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SHARP ESTIMATES OF MULTILINEAR OPERATORS IN POLYDISK AND UNIT BALL

ROMI SHAMOYAN AND RUHAN ZHAO

Bryansk State Uiversity, Russia
E-mail: *shamoyan@tu-bryansk.ru*
SUNY-Brockport, Brockport, NY, USA
E-mail: *rzhao@brockport.edu*

Аннотация. Sharp estimates for some multifunctional analytic expressions and multilinear operators are given, generalizing the corresponding estimates for a single function or for a linear operator.

1. INTRODUCTION

Recent years have seen of study of various multifunctional expressions and multilinear operators in R^n. The purpose of this paper is to develop sharp results for multifunctional analytic expressions and multilinear operators in the area of holomorphic function spaces in the unit disk and in higher dimensions. The papers by Grafakos and Torres [5, 6], Muscalu, Thiele and Tao [9] are the prototypes. Two direct approaches will be indicated in this note. The first leads to new results in higher dimension by modification of the proof of the known "one functional"case; the second yields sharp estimates for multifuntional analytic expressions using classical strong factorization theorems. We note that all our results are known in the particular case of one function and the unit disk on the complex plane. All proofs in higher dimension can be done simultaneously in the polydisk and unit ball in $\mathbb{C}^n$. In the recent work of the second author [11] the problems of this type were considered by purely one dimensional methods in the unit disk using classical properties or inner and outer functions. Some results of this paper extend that results from one to higher dimensions for certain values of the parameters.

In Section 2 we will obtain results on Carleson measures in the polydisk and the unit ball of $\mathbb{C}^n$ by using multiple functions. In Section 3 we apply some factorization theorems to obtain characterizations of certain Carleson measures on the unit disk by multiple functions. In Section 4 we generalize some well-known estimates of the

Poisson transform and the Cauchy-Szegö projection to the multilinear cases. Throughout the paper, C will denote positive constants.

2. SOME SHARP INEQUALITIES FOR MULTIFUNCTIONAL HOLOMORPHIC EXPRESSIONS IN HIGHER DIMENSION

Let $\mathbb{D} = \{z : |z| < 1\}$ be the unit disk, $T = \{z : |z| = 1\}$ be the unit circle. For $0 < p < \infty$, let H^p denote the Hardy space which contains analytic functions f defined on $\mathbb{D}$ such that

$$\|f\|_p^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta < \infty.$$

Let μ be a positive Borel measure on $\mathbb{D}$, X be a Banach space of analytic functions on $\mathbb{D}$. Given $q > 0$, we say that μ is an (X, q)-*Carleson measure*, if there is a constant $C > 0$ such that for any $f \in X$,

$$\int_{\mathbb{D}} |f(z)|^q \, d\mu(z) \leq C \|f\|_X^q.$$

An (H^p, p)-Carleson measure is the classical *Carleson measure*. The Carleson measure was introduced by Carleson [1] in his study of the problem of interpolation by bounded analytic functions and the famous corona problem. It became a very useful tool in the study of function spaces and operator theory. It is known that μ is a Carleson measure if and only if

(i) there is a constant $C_1 > 0$ such that, for every $a \in \mathbb{D}$,

$$\int_{\mathbb{D}} \frac{1 - |a|^2}{|1 - \bar{a}z|^2} \, d\mu(z) \leq C_1;$$

or

(ii) there is a constant $C_2 > 0$ such that for any arc $I \in T$,

$$\mu(S(I)) \leq C_2 |I|,$$

where

$$S(I) = \{z \in \mathbb{D} : 1 - |I| < |z| < 1, \; z/|z| \in I\}$$

is the Carleson box based on I, and $|I|$ is the normalized length measure on T (see, for example, Section 8.2 in [12]).

Let $\mathbb{D}^n = \mathbb{D} \times \mathbb{D} \cdots \times \mathbb{D}$ be the polydisk and $T^n = \{\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{D}^n : |\zeta_j| = 1, j = 1, 2, \ldots, n\}$ the torus. We will write dm_{2n} to denote the Lebesgue measure on $\mathbb{D}^n$ and dm_n to denote the Lebesgue measure on T^n.

Let $\alpha > 1$. Let $\Gamma_\alpha(\zeta) = \{z \in \mathbb{D} : |1 - \bar{\zeta} z| < \alpha(1 - |z|)\}$. Following [4], for $0 < p < \infty$, we define the analytic tent space $A_\infty H^p$ which consists of analytic functions f on $\mathbb{D}^n$ satisfying

$$\|f\|^p_{A_\infty H^p} = \int_{T^n} \sup_{z_1 \in \Gamma_{\alpha_1}(\zeta_1)} \cdots \sup_{z_n \in \Gamma_{\alpha_n}(\zeta_n)} |f(z)|^p \, dm_n(\zeta) < \infty.$$

Theorem 1. *Let $\mu_1, \ldots, \mu_n$ be positive Borel measures on $\mathbb{D}$. If $0 < p_i, q_i < \infty$, $i = 1, \ldots, m$ satisfy*

$$\sum_{i=1}^{m} \left(\frac{q_i}{p_i}\right) = 1,$$

then the following conditions are equivalent:

(i) *There is a constant $C_1 > 0$ such that*

$$\int_{\mathbb{D}^n} \prod_{i=1}^{m} |f_i(z_1, z_2, \ldots, z_n)|^{q_i} d\mu_1(z_1) \cdots d\mu_n(z_n) \le C \prod_{i=1}^{m} \|f_i\|^{q_i}_{A_\infty H^{p_i}}; \tag{2.1}$$

(ii) *There is a constant $C_2 > 0$ such that*

$$\sup_{a \in \mathbb{D}^n} \int_{\mathbb{D}^n} \prod_{k=1}^{n} \frac{1 - |a_k|}{|1 - \bar{a}_k z_k|^2} d\mu_1(z_1) \cdots d\mu_n(z_n) < \infty, \tag{2.2}$$

where $a = (a_1, a_2, \ldots, a_n)$;

(iii) *The measures $\mu_1, \ldots, \mu_n$ are Carleson measures.*

Proof: For simplicity, we give the proof for the case $n = 2$ (the general case is similar). It is clear that conditions (ii) and (iii) are equivalent.

