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# BLASCHKE STRUCTURE FOR A SPECIAL AFFINE IMMERSION

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Аннотация. In this paper we determine a Blaschke structure for affine immersion of Euclidian and Hyperbolic type for plane equi-affine curve. In particular, we consider this structure in the case where the Ricci tensor of affine immersion is constant and give a necessary and sufficient condition for Ricci tensor to be constant.

## 1. INTRODUCTION

The paper studies nondegenerate affine surfaces in affine space $\mathbb{R}^3$. Such surfaces are endowed with an affine connection $\nabla$, a symmetric bilinear form $h$ which is called the affine metric, and a volume form $\theta$. The concept of affine immersion is presented in Section 2. One of the big problems here is to understand those surfaces for which the immersion is Blaschke. The purpose of this paper is to determine Blaschke structure for affine immersion.

Patrick Lehebel investigated in [7] affine surfaces (and hypersurfaces) which are affine rotation surfaces. In 3-space these surfaces can be characterized by the fact that all affine normals (in the Blaschke sense) intersect a fixed straight line (the axis) and the section with planes containing the axis are shadow boundries with respect to parallel light. In case the axis is a proper line (not at infinity) there are three types of surfaces: elliptic, hyperbolic, and parabolic. Friedrich Manhart in [8] investigated the problem with the additional property of vanishing affine Gauss curvature. But there is no classification of affine revolution surfaces, whose generators are equi-affine curves, so in this work we consider those surfaces from this point of view.

This paper consists of five sections. In Section 2 we give some necessary preliminaries. In Section 3 we introduce the concepts of an Euclidian (or Hyperbolic) affine immersion with respect to a curve. Then we compute its Ricci tensor component, second fundamental form, and its shape operator.

In Proposition 2 using the concept of affine arc-length we give a necessary and sufficient condition for Ricci tensor to be constant in terms of the components of the base curve $\alpha$. In Theorem 2 we describe those curves for which their related Ricci tensor of immersion $f$ is constant. Section 4 is devoted to transversal and the second fundamental form, and points at a case where the related affine immersion into the produced transversal is Blaschke .

## 2. PRELIMINARIES

In this section we introduce the general notion of affine immersion and some terminology and definitions of affine differential geometry. We shall always assume that the given affine connections have zero torsion.

We consider two differentiable manifolds with affine connections ($\bar{M}$,$\bar{\nabla}$) and ($M$,$\nabla$) of dimensions $m$ and $n$, respectively. Let $k = m - n$.

**Definition 1.** *A differentiable immersion $f : M \to \bar{M}$ is said to be an affine immersion if the following condition is satisfied. There is a $k$-dimensional differentiable distribution $N$ along $f : x \in M \to N_x$, a subspace of $T_{f(x)}(\bar{M})$, such that*

$$T_{f(x)}(\bar{M}) = f_*(T_x(M)) \oplus N_x, \tag{2.1}$$

$$(\bar{M}r\nabla_X f_*(Y))_x = (f_*(\nabla_X Y))_x + (\alpha(X,Y))_x, \tag{2.2}$$

*at each point $x \in M$ where $X, Y \in \mathfrak{X}(M)$ and $\alpha(X,Y) \in N_x$.*

Since the given distribution $x \in M \to N_x$ is differentiable, each point $x$ has a local basis, namely, a system of $k$ differentiable vectors $\xi_1, \xi_2, \ldots, \xi_k$ on a neighborhood $U$ of $x$ that span $N_y$ at each point $y \in U$. This distribution may be regarded as a bundle of transversal $k-$space. Now we explain the main proposition of the affine immersion. Note that for $k = 1$, $\xi_1$ is called a transversal vector field.

**Proposition 1.** *For a hypersurface immersion $f : M \to \mathbb{R}^{n+1}$, suppose we have a transversal vector field $\xi$ on $M$. Then we have a torsion-free induced connection $\nabla^\xi$ satisfying*

$$D_X f_*(Y) = f_*(\nabla^\xi_X Y) + h^\xi(X,Y)\xi \qquad \textit{(Gauss)}, \tag{2.3}$$

$$D_X \xi = -f_*(S^\xi X) + \tau^\xi(X)\xi \qquad \textit{(Weingarten)}, \tag{2.4}$$

*where $h^\xi$ is a symmetric bilinear function on the tangent space $T_x(M)$ and $D$ is the flat connection of $\mathbb{R}^3$, while $S^\xi$ is a tensor of type $(1,1)$ called the affine shape operator, and $\tau^\xi$ is a 1-form, called the transversal connection form.*

