Известия НАН Армении. Математика, том 43, н. 3, 2008, стр. 13-24

A GENERALIZATION OF 4-DIMENSIONAL LOCALLY CONFORMAL FLAT WALKER MANIFOLDS

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Аннотация. A 4-dimensional Walker metrics with $c = 0$ on a semi-Riemannian manifold M have been investigated by E. García-Río and Y.Matsushita. The case c=constant has been studied in [1]. In this paper we generalize these notions to the case of non–constant c. We find the form of the defining functions that makes this manifold similar to locally conformal flat 4-dimensional Walker manifold.

1. INTRODUCTION

A Walker n-manifold admits a field of parallel null r-planes, with $r \leq \frac{n}{2}$. The canonical form of a metric studied by Walker [5] contains three functions $a(x, y, z, t)$, $b(x, y, z, t)$ and $c(x, y, z, t)$. We consider 4-dimensional Walker manifolds with parallel null 2-planes. In [3] locally conformal flat Walker manifolds were investigated in the restricted form of metric when $c(x, y, z, t) = 0$. The case c= constant is studied in [1]. In this paper following [1] and [3], we focus upon the case of arbitrary non-constant c=$c(x, y, z, t)$, and investigate locally conformal flat 4-dimensional Walker manifolds admitting parallel null 2-planes.

Definition 1. *A Walker manifold is a triad* (M, g, D) *of an n-dimension manifold* M*, an indefinite metric* g *and an r-dimensional parallel distribution* D.

If $\dim M = 4$ and $\dim D = 2$, g has the signature (- -++) and in suitable coordinates

$$g(x,y,z,t) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a(x,y,z,t) & c(x,y,z,t) \\ 0 & 1 & c(x,y,z,t) & b(x,y,z,t) \end{pmatrix} \tag{1.1}$$

for some functions $a(x,y,z,t)$, $b(x,y,z,t)$, $c(x,y,z,t)$ and where $D = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle$ (cf.[5,6]). The plane D is strictly parallel if and only if

$$g(x,y,z,t) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a(z,t) & c(z,t) \\ 0 & 1 & c(z,t) & b(z,t) \end{pmatrix}. \tag{1.2}$$

In this case the coordinate system can be chosen to satisfy $a(z,t) \equiv 0$, $c(z,t) \equiv 0$ (cf.[5]). In this paper, we are interested in a general case of the Walker metric g as in (1.1).

2. CURVATURE TENSOR OF 4-DIMENSIONAL NEUTRAL METRIC ADMITTING A PARALLEL NULL 2-PLANE

In suitable coordinates the canonical form of Walker metric has been obtained by Walker in [5, 6], where the metric expresses as in (1.1).

It follows, after some straightforward calculations, that the Levi Civita connection of a Walker metric (1.1) is given by

$$\begin{aligned}
\nabla_{\partial_x}\partial_z &= \frac{1}{2}a_x\partial_x + \frac{1}{2}c_x\partial_y \\
\nabla_{\partial_x}\partial_t &= \frac{1}{2}c_x\partial_x + \frac{1}{2}b_x\partial_y \\
\nabla_{\partial_y}\partial_z &= \frac{1}{2}a_y\partial_x + \frac{1}{2}c_y\partial_y \\
\nabla_{\partial_y}\partial_t &= \frac{1}{2}c_y\partial_x + \frac{1}{2}b_y\partial_y \\
\nabla_{\partial_z}\partial_z &= \frac{1}{2}(a_z + ca_y + aa_x)\partial_x + \frac{1}{2}(2c_z - a_t + ba_y + ca_x)\partial_y - \frac{1}{2}a_x\partial_z - \frac{1}{2}a_y\partial_t \\
\nabla_{\partial_z}\partial_t &= \frac{1}{2}(a_t + cc_y + ac_x)\partial_x + \frac{1}{2}(bc_y + b_z + cc_x)\partial_y - \frac{1}{2}c_x\partial_z - \frac{1}{2}c_y\partial_t \\
\nabla_{\partial_t}\partial_t &= \frac{1}{2}(2c_t - b_z + cb_y + ab_x)\partial_x + \frac{1}{2}(b_t + bb_y + cb_x)\partial_y - \frac{1}{2}b_x\partial_z - \frac{1}{2}b_y\partial_t,
\end{aligned}$$

where a_x means partial derivative $\frac{\partial}{\partial x}a(x,y,z,t)$ and ∂_k denotes the coordinate vector field $\frac{\partial}{\partial k}$, $k = x,y,z,t$. Hence, if (M,g) admits strictly parallel vector field, then the associated Levi-Civita connection satisfies

$$\begin{aligned}
\nabla_{\partial_z}\partial_z &= \frac{1}{2}a_z\partial_x + \frac{1}{2}(2c_z - a_t)\partial_y \\
\nabla_{\partial_z}\partial_t &= \frac{1}{2}a_t\partial_x + \frac{1}{2}b_z\partial_y
\end{aligned}$$

$$\nabla_{\partial_t}\partial_t = \frac{1}{2}(2c_t - b_z)\partial_x + \frac{1}{2}b_t\partial_y.$$