Let (iii) hold. We write

$$A_\infty(f)(\zeta) = \sup_{z \in \Gamma_\alpha(\zeta)} |f(z)|, \quad C_1(\mu)(\zeta) = \sup_{\zeta \in I} \frac{1}{|I|} \int_{S(I)} d\mu(z).$$

It is well-known that, if μ is a Carleson measure then for any complex-valued function f on $\mathbb{D}$,

$$\int_{\mathbb{D}} |f(z)| \, d\mu(z) \le C \int_T [C_1(\mu)(\zeta)][A_\infty(f)(\zeta)] \, dm_1(\zeta), \tag{2.3}$$

(See Proposition 3 in [4], or Theorem 2.1 in [10]). We have to prove (2.3) in the bidisk $\mathbb{D}^2 = \mathbb{D} \times \mathbb{D}$. First applying (2.3) in z_1, then using Fubini's theorem, and applying (2.3) again in z_2 we get that for any complex-valued function φ on $\mathbb{D}^2$,

$$\begin{aligned}
&\int_{\mathbb{D}^2} |\varphi(z_1, z_2)|\, d\mu_1(z_1)\, d\mu_2(z_2) \\
&\quad \leq C \int_{\mathbb{D}} \int_T (A_\infty)_{z_1}(\varphi)(\zeta_1, z_2) C_1(\mu_1)(\zeta_1)\, dm_1(\zeta_1)\, d\mu_2(z_2) \\
&\quad \leq C \int_T \int_{\mathbb{D}} (A_\infty)_{z_1}(\varphi)(\zeta_1, z_2)\, d\mu_2(z_2) C_1(\mu_1)(\zeta_1)\, dm_1(\zeta_1) \\
&\quad \leq C \left[\int_{T^2} (A_\infty)_{z_1} (A_\infty)_{z_2}(\varphi) C_1(\mu_2)(\zeta_2)\, dm_1(\zeta_2) \right] C_1(\mu_1)(\zeta_1)\, dm_1(\zeta_1) \\
&\quad \leq C \left[\sup_{I \in T} \frac{1}{|I|} \int_{S(I)} d\mu_1(z) \right] \left[\sup_{I \in T} \frac{1}{|I|} \int_{S(I)} d\mu_2(z) \right] \times \\
&\qquad \times \left[\int_{T^2} (A_\infty)_{z_1} (A_\infty)_{z_2}(\varphi)\, dm_1(\zeta_1)\, dm_1(\zeta_2) \right].
\end{aligned}$$

Applying this to $\varphi = |f_1|^{q_1} \cdots |f_m|^{q_m}$ we immediately get that (iii) implies

$$\begin{aligned}
&\int_{\mathbb{D}^2} |f_1(z_1, z_2)|^{q_1} \cdots |f_m(z_1, z_2)|^{q_m}\, d\mu_1(z_1)\, d\mu_2(z_2) \\
&\quad \leq C \int_{T^2} [(A_\infty)_{z_1} (A_\infty)_{z_2} |f_1|^{q_1}] \cdots [(A_\infty)_{z_1} (A_\infty)_{z_2} |f_m|^{q_m}]\, dm_2(\zeta_1, \zeta_2).
\end{aligned}$$

A simple application of Hölder's inequality yields (2.1).

Finally, suppose (i) holds. Let for $i = 1, 2, \ldots, m$

$$f_i(z) = \left(\frac{1 - |a|}{(1 - \bar{a}z)^2} \right)^{1/p_i},$$

where

$$1 - |a| = \prod_{k=1}^{n} (1 - |a_k|), \quad (1 - \bar{a}z)^2 = \prod_{k=1}^{n} (1 - \bar{a}_k z_k)^2.$$

Using the classical theorem on p. 180, Chapter 8 in [8], we get that $f_i \in A_\infty H^{p_i}$. Substituting in (2.1) we get

$$\sup_{a \in \mathbb{D}^n} \int_{\mathbb{D}^n} \frac{\prod_{k=1}^{n} (1 - |a_k|)}{\prod_{k=1}^{n} |1 - \bar{a}_k z_k|^2} d\mu_1(z_1) \cdots d\mu_n(z_n) < \infty.$$

So we get (ii), and the proof is complete.

Remark 1. *A result of this type was first obtained by the second author in* [11], *for the Hardy spaces H^p on the unit disk $\mathbb{D}$. The proof in* [11], *however, is based on properties of outer and inner functions and hence is purely one-dimensional.*

If we define a "mixed-norm"space $M_\infty H^p$, $0 < p < \infty$, as the space of analytic functions f on $\mathbb{D}^n$ such that the norm $\|f\|_{M_\infty H^p}$ is given by the following

$$\left(\int_T \sup_{z_n \in \Gamma_{\alpha_n}(\xi_n)} \cdots \int_T \sup_{z_1 \in \Gamma_{\alpha_1}(\xi_1)} |f(z_1, z_2, \ldots, z_n)|^p \, dm_1(\xi_1) \cdots dm_1(\xi_n)\right)^{1/p},$$

where $\alpha_i > 1$, $i = 1, 2, \ldots, n$, then we get a result similar to Theorem 1:

Theorem 2. *Let $\mu_1, \ldots \mu_n$ be positive Borel measures on $\mathbb{D}$. Let $0 < p_i, q_i < \infty$, $i = 1, \ldots, n$ be such that $\sum_{i=1}^m (q_i/p_i) = 1$. Then the following conditions are equivalent*

(i) *There is a constant $C_1 > 0$ such that*

$$\int_{\mathbb{D}^n} \prod_{i=1}^m |f_i(z_1, z_2, \ldots, z_n)|^{q_i} d\mu_1(z_1) \cdots d\mu_n(z_n) \le C \prod_{i=1}^m \|f_i\|_{M_\infty H^{p_i}}^{q_i};$$

(ii) *There is a constant $C_2 > 0$ such that*

$$\sup_{a \in \mathbb{D}^n} \int_{\mathbb{D}^n} \prod_{k=1}^n \frac{1 - |a_k|}{|1 - \bar{a}_k z_k|^2} d\mu_1(z_1) \cdots d\mu_n(z_n) < \infty,$$

where $a = (a_1, a_2, \ldots, a_n)$;

(iii) *The measures $\mu_1, \ldots, \mu_n$ are Carleson measures.*

Sketchy proof: It is enough to prove that (iii)$\Longrightarrow$(i) $\Longrightarrow$(ii). Let (iii) hold. First applying (2.3) in z_1, then applying (2.3) again in z_2 we get

$$\begin{aligned} &\int_{\mathbb{D}^2} |\varphi(z_1, z_2)| \, d\mu_1(z_1) \, d\mu_2(z_2) \\ &\quad \le C \left[\sup_{I \in T} \frac{1}{|I|} \int_{S(I)} d\mu_1(z)\right] \left[\sup_{I \in T} \frac{1}{|I|} \int_{S(I)} d\mu_2(z)\right] \times \\ &\quad\quad \times \int_T (A_\infty)_{z_2} \int_T (A_\infty)_{z_1}(\varphi) \, dm_1(\zeta_1) \, dm_1(\zeta_2). \end{aligned}$$

Applying this to the function $\varphi = |f_1|^{q_1} \cdots |f_m|^{q_m}$ on $\mathbb{D}^2$, and repeatedly applying Hölder's inequality, we get

$$\int_{\mathbb{D}^2} \prod_{i=1}^m |f_i(z_1, z_2)|^{q_i} \, d\mu(z_1) d\mu(z_2) \le C \prod_{i=1}^m \|f\|_{M_\infty H^{p_i}}^{q_i}.$$

Therefore we get (i). To prove that (i)$\Longrightarrow$(ii), we apply the same testing functions in the proof of Theorem 1.