*Proof:* See [9]. □

**Definition 2.** *For a hypersurface immersion $f : M \to \mathbb{R}^{n+1}$, a transversal vector field $\xi$ is said to be equi-affine if $D_X\xi$ is tangent to $M$ for each $X \in T_xM$, $x \in M$.*

**Definition 3.** *The Ricci tensor of connection $\nabla$ is defined by*

$$Ric(Y, Z) = trace\{X \to R(X, Y)Z\}.$$

Let $f : M \to \mathbb{R}^{n+1}$ be a nondegenerate immersion. If we choose an arbitrary transversal vector field $\xi$, then we obtain on $M$ the affine fundamental form $h$, the induced connection $\nabla$, and the induced volume element $\theta^\xi$. By an appropriate choice of $\xi$ we achieve the following two goals:

(1) $(\nabla, \theta^\xi)$ is an equi-affine structure, that is $\nabla\theta^\xi = 0$;

(2) $\theta^\xi$ coincides with the volume element $\omega_h$ of the nondegenerate metric $h^\xi$.

**Theorem 1.** *Let $f : M \to \mathbb{R}^{n+1}$ be a nondegenerate hypersurface immersion. For each point $x_0 \in M$, there is a transversal vector field defined in a neighborhood of $x_0$ satisfying the conditions* (2.1) *and* (2.2) *above. Such a transversal vector field is unique up to sign.*

*Proof:* See[9]. □

**Definition 4.** *A transversal vector field satisfying* (2.1) *and* (2.2) *is called affine normal field or Blaschke normal field. Locally, it is uniquely determined up to sign. For each point $x \in M$ we take the line through $x$ in the direction of the affine normal vector $\xi_x$. This line is independent of the choice of sign for $\xi$, and is called the affine normal through $x$.*

## 3. EUCLIDIAN (HYPERBOLIC) AFFINE IMMERSION IN $\mathbb{R}^3$

In this section we consider affine immersions $f : M \to \mathbb{R}^3$, where $M$ is two dimensional affine space. In that case $f$ is called a hypersurface immersion and $M$ is called a hypersurface.

Now let $\{u, v\}$ be a flat coordinate system for affine space $M :=\mathbb{R}^2$ with basis $\partial_u := \frac{\partial}{\partial u}$ and $\partial_v := \frac{\partial}{\partial v}$. Let $\xi = (\cos u, \sin u, 0)$ be a transversal vector field for affine immersion $f : M \to \mathbb{R}^3$ such that

$$f(u, v) = \big(\zeta(v)\cos u, \zeta(v)\sin u, \eta(v)\big), \quad \eta' \neq 0.$$

**Definition 5.** *The affine immersion $f$ is called Euclidian affine immersion with generator $\alpha(v) = (\zeta(v), \eta(v))$.*

Now suppose that

$$\xi = (\cosh u, \sinh u, 0)$$

is a transversal vector field for affine immersion $f : M \to \mathbb{R}^3$ with

$$f(u, v) = \big(\zeta(v) \cosh u, \zeta(v) \sinh u, \eta(v)\big).$$

**Definition 6.** *In the above notation $f$ is called Hyperbolic affine immersion with generator $\alpha(v) = (\zeta(v), \eta(v))$.*

**Lemma 1.** *For the Ricci tensor of the induced connection (determined by $f$ and $\xi$) are*

$$R_{11} = \frac{\zeta'\eta'' - \zeta''\eta'}{\zeta\eta'}, \quad R_{12} = R_{21} = 0, \quad R_{22} = 0. \tag{3.1}$$

*Proof:* From Gauss equation we have

$$D_{\partial_u} f_*(\partial_u) = f_*(\nabla^{\xi}_{\partial_u} \partial_u) + h(\partial_u, \partial_u)\xi,$$
$$D_{\partial_u} f_*(\partial_v) = f_*(\nabla^{\xi}_{\partial_u} \partial_v) + h(\partial_u, \partial_v)\xi,$$

and

$$D_{\partial_v} f_*(\partial_v) = f_*(\nabla^{\xi}_{\partial_v} \partial_v) + h(\partial_v, \partial_v)\xi.$$