Let R denote the curvature tensor, taken with the sign convention $R(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$. Then, the nonzero components of the curvature tensor of Walker metric (1.1) are computed as:

$$R_{\partial_z\partial_x}\partial_x = \frac{1}{2}a_{xx}\partial_x + \frac{1}{2}c_{xx}\partial_y$$
$$R_{\partial_z\partial_x}\partial_y = \frac{1}{2}a_{yx}\partial_x + \frac{1}{2}c_{xy}\partial_y$$
$$R_{\partial_z\partial_x}\partial_z = \frac{1}{2}(ca_{yx} + aa_{xx})\partial_x + \frac{1}{4}(2c_{zx} + a_yb_x + 2ba_{yx} + 2ca_{xx} - 2a_{tx} - c_yc_x)\partial_y$$
$$-\frac{1}{2}a_{xx}\partial_z - \frac{1}{2}a_{yx}\partial_t$$
$$R_{\partial_z\partial_x}\partial_t = \frac{1}{4}(2a_{tx} + c_xc_y + 2cc_{yx} - 2c_{xz} + 2ac_{xx} - b_xa_y)\partial_x + \frac{1}{2}(cc_{xx} + bc_{xy})\partial_y$$
$$-\frac{1}{2}c_{xx}\partial_z - \frac{1}{2}c_{yx}\partial_t$$
$$R_{\partial_t}\partial_x\partial_x = \frac{1}{2}c_{xx}\partial_x + \frac{1}{2}b_{xx}\partial_y$$
$$R_{\partial_t\partial_x}\partial_y = \frac{1}{2}c_{xy}\partial_x + \frac{1}{2}b_{yx}\partial_y$$

$$R_{\partial_t\partial_x}\partial_z = \frac{1}{2}(cc_{xy} + ac_{xx})\partial_x + \frac{1}{4}(-2c_{xt} + 2b_{zx} + c_x^2 - c_xb_y + b_xc_y + 2bc_{yx}$$
$$-a_xb_x + 2cc_{xx})\partial_y - \frac{1}{2}c_{xx}\partial_z - \frac{1}{2}c_{xy}\partial_t$$
$$R_{\partial_t}\partial_x\partial_t = \frac{1}{4}(2c_{tx} - 2b_{zx} + a_xb_x + b_yc_x - c_x^2 - b_xc_y + 2cb_{xx} + a_xb_x)\partial_x$$
$$+\frac{1}{2}(bb_{yx} + cb_{xx})\partial_y - \frac{1}{2}b_{xx}\partial_z - \frac{1}{2}b_{yx}\partial_t$$

$$R_{\partial_z}\partial_y\partial_x = \frac{1}{2}a_{xy}\partial_x + \frac{1}{2}c_{xy}\partial_x$$
$$R_{\partial_z\partial_y}\partial_y = \frac{1}{2}a_{yy}\partial_x + \frac{1}{2}c_{yy}\partial_x$$
$$R_{\partial_z\partial_y}\partial_z = \frac{1}{2}(ca_{yy} + aa_{yx})\partial_x + \frac{1}{4}(2c_{zy} - 2a_{ty} + a_yb_y + 2ba_{yy} + 2ca_{xy} + a_xc_y$$
$$-a_yc_x - c_y^2)\partial_y - \frac{1}{2}a_{xy}\partial_z - \frac{1}{2}a_{yy}\partial_t$$

$$R_{\partial_z}\partial_y\partial_t = \frac{1}{4}(2a_{ty} - 2c_{yz} + c_y^2 + 2cc_{yy} + a_yc_x - c_ya_x - b_ya_y + 2ac_{xy})\partial_x$$
$$+\frac{1}{2}(bc_{yy} + cc_{xy})\partial_y - \frac{1}{2}(c_{xy})\partial_z - \frac{1}{2}c_{yy}\partial_t$$

$$R_{\partial_t}\partial_y\partial_x = \frac{1}{2}c_{xy}\partial_x + \frac{1}{2}b_{xy}\partial_y$$

$$R_{\partial_t\partial_y}\partial_y = \frac{1}{2}c_{yy}\partial_y + \frac{1}{2}b_{yy}\partial_y$$

$$R_{\partial_t\partial_y}\partial_z = \frac{1}{2}(cc_{yy} + ac_{xy} + c_y^2)\partial_y + \frac{1}{4}(2b_{zy} - 2c_{yt} + 2bc_{yy} + c_yc_x + 2cc_{xy} \\ -a_yb_x)\partial_y - \frac{1}{2}c_{xy}\partial_z - \frac{1}{2}c_{yy}\partial_t$$