As a consequence of Theorem 1, we give the following result.

Theorem 3. *Let $s > 0$, $p > 0$, $q > 0$ satisfy $s + q/p = 1$. Then the following statements are equivalent:*

(i) $\displaystyle\int_{\mathbb{D}^n} |f(z_1, \dots, z_n)|^q \left[\prod_{k=1}^n \frac{1-|a_k|}{|1-\bar{a}_k z_k|^2}\right]^s d\mu_1(z_1)\cdots d\mu_n(z_n) \le C\|f\|^q_{A_\infty H^p}$;

(ii) $\displaystyle\int_{\mathbb{D}^n} |f(z_1, \dots, z_n)|^q \left[\prod_{k=1}^n \frac{1-|a_k|}{|1-\bar{a}_k z_k|^2}\right]^s d\mu_1(z_1)\cdots d\mu_n(z_n) \le C\|f\|^q_{M_\infty H^p}$;

(iii) *The measures $\mu_1, \dots, \mu_n$ are Carleson measures.*

Proof: The result can be similarly proved as in [11]. Suppose (i) is true, that is, μ_1, $\dots$, μ_n are Carleson measures. In Theorem 1 (i) we let $m = 2$, $q_1 = q$, $q_2 = s$, $p_1 = p$ and $p_2 = 1$, and let $f_1 = f$, and

$$f_2(z_1, \dots, z_n) = \prod_{k=1}^n \left(\frac{(1-|a_k|)}{(1-\bar{a}_k z_k)^2}\right).$$

Then we easily get (iii). The implication (iii)$\Longrightarrow$(i) follows immediately by putting

$$f(z_1, \dots, z_n) = \prod_{k=1}^n \left(\frac{(1-|a_k|)}{(1-\bar{a}_k z_k)^2}\right)^{1/p},$$

and using Theorem 1 (ii). Similarly, applying Theorem 2 we get (ii)$\Longleftrightarrow$(iii). The proof is complete.

Remark 2. *We note that the key result (2.3) holds for the unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$, see, Theorem 2.1 in* [10]. *Therefore, complete analogies of Theorem 1, Theorem 2 and Theorem 3 hold for the unit ball $\mathbb{B}_n$, with similar proofs.*

Let $\mathbb{S}_n$ be the unit sphere in $\mathbb{C}^n$. For $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ in $\mathbb{B}_n$, let $\langle z, w\rangle = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n$ and $|z| = \sqrt{\langle z, z\rangle} = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$. We will denote by $D(a, r)$ the Djrbashian metric ball in $\mathbb{B}_n$ with center at a and radius r. It is well-known that for $z \in D(a, r)$, $|1 - \langle a, z\rangle| \approx 1 - |z|^2 \approx 1 - |a|^2$, see, for example, Lemma 2.20 in [13]. Following the notations in [13] we write dm to denote the volume measure on $\mathbb{B}_n$ and $d\sigma$ to denote the area measure on $\mathbb{S}_n$.

Theorem 4. *Let μ be a positive Borel measure on $\mathbb{B}_n$. Let $0 < p_i, q_i < \infty$, $0 < r \leq 1$, $\alpha > -1$, $i = 1, 2, \ldots, m$. Suppose*

$$\sum_{i=1}^{m} \left(\frac{1}{q_i}\right) = 1.$$

Then there is a sequence of points $\{a_k\}$ in $\mathbb{B}_n$ such that

$$\begin{aligned} &\int_{\mathbb{B}_n} \prod_{i=1}^{m} |f_i(z)|^{p_i}\, d\mu(z) \\ &\leq C \left[\sum_{k=1}^{\infty} \left(\int_{D(a_k,2r)} |f_1(z)|^{p_1} (1-|z|)^{\alpha}\, dm(z)\right)^{q_1}\right]^{1/q_1} \times \\ &\cdots \times \left[\sum_{k=1}^{\infty} \left(\int_{D(a_k,2r)} |f_m(z)|^{p_m} (1-|z|)^{\alpha}\, dm(z)\right)^{q_m}\right]^{1/q_m} \end{aligned} \tag{2.4}$$

if and only if

$$\mu(D(a_k, r)) \leq C(1 - |a_k|)^{(n+1+\alpha)m} \tag{2.5}$$

for every k.

The following lemma can be found in p. 58, [12].

Lemma 1. *There exists a positive integer N such that for any $r \leq 1$, there exists a sequence $\{a_k\}$ in $\mathbb{B}_n$ satisfying the following conditions:*

(1) $\mathbb{B}_n = \bigcup_{k=1}^{\infty} D(a_k, r)$;

(2) $D(a_k, r/4) \cap D(a_m, r/4) = \emptyset$ *if* $k \neq m$;

(3) *Any point in $\mathbb{B}_n$ belongs to at most N of the sets $D(a_k, 2r)$.*

Proof of Theorem 4: Supposing that (2.5) holds we take a sequence $\{a_k\}$ in $\mathbb{B}_n$ satisfying the conditions in Lemma 1. Then by Lemma 1 (1)

$$\begin{aligned} &\int_{\mathbb{B}_n} |f_1(z)|^{p_1} \cdots |f_m(z)|^{p_m}\, d\mu(z) \\ &\leq C \sum_{k=1}^{\infty} \mu(D(a_k, r)) \left(\sup_{z \in D(a_k,r)} |f_1(z)|^{p_1}\right) \cdots \left(\sup_{z \in D(a_k,r)} |f_m(z)|^{p_m}\right). \end{aligned}$$

By Lemma 2.24 in [13] the above term is bounded from above by

$$C\sum_{k=1}^{\infty}\frac{\mu(D(a_k,r))}{(1-|a_k|)^{(n+1+\alpha)m}}\left(\int_{D(a_k,2r)}|f_1(w)|^{p_1}(1-|w|)^{\alpha}\,dm(w)\right)\times \cdots\times\left(\int_{D(a_k,2r)}|f_m(w)|^{p_m}(1-|w|)^{\alpha}\,dm(w)\right)$$

and from (2.5) this is bounded from above by

$$C\prod_{i=1}^{m}\left[\sum_{k=1}^{\infty}\left(\int_{D(a_k,2r)}|f_i(w)|^{p_i}(1-|w|)^{\alpha}\,dm(w)\right)^{q_i}\right]^{1/q_i},$$

so we get (2.4).