Therefore

$$\begin{array}{llll} \Gamma^1_{11} = 0, & \Gamma^1_{12} = \zeta'/\zeta, & \Gamma^2_{11} = 0, & \Gamma^2_{12} = 0, \\ \Gamma^1_{21} = \zeta'/\zeta, & \Gamma^1_{22} = 0, & \Gamma^2_{21} = 0, & \Gamma^2_{22} = \eta''/\eta', \end{array} \tag{3.2}$$

where $\Gamma^k_{ij}$s are the components of $\nabla^{\xi}$, that is,

$$\nabla^{\xi}_{\partial x^i} \partial x^j = \Gamma^k_{ij} \partial x^k$$

where $x^1 := u$ and $x^2 := v$. For calculations concerning $\Gamma^k_{ij}$ we refer to [1, 4]. In coordinates, the affine curvature and Ricci tensors of the connection $\nabla^{\xi}$ are

$$R^l_{ijk} = \frac{\partial \Gamma^i_{lj}}{\partial x^k} - \frac{\partial \Gamma^i_{kj}}{\partial x^l} + \Gamma^m_{lj}\Gamma^i_{km} - \Gamma^m_{kj}\Gamma^i_{lm},$$
$$R_{ij} = R^k_{ikj}. \tag{3.3}$$

andwe have

$$R^1_{122} = -R^2_{121} = \frac{\zeta'\eta'' - \zeta''\eta'}{\zeta\eta'}.$$

For other cases

$$R^i_{jkl} = 0,$$

therefore from Ricci equation (3.3)) we get

$$R_{11} = \frac{\zeta'\eta'' - \zeta''\eta'}{\zeta\eta'}, \quad R_{12} = R_{21} = 0, \quad R_{22} = 0$$

that completes the proof of lemma. □

In terms of Gauss formula for Euclidian immersion we have

$$h^{\xi}(\partial_u, \partial_u) = \zeta \ , \ h^{\xi}(\partial_v, \partial_v) = \frac{\zeta''\eta' - \zeta'\eta''}{\eta'}, \tag{3.4}$$

and

$$h^{\xi}(\partial_u, \partial_v) = h^{\xi}(\partial_v, \partial_u) = 0.$$

For Hyperbolic immersion the components of $h^{\xi}$ are the same as for Euclidian, except $h^{\xi}(\partial_u, \partial_u) = -\zeta$.

By Weingarten formula we have the components of the shape operator:

$$S^{\xi}_{11} = 1/\zeta, \quad S^{\xi}_{22} = 0, \tag{3.5}$$

and

$$S^{\xi}_{12} = S^{\xi}_{21} = 0. \tag{3.6}$$

*Corollary* 1. *The transversal vector field $\xi$ both for Euclidian and Hyperbolic affine immersions is equi-affine.*

**Definition 7.** *The parameter of differential curve $\alpha : I \to \mathbb{R}^2$ is called affine arc-length if $|\alpha' \wedge \alpha''| = 1$ [2, 3].*

In other words, for $\alpha(t) = (x(t), y(t))$, then the parameter $t$ is affine arc-length if

$$x'y'' - y'x'' = 1.$$

**Definition 8.** *A curve $\alpha$ is called equi-affine, if its parameter is arc-length parameter* [2, 5].

**Proposition 2.** *Let $f$ be an Euclidian affine immersion with equi-affine curve generator $\alpha(v) = (\zeta(v), \eta(v))$. Then the components of Ricci tensor of immersion $f$ are constant if and only if $\zeta$ and $\eta$ satisfy the equation*

$$\zeta\eta' = c \neq 0. \tag{3.7}$$

*Proof:* The parameter of the generator $\alpha$ for affine immersion $f$ is the arc-length, therefore $\zeta'\eta'' - \zeta''\eta' = 1$. From equation (3.1) if $R_{11} = k$ then $\zeta\eta' = 1/k = c$, and vice versa. □

By equation (3.7) and the fact that $\alpha$ has an affine arc-length parameter, we have:

$$-c(\zeta')^2 - c\zeta''\zeta = \zeta^2. \tag{3.8}$$

**Theorem 2.** *If the components of Ricci tensor of Euclidian (or Hyperbolic) affine immersion are constant, then we have*

$$\zeta = \pm\sqrt{\varphi}, \quad \eta = \pm\int k^{-1}\varphi^{-1/2}\,dv. \tag{3.9}$$

*Proof:* Setting in the equation (3.8) $\varphi = \zeta^2$, we get

$$c\varphi'' + 2\varphi = 0.$$

Under the assumption $c = 1/k$, solving the above differential equations yields

$$\begin{aligned} \varphi &= c_1 \cos\sqrt{2k}\,v + c_2 \sin\sqrt{2k}\,v, && k > 0,\\ \varphi &= c_1 \cosh\sqrt{-2k}\,v + c_2 \sinh\sqrt{-2k}\,v, && k < 0. \end{aligned}$$

It is easy to show that

$$\zeta = \pm\sqrt{\varphi}, \quad \eta = \pm\int k^{-1}\varphi^{-1/2}\,dv. \tag{3.10}$$

□

## 4. BLASCHKE STRUCTURE

A well-known choice of relative normals, concerning the induced volume element $\theta^\xi$ and the second fundamental form $h^\xi$, comes from the fact, that there exists a unique choice (up to a sign) of a relative normal $\xi$ that satisfies

$$\theta^\xi = \omega_h,$$

where $\omega_h$ is the volume element with respect to $h^\xi$. In this case, one calls $\xi$ the affine normal, $h^\xi$ the affine metric, and $f$ a Blaschke immersion.