$$\begin{aligned}
R_{\partial_t}\partial_y\partial_t =& \frac{1}{4}(2c_{ty} - 2b_{zy} - c_yc_x + 2cb_{yy} + a_yb_x + 2ab_{xy})\partial_x + \frac{1}{2}(bb_{yy} + cb_{xy})\partial_y \\
& - \frac{1}{2}b_{xy}\partial_z - \frac{1}{2}b_{yy}\partial_t \\
R_{\partial_t}\partial_z\partial_x =& \frac{1}{4}(2c_{xz} + b_xa_y - 2a_{tx} - c_xc_y)\partial_x + \frac{1}{4}(c_x^2 + 2b_{xz} + b_xc_y - a_xb_x \\
& - 2c_{xt} - c_xb_y)\partial_y \\
R_{\partial_t\partial_z}\partial_y =& \frac{1}{4}(2c_{yz} + b_ya_y - 2a_{yt} - c_y^2 + c_ya_x - a_yc_x + c_yc_x)\partial_x \\
& + \frac{1}{4}(2b_{yz} - a_yb_x - 2c_{yt})\partial_y \\
R_{\partial_t\partial_z}\partial_z =& \frac{1}{4}(ca_xc_y - cc_xa_y - cc_y^2 - 2ca_{yt} - 2aa_{xt} + ca_yb_y + aa_yb_x + 2cc_{yz} \\
& + 2ac_{xz} - ac_yc_x)\partial_x + \frac{1}{4}(2b_{zz} + 2a_{tt} - 2c_ta_x + a_xb_z - b_xa_z \\
& - a_yb_t + b_ya_t - 2a_{yt}b - 2ca_{xt} + 2b_zc_y - aa_xb_x - ca_xb_y \\
& + 2bc_{yz} + 2cc_{xz} - 4c_{zt} + 2c_xa_t + cc_xc_y + ac_x^2 - bc_xa_y - 2b_yc_z \\
& + a_xbc_y)\partial_y + \frac{1}{4}(2a_{xt} - a_yb_x - 2c_{xz})\partial_z \\
& + \frac{1}{4}(2a_{yt} - a_yb_y + a_yc_x - 2c_{yz} - a_xc_y + c_y^2)\partial_t \\
R_{\partial_t\partial_z}\partial_t =& \frac{1}{4}(4c_{tz} - 2b_{zz} + a_zb_x + 2ab_{xz} - 2a_{tt} - b_za_x + 2cb_{yz} + b_ta_y \\
& - b_ya_t + bb_ya_y + 2c_zb_y + 2a_xc_t + ca_xb_y - ab_yc_x - 2cc_{yt} \\
& - 2a_tc_x - 2ac_{xt} - 2c_yb_z - bc_y^2 - cc_xc_y + ac_yb_x)\partial_x \\
& + \frac{1}{4}(cc_yb_x - 2bc_{yt} - 2cc_{xt} + bc_xc_y + cc_x^2 + 2bb_{yz} + 2cb_{xz} - b_xba_y \\
& - cb_xa_x - b_ycc_x)\partial_y + \frac{1}{4}(-2b_{xz} + b_xa_x + b_yc_x + 2c_{xt} - c_x^2 - b_xc_y)\partial_z \\
& + \frac{1}{4}(b_xa_y - c_xc_y + 2c_{yt} - 2b_{yz})\partial_t. \qquad (2.1)
\end{aligned}$$

Further note that the existence of strictly parallel vector field simplifies (2.1) as follows:

$$R_{\partial_t\partial_z}\partial_z = \frac{1}{2}(b_{zz} + a_{tt} - 2c_{zt})\partial_y$$

$$R_{\partial_t\partial_z}\partial_t = \frac{1}{2}(2c_{tz} - b_{zz} - a_{tt})\partial_x. \tag{2.2}$$

3. LOCALLY CONFORMAL FLAT METRICS

A semi-Riemannian manifold is locally conformal flat if and only if its Weyl tensor, given by:

$$W(X,Y,Z,V) = R(X,Y,Z,V) - \frac{Sc}{(n-1)(n-2)}\big(g(X,Z)g(Y,V) - g(X,V)g(Y,Z)\big)$$

$$-\frac{1}{(n-2)}\big(Ric(X,Z)g(Y,V) - Ric(Y,Z)g(X,V) + Ric(Y,V)g(X,Z)g(X,V)g(Y,Z)\big)$$

vanishes. The nonzero components of Weyl tensor of a Walker metric (1.1) are calculated as:

$$\begin{aligned}
W(\partial_x,\partial_y,\partial_z,\partial_t) &= -\frac{1}{12}a_{xx} - \frac{1}{6}c_{xy} - \frac{1}{12}b_{yy}\\
W(\partial_x,\partial_z,\partial_x,\partial_z) &= -\frac{1}{6}a_{xx} + \frac{1}{6}c_{xy} - \frac{1}{6}b_{yy}\\
W(\partial_x,\partial_z,\partial_x,\partial_t) &= -\frac{1}{4}c_{xx} + \frac{1}{4}b_{xy}\\
W(\partial_x,\partial_z,\partial_y,\partial_z) &= -\frac{1}{4}a_{xy} + \frac{1}{4}c_{yy}\\
W(\partial_x,\partial_z,\partial_y,\partial_t) &= -\frac{1}{2}c_{yx}\\
W(\partial_x,\partial_z,\partial_z,\partial_t) &= \frac{1}{4}a_{xt} - \frac{1}{4}c_{zx} - \frac{5}{12}cc_{yx} - \frac{1}{4}bc_{yy} + \frac{1}{4}c_{yt} - \frac{1}{4}b_{zy} - \frac{1}{12}ca_{xx}\\
&\quad + \frac{1}{4}ab_{xy} + \frac{1}{6}cb_{yy}\\
W(\partial_x,\partial_t,\partial_x,\partial_t) &= -\frac{1}{2}b_{xx}\\
W(\partial_x,\partial_t,\partial_y,\partial_z) &= -\frac{1}{3}c_{yx} + \frac{1}{12}b_{yy} + \frac{1}{12}a_{xx}\\
W(\partial_x,\partial_t,\partial_y,\partial_t) &= -\frac{1}{4}b_{xy} + \frac{1}{4}c_{xx}\\
W(\partial_x,\partial_t,\partial_z,\partial_t) &= -\frac{1}{4}ab_{xx} - \frac{1}{4}cb_{xy} - \frac{1}{12}bb_{yy} - \frac{1}{12}ba_{xx} + \frac{1}{12}bc_{yx} + \frac{1}{4}cc_{xx}
\end{aligned}$$