Conversely, let (2.4) be true. Let

$$f_i(z)=\left(\frac{(1-|a_k|^2)^{n+1+\alpha}}{(1-\langle a_k,z\rangle)^{2(n+1+\alpha)}}\right)^{1/p_i},$$

$i=1,2,\dots,m$. Then $|1-\langle a_k,z\rangle|\approx 1-|z|^2\approx 1-|a_k|^2$ for every $z\in D(a_k,r)$ implies

$$\begin{aligned}\int_{\mathbb{B}_n}|f_1(z)|^{p_1}\cdots|f_m(z)|^{p_m}\,d\mu(z)&\geq\int_{D(a_k,r)}\frac{(1-|a_k|^2)^{m(n+1+\alpha)}}{|1-\langle a_k,z\rangle|^{2m(n+1+\alpha)}}\,d\mu(z)\\&\geq\frac{\mu(D(a_k,r))}{(1-|a_k|^2)^{m(n+1+\alpha)}}.\end{aligned}$$

Using properties of $\{a_k\}$ and Theorem 1.12 in [13] we can show that the right-hand side of (2.4) is less than or equal to

$$C\left(\int_{\mathbb{B}_n}\frac{(1-|a_k|)^{n+1+\alpha}(1-|z|)^{\alpha}}{|1-\langle a_k,z\rangle|^{2(n+1+\alpha)}}\,dm(z)\right)^m\leq C<\infty.$$

Combining this with the above inequality we get (2.5).

Remark 3. *A result similar to Theorem 4 holds for $\mathbb{D}^n$. We leave the proof to the readers.*

Proposition 1. *Given $0<q_i<p_i<\infty$, let $\lambda=\sum_{i=1}^{m}(q_i/p_i)<1$. For any positive Borel measures μ_i on $\mathbb{D}$, $i=1,\dots,n$ we have*

$$\int_{\mathbb{D}^n}\prod_{i=1}^m |f_i(z_1,z_2,\dots,z_n)|^{q_i}\,d\mu_1(z_1)\cdots d\mu_n(z_n)$$
$$\le C\left(\prod_{i=1}^m \|f_i\|_{A_\infty H^{p_i}}^{q_i}\right)\left[\int_{T^n}\left(\prod_{k=1}^n\int_{\Gamma_\alpha(\xi_k)}\frac{d\mu_k(z_k)}{1-|z_k|}\right)^{\frac{1}{1-\lambda}}dm_n(\xi)\right]^{1-\lambda}.$$

Proof: Recall that, for $\xi\in T$, $\Gamma_\alpha(\xi)=\{z\in\mathbb{D}\ :\ |1-\bar{\xi}z|<\alpha(1-|z|)\}$. Now for a fixed $z\in\mathbb{D}$, let $I_\alpha(z)=\{\xi\in T\ :\ z\in\Gamma_\alpha(\xi)\}$. So it is easy to see that $|I_\alpha(z)|=\int_{I_\alpha(z)}dm_1(\xi)\approx 1-|z|^2$. Therefore, by Fubini's theorem we get

$$\begin{aligned}\int_{\mathbb{D}}\varphi(z)\,d\mu(z)&\le C\int_{\mathbb{D}}\frac{\varphi(z)}{1-|z|^2}\int_{I_\alpha(z)}dm_1(\xi)\,d\mu(z)\\ &=C\int_T\int_{\Gamma_\alpha(\xi)}\frac{|\varphi(z)|\,d\mu(z)}{(1-|z|)}\,dm_1(\xi),\end{aligned}\tag{2.6}$$

Let $z=(z_1,\dots,z_n)$ and $d\mu(z)=d\mu_1(z_1)\cdots d\mu_n(z_n)$. Applying (2.6) to each variable $z_1,\dots,z_n$, we get

$$\begin{aligned}&\int_{\mathbb{D}^n}\prod_{i=1}^m|f_i(z)|^{q_i}\,d\mu(z)\\
&\le C\int_{T^n}\int_{\Gamma_\alpha(\xi_1)}\cdots\int_{\Gamma_\alpha(\xi_n)}\frac{\prod_{i=1}^m|f_i(z)|^{q_i}\,d\mu(z)}{(1-|z_1|)\cdots(1-|z_n|)}\,dm_n(\xi)\\
&\le C\int_{T^n}\left(\sup_{z_1\in\Gamma_\alpha(\xi_1)}\cdots\sup_{z_n\in\Gamma_\alpha(\xi_n)}\prod_{i=1}^m|f_i(z)|^{q_i}\right)\\
&\qquad\times\int_{\Gamma_\alpha(\xi_1)}\cdots\int_{\Gamma_\alpha(\xi_n)}\frac{d\mu_1(z_1)\cdots d\mu_n(z_n)}{(1-|z_1|)\cdots(1-|z_n|)}\,dm_n(\xi)\\
&\le C\int_{T^n}\prod_{i=1}^m\left(\sup_{z_1\in\Gamma_\alpha(\xi_1)}\cdots\sup_{z_n\in\Gamma_\alpha(\xi_n)}|f_i(z)|\right)^{q_i}\\
&\qquad\times\left(\prod_{k=1}^n\int_{\Gamma_\alpha(\xi_k)}\frac{d\mu_k(z_k)}{(1-|z_k|)}\right)dm_n(\xi).\end{aligned}$$

Because $(1-\lambda)+\sum_{i=1}^m q_i/p_i=1$, it remains to use Hölder's inequality to get what we need. The proof is complete.