In this section we introduce the Blaschke structure for Euclidian and Hyperbolic immersions that we defined in previous section. Accordingly to [9], we have to find the affine normal field. We first do that procedure for Euclidian immersion.

**Step 1.:** We choose $\xi = (\cos u, \sin u, 0)$ for a tentative transversal vector field. As we computed in the previous section and from Proposition 1, necessarily $\tau^\xi = 0$.

**Step 2.:** From the computations of Section 2 and from the equation (3.4) we conclude that the second fundamental form $h$ is nondegenerate.

**Step 3.:** For $\xi$ as in Step 1, the induced volume element $\theta^{\xi}$ we write as

$$\theta^{\xi}\big(\partial_u, \partial_v\big) = \omega\big(f_*(\partial_u), f_*(\partial_v), \xi\big),$$

hence

$$\theta^{\xi}\big(\partial_u, \partial_v\big) = \eta'(v)\zeta(v).$$

**Step 4.:** We introduce a unimodular basis $\{X_1, X_2\}$ with

$$\theta^{\xi}(X_1, X_2) = 1.$$

(special case of the statement that we have established for Ricci tensor in Proposition 2). This choice implies

**Proposition 3.** *If*

$$\eta'(v)\zeta(v) = 1, \tag{4.1}$$

*then the basis* $\{\partial_u, \partial_v\}$ *is unimodular.*

*Proof:* follows from the definition of unimodular basis. □

**Step 5.:** Taking $\phi = |\det_{\theta^{\xi}} h|^{\frac{1}{4}}$, we obtain

$$\phi = \sqrt[4]{\frac{\zeta(\zeta''\eta' - \zeta'\eta'')}{\eta'}}.$$

Now let $\bar{\xi} = \phi\xi + Z$, where $Z$ is to be determined from

$$\tau^{\xi} + \frac{1}{\phi}h^{\xi}(Z, \cdot) + d\log\phi = 0.$$

From the previous section $\tau^{\xi} = 0$, therefore this equation is simply $h(Z, X) = -X\phi$ for every $X$, so we choose a unimodular basis of Step 4 by taking $Z = a\partial_u + b\partial_v$ In this case we have

$$h^{\xi}(a\partial_u + b\partial_v, X) = -X\phi.$$

First by taking

$$X = \partial_u,$$

we obtain

$$ah^{\xi}(\partial_u, \partial_u) = a\zeta = 0,$$

therefore,

$$a = 0.$$

Secondly,

$$X = \partial_v,$$

so,

$$h^{\xi}(a\partial_u + b\partial_v, \partial_v) = -\partial_v\phi,$$

$$bh^{\xi}(\partial_v, \partial_v) = b\frac{\zeta''\eta' - \zeta'\eta''}{\zeta},$$

$$b = -\frac{\zeta}{4}\left(\frac{\zeta}{\eta'}\right)^{-\frac{3}{4}}\left(\frac{\zeta'\eta' - \eta''\zeta}{\eta'^2}\right),$$

therefore

$$Z = b\partial_v.$$

**Step 6.:** Once we get the affine normal field $\bar{\xi}$, it is easy to compute the affine metric $\bar{h} = h^{\xi}/\phi$, the affine shape operator $S$, and the induced connection $\nabla^{\xi}$.

**Theorem 3.** *Let $f$ be an Euclidian(Hyperbolic) affine immersion with equi-affine curve generator $\alpha$, and $\xi$, $h^{\xi}$ be transversal and second fundamental form respectively. The choice of $\phi$, $Z$ and $\bar{\xi} = \phi\xi + Z$ renders $f$ Blaschke immersion.*

*Proof:* By Steps $1 - 6$ and the fact that $\alpha$ is an equi-affine curve we obtain

$$\phi = \sqrt[4]{\frac{\zeta}{\eta'}}.$$

From Step 5 by a simple calculation we get $Z = b\partial_v$, where

$$b = -\frac{\zeta}{2}\left(\frac{\zeta}{\eta'}\right)^{-\frac{3}{4}}\left(\frac{\zeta'}{\eta'^2}\right).$$

As we saw in Step 6,

$$\bar{\xi} = \sqrt[4]{\frac{\zeta}{\eta'}}\,\xi + Z.$$

By taking $\bar{\xi}$ as above, $f$ becomes a Blaschke immersion. □

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