$$W(\partial_y,\partial_z,\partial_y,\partial_z) = -\frac{1}{2}a_{yy}$$

$$W(\partial_y,\partial_z,\partial_y,\partial_t) = -\frac{1}{4}c_{yy} + \frac{1}{4}a_{xy}$$

$$W(\partial_y, \partial_z, \partial_z, \partial_t) = \frac{1}{12}aa_{xx} + \frac{1}{4}ca_{xy} + \frac{1}{4}ba_{yy} - \frac{1}{4}cc_{yy} - \frac{1}{12}ac_{yx} + \frac{1}{12}ab_{yy}$$

$$W(\partial_y, \partial_t, \partial_y, \partial_t) = -\frac{1}{6}a_{xx} + \frac{1}{6}c_{xy} - \frac{1}{6}b_{yy}$$

$$W(\partial_y, \partial_t, \partial_z, \partial_t) = \frac{1}{4}c_{yt} - \frac{1}{4}b_{zy} + \frac{1}{4}ac_{xx} + \frac{5}{12}cc_{yx} + \frac{1}{4}a_{xt} - \frac{1}{4}c_{zx} - \frac{1}{4}ba_{xy} + \frac{1}{12}cb_{yy} - \frac{1}{6}ca_{xx}$$

$$\begin{aligned} W(\partial_z, \partial_t, \partial_z, \partial_t) = {} & \frac{1}{4}bc_xa_y - \frac{1}{4}ba_xc_y + \frac{1}{4}ab_xc_y - \frac{1}{4}ac_xb_y + \frac{1}{2}b_yc_z + \frac{1}{4}b_xa_z \\ & - \frac{1}{4}a_xb_z - \frac{1}{12}baa_{xx} - \frac{1}{2}bca_{yx} + \frac{1}{2}bcc_{yy} - \frac{1}{2}c_yb_z - \frac{1}{2}cc_{yt} \\ & + \frac{1}{2}cb_{zy} - \frac{1}{6}c^2b_{yy} - \frac{1}{6}c^2a_{xx} - \frac{1}{2}acb_{yx} - \frac{1}{2}cc_{zx} - \frac{1}{2}bc_{zy} \\ & + \frac{1}{2}acc_{xx} + \frac{1}{3}bac_{yx} + \frac{2}{3}c^2c_{yx} - \frac{1}{2}ac_{xt} + \frac{1}{2}ba_{yt} - \frac{1}{4}b^2a_{yy} \\ & + \frac{1}{2}ab_{zx} + c_{zt} - \frac{1}{4}a^2b_{xx} - \frac{1}{12}bab_{yy} + \frac{1}{2}ca_{xt} - \frac{1}{2}b_{zz} - \frac{1}{2}a_{tt} \\ & + \frac{1}{4}ca_xb_y - \frac{1}{4}cb_xa_y + \frac{1}{2}a_xc_t - \frac{1}{4}b_ya_t + \frac{1}{4}a_yb_t - \frac{1}{2}c_xa_t. \end{aligned} \tag{3.1}$$

Now it is possible to obtain the form of a locally conformal flat Walker metric as follows.

Theorem 1. *A Walker metric* (1.1) *is locally conformal flat if and only if the functions* $a(x, y, z, t)$*,* $b(x, y, z, t)$ *and* $c(x, y, z, t)$ *satisfy*

$$\begin{aligned} a(x, y, z, t) &= F(z, t)xy + R(z, t)y - \frac{1}{2}S(z, t)x^2 + L(z, t)x + \xi(z, t) \\ b(x, y, z, t) &= xyA(z, t) + xB(z, t) + \frac{1}{2}S(z, t)y^2 + N(z, t)y + \eta(z, t) \\ c(x, y, z, t) &= \frac{1}{2}A(z, t)x^2 + M(z, t)x + \frac{1}{2}F(z, t)y^2 + H(z, t)y + \theta(z, t), \end{aligned} \tag{3.2}$$

where the functions $R(z, t)$*,* $A(z, t)$*,* $B(z, t)$*,* $S(z, t)$*,* $Q(z, t)$*,* $\xi(z, t), \eta(z, t)$ *and* $\theta(z, t)$ *satisfy the following relations:*