Proposition 2. *Given $0 < q_i < p_i < \infty$, let $\lambda = \sum_{i=1}^{m}(q_i/p_i) < 1$. For any positive Borel measures μ_i on $\mathbb{D}$, $i = 1, \dots, n$ we have*

$$\int_{\mathbb{D}^n} \prod_{i=1}^{m} |f_i(z_1, z_2, \dots, z_n)|^{q_i}\, d\mu_1(z_1) \cdots d\mu_n(z_n)$$
$$\leq C \left(\prod_{i=1}^{m} \|f_i\|_{M_\infty H^{p_i}}^{q_i}\right) \left[\int_{T^n} \left(\prod_{k=1}^{n} \int_{\Gamma_\alpha(\xi_k)} \frac{d\mu_k(z_k)}{1-|z_k|}\right)^{\frac{1}{1-\lambda}} dm_n(\xi)\right]^{1-\lambda}.$$

Proof for the case $n = 2$: (for general n the proof is similar). Let $z = (z_1, z_2)$ and $\xi = (\xi_1, \xi_2)$. Because $(1-\lambda) + \sum_{i=1}^{m} q_i/p_i = 1$, by repeated application of (2.6) and of Hölder's inequality we get

$$\int_{\mathbb{D}^2} \prod_{i=1}^{m} |f_i(z_1, z_2)|^{q_i}\, d\mu_1(z_1)\, d\mu_2(z_2)$$
$$\leq C \int_T (A_\infty)_{z_2} \int_T (A_\infty)_{z_1} \left(\prod_{i=1}^{m} |f_i|^{q_i}\right) \times$$
$$\times \left(\prod_{k=1}^{2} \int_{\Gamma_k(\xi_k)} \frac{d\mu_k(z_k)}{1-|z_k|}\right) dm_1(\xi_1)\, dm_1(\xi_2)$$
$$\leq C \int_T (A_\infty)_{z_2} \prod_{i=1}^{m} \left(\int_T (A_\infty)_{z_1} |f_i|^{p_i}\, dm_1(\xi_1)\right)^{q_i/p_i} \times$$
$$\times \left(\int_T \left(\prod_{k=1}^{2} \int_{\Gamma_k(\xi_k)} \frac{d\mu_k(z_k)}{1-|z_k|}\right)^{1/(1-\lambda)} dm_1(\xi_1)\right)^{1-\lambda} dm_1(\xi_2)$$
$$\leq C \prod_{i=1}^{m} \left(\int_T (A_\infty)_{z_2} \int_T (A_\infty)_{z_1} |f_i|^{p_i}\, dm_1(\xi_1)\, dm_1(\xi_2)\right)^{q_i/p_i} \times$$
$$\times \left(\int_T \int_T \left(\prod_{k=1}^{2} \int_{\Gamma_k(\xi_k)} \frac{d\mu_k(z_k)}{1-|z_k|}\right)^{1/(1-\lambda)} dm_1(\xi_1)\, dm_1(\xi_2)\right)^{1-\lambda}$$
$$\leq C \left(\prod_{i=1}^{m} \|f_i\|_{M_\infty H^{p_i}}^{q_i}\right) \left[\int_{T^2} \left(\prod_{k=1}^{2} \int_{\Gamma_\alpha(\xi_k)} \frac{d\mu_k(z_k)}{1-|z_k|}\right)^{\frac{1}{1-\lambda}} dm_2(\xi)\right]^{1-\lambda}.$$

The proof is complete.

Remark 4. *(2.6) is in fact true for the unit ball $\mathbb{B}_n$ (see p. 138 of [2]). Therefore Propositions 1 and 2 hold for the unit ball $\mathbb{B}_n$.*

3. SHARP ESTIMATES FOR MULTIFUNCTIONAL ANALYTIC EXPRESSIONS VIA STRONG FACTORIZATION THEOREMS IN THE UNIT DISK

Let X, X_1 and X_2 be subspaces of $H(\mathbb{D})$, the space of analytic functions on $\mathbb{D}$, with quasinorms $\|\cdot\|_X$, $\|\cdot\|_{X_1}$ and $\|\cdot\|_{X_2}$. We say that $X \subset H(\mathbb{D})$ admits strong factorization if $X = X_1 \cdot X_2$, that is, for any $f \in X$, there exist functions $f_1 \in X_1$ and $f_2 \in X_2$ such that $f = f_1 f_2$; and conversely, for any $f_1 \in X_1$ and $f_2 \in X_2$ we have $f = f_1 f_2 \in X$. Therefore, if $X = X_1 \cdot X_2$, then

$$\int_{\mathbb{D}} |f|^q \, d\mu(z) \le C\|f\|_X^q$$

if and only if

$$\int_{\mathbb{D}} |f_1|^q |f_2|^q \, d\mu(z) \le C\|f_1\|_{X_1}^q \|f_2\|_{X_2}^q,$$

where $0 < q < \infty$. For example, it is known that for the Hardy spaces, if $0 < p, q, p_1, p_2 < \infty$, and

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$$

then $H^p = H^{p_1} \cdot H^{p_2}$. Thus if μ is a (H^p, q)-Carleson measure, i.e., if

$$\int_{\mathbb{D}} |f(z)|^q \, d\mu(z) \le C\|f\|_{H^p},$$

then

$$\int_{\mathbb{D}} |f_1(z)|^q |f_2(z)|^q \, d\mu(z) \le C\|f_1\|_{H^{p_1}}^q \|f_2\|_{H^{p_2}}^q.$$

By similar steps we can get that, if

$$\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_n},$$

then μ is a (H^p, q)-Carleson measure if and only if

$$\int_{\mathbb{D}} |f_1|^q \cdots |f_n|^q \, d\mu \le C\|f_1\|_{H^{p_1}} \cdots \|f_n\|_{H^{p_n}},$$

where $p, q, p_1, \dots, p_n < \infty$. Therefore, we get an easy proof of a special case of Theorem 1 in [11].

This approach can be developed further following [3], where several strong factorization theorems were given.

For $f(z) = \sum_{k=0}^{\infty} a_k z^k$, we denote by $D^s f(z) = \sum_{k=0}^{\infty} (k+1)^s a_k z^k$ the fractional derivative of f. For $s > 0$, $0 < p < \infty$, let $H_s^p = \{f \in H^p : D^s f \in H^p\}$ be the Hardy-Sobolev space, and $H_{-s}^p = \{D^s f : f \in H^p\}$ be the space of fractional derivatives of functions in H^p. We write $BMOA_s = \{f : D^s f \in BMOA\}$ and $BMOA_{-s} = \{D^s f : f \in BMOA\}$. For $s = 0$, we denote $H_0^p = H^p$, and $BMOA_0 = BMOA$.

