(\Omega)$ whenever*

$$\int_0 \log\log\frac{1}{w(g)}\,dg = +\infty. \tag{4.2}$$

If, conversely, the integral in (4.2) *converges and $\partial_i\Omega$ contains an isolated smooth arc lying in $\bar{\Omega}$, then the polynomials are not dense in $A_w(\Omega)$.*

Proof (outline) Assume that the integral in (4.2) diverges. Let μ be any bounded Borel measure on Ω such that

$$\int_\Omega Pw\,d\mu = 0$$

for all polynomials P. Thus, the Cauchy integral

$$f(\zeta) = \int_\Omega \frac{w(z)}{z-\zeta}\,d\mu(z)$$

vanishes identically in Ω_∞ and converges absolutely at every point $\zeta \in \partial\Omega$. The essential step is to verify that $f \equiv 0$ on $\partial\Omega$, then the proof proceeds exactly as in Theorem 1. To accomplish that task let

$$f_\epsilon(\zeta) = \int_{\Omega_{2\epsilon}} \frac{w(z)}{z-\zeta}\,d\mu(z),$$

where as before $\Omega_{2\epsilon} = \{z : \ g(z) > 2\epsilon\}$. It is a simple matter to check that for some absolute constant C

(i) $|f(\zeta) - f_\epsilon(\zeta)| \le C\,\dfrac{w(2\epsilon)}{\epsilon^2}$ if $\zeta \in \partial\Omega$

(ii) $|f_\epsilon(\zeta)| \le \dfrac{C}{\epsilon^2}$ if $g(\zeta) < \epsilon$.

Both inequalities are consequences of the Koebe distortion theorem as embodied in Lemmas 4 and 5. The first has been noted in (4.1) above. To arrive at the second let $z \in \Omega_{2\epsilon}$, set $D_\epsilon = \phi(\Omega_\epsilon)$, and recall that by Lemma 5

$$\text{dist}\,(z, \partial\Omega_\epsilon) \ge C\,\text{dist}\,(\phi(z), \partial D_\epsilon)^2 > C\epsilon^2$$

for all ϵ sufficiently small, $\epsilon < 1/2$ say. This in effect is (ii).

Letting $\epsilon \to 0$ in (i) we see that $f_\epsilon \to f$ uniformly on $\partial\Omega$, and so f is continuous there. Under the conformal map we obtain the functions $F = f(\psi)$ and $F_\epsilon = f_\epsilon(\psi)$ where F is continuous on ∂D while F_ϵ is analytic in the region $D \setminus D_\epsilon$ abutting ∂D. Our assumptions allow (i) and (ii) to be expressed in the form

(iii) $|F - F_\epsilon| \le e^{-ch(\epsilon)}$ on ∂D

(iv) $|F_\epsilon| \le \dfrac{c}{\epsilon^2}$ in $D \setminus D_\epsilon$,

where $h(\epsilon) \uparrow +\infty$ as $\epsilon \downarrow 0$ and $\int_0 \log\,h(t)\,dt = +\infty$. Since $F = 0$ on a nontrivial subarc of ∂D, we can reason as in [6], p. 44, to infer from (4.2) that $F \equiv 0$ on ∂D.

The argument here goes back to Beurling [1], and even earlier to a series of lectures he presented during the summer of 1961 at Stanford University. The result is that $f \equiv 0$ on $\partial\Omega$ and the density of the polynomials follows. For the proof in the converse direction see [6], p. 46. $\square$

In the case of a slightly more regular manner in which $w \to 0$ at $\partial\Omega$, we are able to obtain a result similar to Theorem 2 valid for every bounded simply connected domain (cf. [8]).

Theorem 3. *If* $g \log w(g) \downarrow -\infty$ *as* $g \downarrow 0$ *and if*

$$\int_0 \log \log \frac{1}{w(g)}\, dg = +\infty,$$

then necessarily

(1) $H^p(\Omega, wdA) = L^p_a(\Omega, wdA)$ *for all* p, $1 \le p < \infty$

(2) *the polynomials are dense in* $A_w(\Omega)$.

In addition to the ideas of Beurling already mentioned, the proof makes use of concepts from the theory of asymptotically holomorphic functions begun by Vol'berg and further developed by the author [8] (cf. also [9]).

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