$$LA + MS - BF - 2A_z - S_t = 0,$$

$$2F_t - S_z - NF + HS + RA = 0,$$

$$L_t - M_z - \eta F + H_t - N_z + \theta S + \xi A = 0,$$

$$-\theta S_t - MH_t + F\eta_t + RB_t + S\eta_z - 2\xi A_t + LB_z - 2B_zH + NM_z$$

$$-2S\theta_t - 2BH_z - 3MM_z + BL_z + 2BR_t + 2A\theta_z - 3A\xi_t - 2L_{tt}$$
$$+4M_{zt} - 2B_{zz} + gS_z + NH_t - NN_z + ML_t + 2hA_z = 0,$$

$$2\xi A_z - 2RM_t - 3F\eta_z - 2HH_t + 2HL_t + 2\theta S_z + RN_t + NR_t - S\xi_t$$
$$+2RB_z + 2F\theta_t - 2MR_t + BR_z + 2S\theta_z + 2LH_t - LN_z + A\xi_z$$
$$-2HM_z - 2\eta F_z - \eta HF - \eta FL - \xi AH + \xi BF + \theta NF + \theta SL$$
$$-\xi MS - \theta AR - 2N_{zz} + 4H_{zt} - 2R_{tt} = 0,$$

$$\begin{aligned} &\eta MR + \xi BH - N\xi_t + 2\xi B_z - \xi MN + \theta NL - 2\xi M_t - \eta HL - \theta BR \\ &- 2M\xi_t + R\eta_t + 2\eta R_t + 2L\theta_t + 2\theta L_t + 2N\theta_z + B\xi_z - L\eta_z - 2\eta_z H \\ &- 2\theta H_t - 2\theta M_z + 2\theta N_z - 2\eta H_z + 4\theta_{zt} - 2\eta_{zz} - 2\xi_{tt} = 0. \end{aligned} \tag{3.3}$$

Proof: Since a four-dimensional manifold is locally flat if and only if the Weyl tensor vanishes, we consider (3.1) as a system of PDEs.

As regards the components of the Weyl tensor (3.1) we have

$$W(\partial_x,\partial_y,\partial_z,\partial_t) = -\frac{1}{12}a_{xx} - \frac{1}{6}c_{xy} - \frac{1}{12}b_{yy} = 0$$
$$W(\partial_x,\partial_z,\partial_x,\partial_z) = -\frac{1}{6}a_{xx} + \frac{1}{6}c_{xy} - \frac{1}{6}b_{yy} = 0$$
$$W(\partial_x,\partial_z,\partial_x,\partial_t) = -\frac{1}{4}c_{xx} + \frac{1}{4}b_{xy} = 0$$
$$W(\partial_x,\partial_z,\partial_y,\partial_z) = -\frac{1}{4}a_{xy} + \frac{1}{4}c_{yy} = 0$$
$$W(\partial_x,\partial_z,\partial_y,\partial_t) = -\frac{1}{2}c_{yx} = 0$$
$$W(\partial_x,\partial_t,\partial_x,\partial_t) = -\frac{1}{2}b_{xx} = 0$$
$$W(\partial_x,\partial_t,\partial_y,\partial_z) = -\frac{1}{3}c_{xy} + \frac{1}{12}b_{yy} + \frac{1}{12}a_{xx} = 0$$
$$W(\partial_x,\partial_t,\partial_y,\partial_t) = -\frac{1}{4}b_{xy} + \frac{1}{4}c_{xx} = 0$$
$$W(\partial_x,\partial_t,\partial_z,\partial_t) = -\frac{1}{4}ab_{xx} - \frac{1}{4}cb_{xy} - \frac{1}{12}bb_{yy} - \frac{1}{12}ba_{xx} + \frac{1}{12}bc_{yx} + \frac{1}{4}cc_{xx} = 0$$
$$W(\partial_y,\partial_z,\partial_y,\partial_z) = -\frac{1}{2}a_{yy} = 0$$

$$
\begin{aligned}
&W(\partial_y,\partial_z,\partial_y,\partial_t) = -\frac{1}{4}c_{yy} + \frac{1}{4}a_{xy} = 0\\
&W(\partial_y,\partial_z,\partial_z,\partial_t) = \frac{1}{12}aa_{xx} + \frac{1}{4}ca_{xy} + \frac{1}{4}ba_{yy} - \frac{1}{4}cc_{yy} - \frac{1}{12}ac_{yx} + \frac{1}{12}ab_{yy} = 0\\
&W(\partial_y,\partial_t,\partial_y,\partial_t) = -\frac{1}{6}a_{xx} + \frac{1}{6}c_{xy} - \frac{1}{6}b_{yy} = 0. \qquad (3.4)
\end{aligned}
$$

The above equations imply

(1) $c_{yx} = 0$
(2) $b_{xx} = 0$
(3) $a_{yy} = 0$
(4) $b_{yyx} = 0$
(5) $b_{yyy} = 0$
(6) $c_{xx} = b_{yx}$
(7) $a_{yx} = c_{yy}$
(8) $a_{xx} = -b_{yy}$
(9) $c_{yyy} = 0$.