For $s < 0$ and $0 < p, q < \infty$, let $F_{s,q}^p$ be the holomorphic Triebel-Lizorkin space which consists of holomorphic functions f on $\mathbb{D}$ such that

$$\int_{\mathbb{T}} \left(\int_0^1 |f(r\zeta)|^q (1-r^2)^{-qs-1}\, dr \right)^{p/q} dm_1(\zeta).$$

It is known that for $s < 0$, $F_{s,2}^p = H_s^p$ and $F_{s,2}^\infty = BMOA_s$.

In [3] it has been proved that, for $s < 0$,

$$H_s^p = H_s^r \cdot H^t, \quad \frac{1}{p} = \frac{1}{r} + \frac{1}{t},$$

$$H_s^p = H^p \cdot BMOA_s,$$

and more generally,

$$F_{s,q}^p = F_{s,q}^\infty \cdot H^p.$$

The following result is a consequence of the first two strong factorizations and $H^p = H^{p_1} \cdot H^{p_2}$ $(1/p = 1/p_1 + 1/p_2)$.

Proposition 3. *Let $s < 0$, let $0 < p, q < \infty$. Let μ be a positive measure on $\mathbb{D}$. Then the following statements are equivalent.*

(i) *μ is an (H_s^p, q)-Carleson measure;*

(ii) *For $0 < r < \infty$ and $0 < p_i < \infty$, $i = 1, \dots, m$ satisfying $1/p = 1/r + 1/p_1 + \cdots + 1/p_m$,*

$$\int_{\mathbb{D}} |f_1(z)|^q \cdots |f_m(z)|^q |f(z)|^q \, d\mu(z) \le C \|f_1\|_{H^{p_1}}^q \cdots \|f_m\|_{H^{p_m}}^q \|f\|_{H_s^r}^q,$$

for any $f_i \in H^{p_i}$, $i = 1, \dots, m$, and any $f \in H_s^r$;

(iii) *For $0 < p_i < \infty$, $i = 1, \dots, m$, satisfying $1/p = 1/p_1 + \cdots + 1/p_m$,*

$$\int_{\mathbb{D}} |f_1(z)|^q \cdots |f_m(z)|^q |f(z)|^q \, d\mu(z) \le C \|f_1\|_{H^{p_1}}^q \cdots \|f_m\|_{H^{p_m}}^q \|f\|_{BMOA_s}^q.$$

for any $f_i \in H^{p_i}$ $(i = 1, \dots, m)$ and any $f \in BMOA_s$.

The same result can be obtained using $F_{s,q}^p = F_{s,q}^\infty \cdot H^p$ and other factorizations.

For the weighted Djrbashian space A_α^p, consisting of analytic functions f on $\mathbb{D}$ satisfying

$$\|f\|_{A_\alpha^p}^p = \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha \, dm_2(z) < \infty,$$

where $0 < p < \infty$ and $-1 < \alpha < \infty$, Horowitz [7] obtained the following strong factorization result. For $\alpha \geq 0$ and $0 < p, p_1, \dots, p_m < \infty$, with $1/p = 1/p_1 + \cdots + 1/p_m$,

$$A_\alpha^p = \prod_{i=1}^m A_\alpha^{p_i}.$$

Using this, we get a result on Carleson measures on weighted Djrbashian spaces.

Proposition 4. *Let $0 < p, p_1, \dots, p_m, q < \infty$ satisfy $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, let $0 \leq \alpha < \infty$. Let μ be a positive measure on $\mathbb{D}$. Then μ is an (A_α^p, q)-Carleson measure if and only if*

$$\int_{\mathbb{D}} |f_1(z)|^q \cdots |f_m(z)|^q \, d\mu(z) \leq C \|f_1\|_{A_\alpha^{p_1}}^q \cdots \|f_m\|_{A_\alpha^{p_m}}^q.$$

for any $f_i \in A_\alpha^{p_i}$ $(i = 1, \dots, m)$.

4. INEQUALITIES FOR MULTILINERAR POISSON AND CAUCHY–SZEGÖ TRANSFORMS

Our intention now is to prove some multilinear results for multilinear Poisson and Cauchy-Szegö transforms and also for operators of multilinear type in the unit ball $\mathbb{B}_n$. More precisely, we estimate in $\mathbb{S}_n$ and $\mathbb{B}_n$ the multilinear operator

$$T(f_1, f_2, \dots, f_m)(z) = \int_{\mathbb{S}_n} \frac{f_1(\xi) \cdots f_m(\xi)}{|1 - \langle z, \xi \rangle|^n} \, d\sigma(\xi), \quad z \in \mathbb{B}_n,$$

where $f_i \in L^1(\mathbb{S}_n, d\sigma)$ and then follow the path proposed in Zhu's book [13] to obtain estimates from above for the corresponding multilinear Poisson transform

$$P(f_1, f_2, \dots, f_m)(z) = \int_{\mathbb{S}_n} f_1(\xi) \cdots f_m(\xi) P(z, \xi) \, d\sigma(\xi),$$

and Cauchy-Szegö projection

$$C(f_1, f_2, \dots, f_m)(z) = \int_{\mathbb{S}_n} f_1(\xi) \cdots f_m(\xi) C(z, \xi) \, d\sigma(\xi),$$

Recall that

$$P(z, \xi) = \frac{(1-|z|^2)^n}{|1 - \langle z, \xi \rangle|^{2n}}$$

and

$$C(z,\xi)=\frac{1}{(1-\langle z,\xi\rangle)^n}$$

are Poisson and Cauchy-Szegö kernels.

Following Chapter 4 in Zhu's book [13], let μ be a complex Borel measure on $\mathbb{S}_n$. Let $|\mu|$ denote the total variation of μ, so $|\mu|$ becomes a positive Borel measure on $\mathbb{S}_n$. For $z,w\in\overline{\mathbb{B}_n}$, let $d(z,w)=|1-\langle z,w\rangle|^{1/2}$. For $\zeta\in\mathbb{S}_n$ and $\delta>0$, let $Q=Q(\zeta,\delta)=\{\eta\in\mathbb{S}_n : d(\zeta,\eta)<\delta\}$. Thus $Q(\zeta,\eta)$ is the nonisotropic metric ball at ζ with radius δ. We will call them d-balls. We define the following maximal functions.

$$(Mf)(\zeta)=\sup_{\delta>0}\frac{1}{\sigma(Q(\zeta,\delta))}\int_{Q(\zeta,\delta)}f(z)\,d\sigma(z),$$

and

$$M_\alpha(f)(\zeta)=\sup_{z\in D_\alpha(\zeta)}|f(z)|,$$

where

$$D_\alpha(\xi)=\{z:\,|1-\langle z,\xi\rangle|<\frac{\alpha}{2}(1-|z|)\}.$$

The following results can be found in Chapter 4 and Chapter 5 in Zhu's book [13].