>From the first equation we have $c(x,y,z,t) = \overline{c}(x,z,t) + \widehat{c}(y,z,t)$, and from the second equation $b(x,y,z,t) = xM_1(y,z,t) + N_1(y,z,t)$, while the third equation we get $a(x,y,z,t) = yF_1(x,z,t) + G_1(x,z,t)$.

>From the fourth and fifth equations we conclude that $M_{1yy} = 0$ and $N_{1yyy} = 0$. Therefore $M_1(y,z,t) = yA(z,t) + B(z,t)$ and

$$N_1(y,z,t) = \frac{1}{2}S(z,t)y^2 + N(z,t)y + \eta(z,t),$$

so,

$$b(x,y,z,t) = xyA(z,t) + xB(z,t) + \frac{1}{2}S(z,t)y^2 + N(z,t)y + \eta(z,t).$$

Since $c_{yyy} = 0$, we have $\widehat{c}_{yyy}(y,z,t) = 0$, therefore

$$\widehat{c}(y,z,t) = \frac{1}{2}F(z,t)y^2 + H(z,t)y + K_1(z,t).$$

>From $c_{xx} = b_{xy}$, we have $\overline{c}_{xx}(x,z,t) = A(z,t)$, hence,

$$\overline{c}(x,z,t) = \frac{1}{2}A(z,t)x^2 + M(z,t)x + N_2(z,t),$$

and

$$c(x,y,z,t) = \frac{1}{2}A(z,t)x^2 + M(z,t)x + \frac{1}{2}F(z,t)y^2 + H(z,t)y + \theta(z,t).$$

Since $a_{xx} = -b_{yy}$, we have $yF_{1xx}(x,z,t) + G_{1xx} = -S(z,t)$, therefore $F_{1xx} = 0$ and

$$G_1(x,z,t) = -\frac{1}{2}S(z,t)x^2 + L(z,t)x + \xi(z,t).$$

By $a_{xy} = F_{1x} = c_{yy} = F(z,t)$, we have $F_1(x,z,t) = F(z,t)x + R(z,t)$, so

$$a(x,y,z,t) = F(z,t)xy + R(z,t)y - \frac{1}{2}S(z,t)x^2 + L(z,t)x + \xi(z,t).$$

By sixth equation of (3.1)

$$W_{1334} = \frac{1}{4}a_{xt} - \frac{1}{4}c_{zx} - \frac{5}{12}cc_{yx} - \frac{1}{4}bc_{yy} + \frac{1}{4}c_{yt} - \frac{1}{4}b_{zy} - \frac{1}{12}ca_{xx} + \frac{1}{4}ab_{yx} + \frac{1}{6}cb_{yy} = 0,$$

so from the above relations we get

$$\begin{aligned}
&\frac{1}{2}yF_t(z,t) + \frac{1}{4}L_t(z,t) - \frac{1}{4}xS_t(z,t) - \frac{1}{2}xA_z(z,t) - \frac{1}{4}M_z(z,t)\\
&- \frac{1}{4}yN(z,t)F(z,t) - \frac{1}{4}xB(z,t)F(z,t) - \frac{1}{4}\eta(z,t)F(z,t) + \frac{1}{4}H_t(z,t)\\
&- \frac{1}{4}N_z(z,t) - \frac{1}{4}yS_z(z,t) + \frac{1}{4}xM(z,t)S(z,t) + \frac{1}{4}yH(z,t)S(z,t)\\
&+ \frac{1}{4}\theta(z,t)S(z,t) + \frac{1}{4}xL(z,t)A(z,t) + \frac{1}{4}yR(z,t)A(z,t) + \frac{1}{4}\xi(z,t)A(z,t) \equiv 0,
\end{aligned}$$

and

$$\begin{aligned}
W_{1334} =& \frac{1}{4}y\big(2F_t(z,t) - S_z(z,t) - N(z,t)F(z,t) + H(z,t)S(z,t)\\
&+ R(z,t)A(z,t)\big) + \frac{1}{4}x\big(L(z,t)A(z,t) + M(z,t)S(z,t)\\
&- B(z,t)F(z,t) - 2A_z(z,t) - S_t(z,t)\big) + L_t(z,t) - M_z(z,t)\\
&- \eta(z,t)F(z,t) + H_t(z,t) - N_z(z,t) + \theta(z,t)S(z,t) + \xi(z,t)A(z,t) \equiv 0,
\end{aligned}$$

implying the first three equations of (3.3). Similarly, from the fifteenth equation of (3.1) follow the first three equations of (3.3).