(A) For $0<p<\infty$, $\alpha>1$ and $f\in H^p$,

$$\int_{\mathbb{S}_n}|(M_\alpha f)(z)|^p\,d\sigma(z)\le C\|f\|^p_{H^p}.$$

(B) For $1<p<\infty$ and $f\in L^p(\mathbb{S}_n,\sigma)$,

$$\int_{\mathbb{S}_n}|(Mf)(z)|^p\,d\sigma(z)\le C\|f\|^p_{L^p}.$$

(C) For $1<p<\infty$, $\alpha>1$ and $f\in L^p(\mathbb{S}_n,\sigma)$,

$$\int_{\mathbb{S}_n}|(M_\alpha P(f))(z)|^p\,d\sigma(z)\le C\|f\|^p_{L^p}.$$

(D) For $1<p<\infty$, $\alpha>1$ and $f\in L^p(\mathbb{S}_n,\sigma)$,

$$\int_{\mathbb{S}_n}(M_\alpha C(f)(z))^p\;d\sigma(z)\le C\|f\|^p_{L^p}.$$

(E) For $z\in\mathbb{B}_n$, $z\neq 0$, let

$$Q_z=Q(z/|z|,\sqrt{1-|z|})=\{\zeta\in\mathbb{S}_n\,|1-\langle z/|z|,\zeta\rangle|<1-|z|\}.$$

If μ is a Carleson measure, then for any $1<p<\infty$ and $f\in L^p(\mathbb{S}_n,\sigma)$,

$$\int_{\mathbb{B}_n}\left(\sup_Q\frac{1}{\sigma(Q)}\int_Q|f|\,d\sigma\right)^p\,d\mu(z)\le C\|f\|^p_{L^p}.$$

where the supremum is taken over all d-balls Q in $\mathbb{S}_n$ such that $Q_z\subset Q$.

We now get the multifunctional versions of these inequalities. First we generalize (A) in the following way.

Proposition 5. *Let* $0 < p_i, q_i < \infty$, $i = 1, 2, \ldots, m$ *and* $\alpha > 1$. *If* $0 < s < \infty$ *satisfies* $1/s = \sum_{i=1}^m (q_i/p_i)$, *then there is a positive constant* C *such that, for* $f_i \in H^{p_i}$, $i = 1, \ldots, m$,

$$\left\| M_\alpha \left(\prod_{i=1}^m |f_i|^{q_i} \right) \right\|_{L^s} \le C \prod_{i=1}^m \|f_i\|_{H^{p_i}}^{q_i}.$$

Proof: For $m = 1$, the result is known from Theorem 4.24 in [13]. Let $m \ge 2$. Then

$$\frac{p_i}{sq_i} = \frac{p_i}{q_i} \sum_{i=1}^m \frac{q_i}{p_i} > 1.$$

Hence, we can apply Hölder's inequality and Theorem 4.24 in [13] to obtain

$$\begin{aligned}
\int_{\mathbb{S}_n} \left[M_\alpha \left(\prod_{i=1}^m |f_i(z)|^{q_i} \right) \right]^s d\sigma(z) &\le \int_{\mathbb{S}_n} \prod_{i=1}^m M_\alpha(|f_i(z)|^{sq_i})\, d\sigma(z) \\
&\le \prod_{i=1}^m \left(\int_{\mathbb{S}_n} (M_\alpha |f_i(z)|)^{(pq_i/q)[(p_i)/(sq_i)]}\, d\sigma(z) \right)^{(sq_i)/(p_i)} \\
&= \prod_{i=1}^m \left(\int_{\mathbb{S}_n} M_\alpha(|f_i(z)|^{p_i})\, d\sigma(z) \right)^{(sq_i)/(p_i)} \le C \prod_{i=1}^m (\|f_i\|_{H^{p_i}}^{q_i})^s,
\end{aligned}$$

implying the proposition.

Here we point out some special cases of the above result.

1. As $\sum_{i=1}^m (q_i/p_i) = 1$, then $s = 1$ and so we have

$$\int_{\mathbb{S}_n} M_\alpha \left(\prod_{i=1}^m |f_i(z)|^{q_i} \right) d\sigma(z) \le C \prod_{i=1}^m \|f_i\|_{H^{p_i}}^{q_i}.$$

2. As $q_i = p_i$, then $1/s = \sum_{i=1}^m 1 = m$. Thus

$$\left\| M_\alpha \left(\prod_{i=1}^m |f_i|^{q_i} \right) \right\|_{L^{1/m}} \le C \prod_{i=1}^m \|f_i\|_{H^{p_i}}^{p_i}.$$

3. If $1/\tilde{p} = \sum_{i=1}^m (1/p_i)$ and $q_i = \tilde{p}$, $i = 1, \ldots, m$, then $1/s = \sum_{i=1}^m (q_i/p_i) = \tilde{p} \sum_{i=1}^m (1/p_i) = 1$, and so

$$\int_{\mathbb{S}_n} M_\alpha \left(\prod_{i=1}^m |f_i(z)|^{\tilde{p}} \right) d\sigma(z) \le C \prod_{i=1}^m \|f_i\|_{H^{p_i}}^{\tilde{p}}.$$

Using (B), we can get the multifunctional analog of (B).

Proposition 6. *Let $1 < p_i < \infty$ and $0 < q_i < \infty$, $i = 1, 2, \ldots, m$. If $0 < s < \infty$ satisfies $1/s = \sum_{i=1}^m (q_i/p_i)$, then there exists a positive constant C such that, for $f_i \in H^{p_i}$, $i = 1, \ldots, m$,*

$$\left\| M\left(\prod_{i=1}^m |f_i|^{q_i}\right)\right\|_{L^s} \le C \prod_{i=1}^m \|f_i\|_{H^{p_i}}^{q_i}.$$

Proposition 7. *(multifunctional versions of (C) and (D)). Let $1 < \alpha, p_i < \infty$, $i = 1, 2, \ldots, m$ and $1/p = \sum_{i=1}^m (1/p_i)$. If $1 < p < \infty$, then*

(i) *there is a positive constant C_1 such that, for $f_i \in L^{p_i}(\mathbb{S}_n, \sigma)$, $i = 1, \ldots, m$,*

$$\|M_\alpha P(f_1, \ldots, f_m)\|_{L^p} \le C_1 \prod_{i=1}^m \|f_i\|_{L^{p_i}}^{q_i}.$$

(ii) *there is a positive constant C_2 such that, for $f_i \in L^{p_i}(\mathbb{S}_n, d\sigma)$, $i = 1, \ldots, m$,*

$$\|M_\alpha C(f_1, \ldots, f_m)\|_{L^p} \le C_2 \prod_{i=1}^m \|f_i\|_{L^{p_i}}^{q_i}.$$

Proof: Both of these inequalities follow from the boundedness of the operators $M_\alpha P$ and $M_\alpha C$ on $L^p(\mathbb{S}_n, d\sigma)$ (see, Corollary 4.11 and Theorem 4.35 in [13]) and the Hölder inequality. We omit the details.