By the last equation of (3.1)

$$\begin{aligned}
W_{3434} =& \frac{1}{4}bc_xa_y - \frac{1}{4}ba_xc_y + \frac{1}{4}ab_xc_y - \frac{1}{4}ac_xb_y + \frac{1}{2}b_yc_z + \frac{1}{4}b_xa_z - \frac{1}{4}a_xb_z\\
&-\frac{1}{12}baa_{xx} - \frac{1}{2}bca_{yx} + \frac{1}{2}bcc_{yy} - \frac{1}{2}c_yb_z - \frac{1}{2}cc_{yt} + \frac{1}{2}cb_{zy} - \frac{1}{6}c^2b_{yy}\\
&-\frac{1}{6}c^2a_{xx} - \frac{1}{2}acb_{yx} - \frac{1}{2}cc_{zx} - \frac{1}{2}bc_{zy} + \frac{1}{2}acc_{xx} + \frac{1}{3}bac_{yx} + \frac{2}{3}c^2c_{yx}\\
&-\frac{1}{2}ac_{xt} + \frac{1}{2}ba_{yt} - \frac{1}{4}b^2a_{yy} + \frac{1}{2}ab_{zx} + c_{zt} - \frac{1}{4}a^2b_{xx} - \frac{1}{12}bab_{yy}\\
&+\frac{1}{2}ca_{xt} - \frac{1}{2}b_{zz} - \frac{1}{2}a_{tt} + \frac{1}{4}ca_xb_y - \frac{1}{4}cb_xa_y + \frac{1}{2}a_xc_t - \frac{1}{4}b_ya_t
\end{aligned}$$

$$+\frac{1}{4}a_y b_t - \frac{1}{2}c_x a_t \equiv 0,$$

implying

$$x^3\Big(\frac{1}{2}AS_t - \frac{1}{2}LA^2 + AA_z + \frac{1}{2}ABF - \frac{1}{2}SMA\Big) + x^2\Big(- MS_t + 2BF_t$$
$$+ NA_z + \frac{1}{2}ABR - \frac{1}{2}SMN - \frac{1}{2}LAN + \frac{1}{2}SBH - \theta AS + \eta AF - LA_t - AH_t$$
$$- 2AL_t - \xi A^2 - SM_t + AM_z + FB_t + AN_z + S_{tt} + 2A_{zt} - \frac{1}{2}S_z B + \frac{1}{2}S_t N\Big)$$
$$+ x\Big(- 2\theta S_t - \theta NS - \theta BF + \eta AR + \theta AL + \eta MF + \eta HS - \xi MA - \xi AN$$
$$+ 2\eta F_t - 2MH_t + F\eta_t + RB_t + S\eta_z - 2\xi A_t + LB_z - 2B_z H + 2NM_z$$
$$- 2S\theta_t - NL_t + 2MN_z - 2BH_z - 2MM_z + BL_z + 2BR_t + 2A\theta_z - 3A\xi_t$$
$$- 2L_{tt} + 4M_{zt} - 2B_{zz}\Big) + y^3\Big(\frac{1}{2}SHF - \frac{1}{2}NF^2 + \frac{1}{2}RAF - \frac{1}{2}FS_z + FF_t\Big)$$
$$+ y^2\Big(- FM_z + 2RA_z - \eta F^2 + FH_t + LF_t + FL_t + AR_z - 2FN_z - \frac{1}{2}LS_z$$
$$+ \frac{1}{2}RS_t - NF_z + SH_z + S_z H + \theta SF + \frac{1}{2}SHL - \frac{1}{2}NFL + \xi AF + \frac{1}{2}RBF$$
$$- \frac{1}{2}RMC + 2F_{zt} - S_{zz}\Big) + y\Big(2\xi A_z - 2RM_t - 3F\eta_z - 2HH_t + 2HL_t + 2\theta S_z$$
$$+ RN_t + NR_t - S\xi_t + 2RB_z + 2F\theta_t - 2MR_t + BR_z + 2S\theta_z + 2LH_t$$
$$- LN_z + A\xi_z - 2HM_z - 2\eta F_z - \eta HF - \eta FL - \xi AH + \xi BF + \theta NF + \theta SL$$
$$- \xi MS - \theta AR - 2N_{zz} + 4H_{zt} - 2R_{tt}) + xy(-\theta S^2 + AL_z + SN_z + FN_t$$
$$- BF_z - 2A_z H + 2MS_z - 2MF_t + LA_z + NF_t - 2SH_t - 2HS_t - AR_t - SL_t$$
$$- FB_z - RA_t + 2SM_z + FMN + \eta FS - FBH - fAS - RMA + LAH$$
$$- 2F_{tt} - 2A_{zz}\Big) + xy^2\Big(\frac{1}{2}SS_z - \frac{1}{2}FS_t - SF_t - FA_z - \frac{1}{2}S^2 H - \frac{1}{2}F^2 B$$
$$+ \frac{1}{2}LAF - \frac{1}{2}RAS + \frac{1}{2}FMS + \frac{1}{2}NFS\Big) + x^2 y\Big(- \frac{1}{2}SAH + SA_z + \frac{1}{2}SS_t$$
$$- AF_t - \frac{1}{2}S^2 M + \frac{1}{2}AS_z - \frac{1}{2}RA^2 + \frac{1}{2}SBF - \frac{1}{2}LAS + \frac{1}{2}FAN\Big) + \eta MR$$
$$+ \xi BH - N\xi_t + 2\xi B_z - \xi MN + \theta NL - 2\xi M_t - \eta HL - \theta BR - 2M\xi_t + R\eta_t$$
$$+ 2\eta R_t + 2L\theta_t + 2\theta L_t + 2N\theta_z + B\xi_z - L\eta_z - 2\eta_z H - 2\theta H_t - 2\theta M_z + 2\theta N_z$$
$$- 2\eta H_z + 4\theta_{zt} - 2\eta_{zz} - 2\xi_{tt} \equiv 0,$$

from where, the three equations of (3.3) follow.