Proposition 8. *(multifunctional version of (E). Let $1 < p_i < \infty$ and $0 < q_i < \infty$, $i = 1, 2, \ldots, m$. Let $0 < s < \infty$ satisfy $1/s = \sum_{i=1}^m (q_i/p_i)$. Let μ be a Carleson measure. Then there is a positive constant C such that, for $f_i \in L^{p_i}(\mathbb{S}_n, d\sigma)$, $i = 1, \ldots, m$,*

$$\left(\int_{\mathbb{B}_n} \left(\sup_Q \frac{1}{\sigma(Q)} \int_Q \prod_{i=1}^m |f_i|^{q_i}\, d\sigma\right)^s d\mu(z)\right)^{1/s} \le C \prod_{i=1}^m \|f_i\|_{L^{p_i}}^{q_i},$$

where the supremum is taken over all d-balls Q in $\mathbb{S}_n$ such that $Q_z \subset Q$.

Proof is similar to that of Proposition 5.

Now let us go back to the multilinear operator $T(f_1, \ldots, f_m)(z)$ defined in the beginning of the section. For the case of one function, the following estimate is known, see, for example, Lemma 4.44 in [13].

Lemma 2. *Let $1 \le p < q < \infty$. There exists a positive constant C such that*

$$\left(\int_{\mathbb{S}_n} |T(f)(r\eta)|^q\, d\sigma(\eta)\right)^{1/q} \le C(1 - r^2)^{n(1/q - 1/p)} \|f\|_p$$

for all $f \in L^p(\mathbb{S}_n, d\sigma)$ and $0 < r < 1$.

Proposition 9. *Let $1 \le p_i < \infty$, $i = 1, \ldots, m$, let $1/\tilde{p} = \sum_{i=1}^m (1/p_i)$. If $1 < \tilde{p} < q < \infty$, then there exists a positive constant C such that*

$$\left(\int_{\mathbb{S}_n} |T(f_1, \ldots, f_m)(r\eta)|^q \, d\sigma(\eta) \right)^{1/q} \le C(1 - r^2)^{n(1/q - 1/\tilde{p})} \prod_{i=1}^m \|f_i\|_{p_i}$$

for all $f_i \in L^{p_i}(\mathbb{S}_n, d\sigma)$, $i = 1, \ldots, m$, and $0 < r < 1$.

Proof: Let $\tilde{f} = f_1 \cdots f_n$. By Hölder's inequality $\tilde{f} \in H^{\tilde{p}}$, and

$$\|\tilde{f}\|_{H^{\tilde{p}}} \le \prod_{i=1}^m \|f_i\|_{H^{p_i}}. \tag{4.1}$$

By Lemma 2

$$\left(\int_{\mathbb{S}_n} |T(\tilde{f})(r\eta)|^q \, d\sigma(\eta) \right)^{1/q} \le C(1 - r^2)^{n(1/q - 1/\tilde{p})} \|\tilde{f}\|_{\tilde{p}}.$$

Combining with (4.1), we get the proposition.

Clearly, for multilinear Poisson transform and Cauchy-Szegö projection we have

$$|P(f_1, \ldots, f_n)(z)| \le 2^n |T(f_1, \ldots, f_n)(z)| \tag{4.2}$$

and

$$|C(f_1, \ldots, f_n)(z)| \le |T(f_1, \ldots, f_n)(z)|. \tag{4.3}$$

Hence the following result.

Corollary 1. Let $1 \le p_i < \infty$, $i = 1, \ldots, m$, and $1/\tilde{p} = \sum_{i=1}^m (1/p_i)$. If $1 < \tilde{p} < q < \infty$, then there exists a positive constant C such that

$$\left(\int_{\mathbb{S}_n} |P(f_1, \ldots, f_m)(r\eta)|^q \, d\sigma(\eta) \right)^{1/q} \le C(1 - r^2)^{n(1/q - 1/\tilde{p})} \prod_{i=1}^m \|f_i\|_{p_i}$$

and

$$\left(\int_{\mathbb{S}_n} |C(f_1, \ldots, f_m)(r\eta)|^q \, d\sigma(\eta) \right)^{1/q} \le C(1 - r^2)^{n(1/q - 1/\tilde{p})} \prod_{i=1}^m \|f_i\|_{p_i}$$

for all $f_i \in L^{p_i}(\mathbb{S}_n, d\sigma)$, $i = 1, \ldots, m$, and $0 < r < 1$.

In fact, some better estimates are known as, for example, Theorem 4.46 and Corollary 4.47 in [13].

Proposition 10. *Let $1 < p < q < \infty$. Then there exists a positive constant C such that*

$$\left(\int_{\mathbb{B}_n} |T(f)(z)|^q (1-|z|^2)^{qn(1/p-1/q)-1}\, dv(z)\right)^{1/q} \leq C\|f\|_p$$

for all $f \in L^p(\mathbb{S}_n, d\sigma)$.

Applying Proposition 7, we get by (4.1):

Proposition 11. *Let* $1 \leq p_i < \infty$, $i = 1, \ldots, m$ *and* $1/\tilde{p} = \sum_{i=1}^m (1/p_i)$. *If* $1 < \tilde{p} < q < \infty$, *then there exists a positive constant* C *such that*

$$\left(\int_{\mathbb{B}_n} |T(f_1, \ldots, f_m)(z)|^q (1-|z|^2)^{qn(1/\tilde{p}-1/q)-1}\, dv(z)\right)^{1/q} \leq C \prod_{i=1}^m \|f_i\|_{p_i}$$

for all $f_i \in L^{p_i}(\mathbb{S}_n, d\sigma)$, $i = 1, \ldots, m$.

>From this result, using (4.2) and (4.3) we can get the corresponding estimates for the Poisson transform $P(f_1, \ldots, f_m)$ and Cauchy-Szegö projection $C(f_1, \ldots, f_m)$.

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