Definition 2. *Let Ric and Sc denote Ricci tensor and scalar curvature of (M,g), defined by $Ric = trace\{Z \to R(X,Z)Y\}$ and $Sc = trace\ Ric$, respectively.*

Lemma 1. *The scalar curvature of a Walker metric* (1.1) *is given by*

$$Sc = b_{yy} + a_{xx} + 2c_{xy} \tag{3.5}$$

Proof: The nonzero components of the Ricci tensor of any Walker metric (1.1) are as follows

$$\begin{aligned}
Ric(X_1,X_3) &= \frac{1}{2}a_{xx} + \frac{1}{2}c_{xy}\\
Ric(X_1,X_4) &= \frac{1}{2}c_{xx} + \frac{1}{2}b_{xy}\\
Ric(X_2,X_3) &= \frac{1}{2}a_{xy} + \frac{1}{2}c_{yy}\\
Ric(X_2,X_4) &= \frac{1}{2}c_{xy} + \frac{1}{2}b_{yy}\\
Ric(X_3,X_1) &= \frac{1}{2}a_{xx} + \frac{1}{2}c_{xy}\\
Ric(X_3,X_2) &= \frac{1}{2}a_{xy} + \frac{1}{2}c_{yy}\\
Ric(X_3,X_3) &= \frac{1}{2}\Bigg(2ca_{xy} + aa_{xx} + ba_{yy} + a_yb_y - 2a_{yt} + a_xc_y\\
&\quad - a_yc_x + 2c_{zy} - c_y^2\Bigg)
\end{aligned}$$

$$\begin{aligned}
Ric(X_3,X_4) &= \frac{1}{2}\Bigg(a_{tx} - a_yb_x + b_{yz} + ac_{xx} + 2cc_{xy} - c_{xz} + bc_{yy} - c_{yt} + c_xc_y\Bigg)\\
Ric(X_4,X_1) &= \frac{1}{2}c_{xx} + \frac{1}{2}b_{xy}\\
Ric(X_4,X_2) &= \frac{1}{2}c_{xy} + \frac{1}{2}b_{yy}\\
Ric(X_4,X_3) &= \frac{1}{2}\Bigg(ac_{xx} + 2cc_{xy} + a_{tx} - c_{xz} + c_xc_y + bc_{yy} - c_{yt} - a_yb_x + b_{yz}\Bigg)\\
Ric(X_4,X_4) &= \frac{1}{2}\Bigg(-2b_{zx} + a_xb_x + cb_{xy} + cb_{xx} + bb_{yy} + 2c_{xt} - c_x^2 - b_xc_y + c_xb_y\Bigg).
\end{aligned}$$

The equations expressing the components of the curvature tensor ensure that the Ricci operator $\langle \widehat{Ric}(X), Y\rangle = Ric(X,Y)$ satisfies

$$\widehat{Ric}(\partial_x) = \left(\frac{1}{2}a_{xx} + \frac{1}{2}c_{xy}\right)\partial_x + \left(\frac{1}{2}a_{xy} + \frac{1}{2}c_{yy}\right)\partial_y$$

$$\widehat{Ric}(\partial_y) = \left(\frac{1}{2}a_{xy} + \frac{1}{2}c_{yy}\right)\partial_x + \left(\frac{1}{2}c_{xy} + \frac{1}{2}b_{yy}\right)\partial_y$$

$$\begin{aligned}\widehat{Ric}(\partial_z) &= \frac{1}{2}\Big(2ca_{xy} + aa_{xx} + ba_{yy} + a_yb_y - 2a_{yt} + a_xc_y - a_yc_x + 2c_{zy} - c_y^2\Big)\partial_x \\ &+ \frac{1}{2}\Big(a_{tx} - a_yb_x + b_{yz} + ac_{xx} + 2cc_{xy} - c_{xz} + bc_{yy} - c_{yt} + c_xc_y\Big)\partial_y \\ &+ \left(\frac{1}{2}a_{xx} + \frac{1}{2}c_{xy}\right)\partial_z + \left(\frac{1}{2}a_{xy} + \frac{1}{2}c_{yy}\right)\partial_t \\ \widehat{Ric}(\partial_t) &= \frac{1}{2}\Big(ac_{xx} + 2cc_{xy} + a_{tx} - c_{xz} + c_xc_y + bc_{yy} - c_{yt} - a_yb_x + b_{yz}\Big)\partial_x \\ &+ \frac{1}{2}\Big(- 2b_{zx} + a_xb_x + cb_{xy} + cb_{xx} + bb_{yy} + 2c_{xt} - c_x^2 - b_xc_y + c_xb_y\Big)\partial_y \\ &+ \left(\frac{1}{2}c_{xx} + \frac{1}{2}b_{xy}\right)\partial_z + \left(\frac{1}{2}a_{xx} + \frac{1}{2}c_{xy}\right)\partial_t.\end{aligned}$$

Now the result follows by a straightforward calculation.

Remark 1. *It follows from Lemma 1 that any locally conformal flat Walker metric has vanishing scaler curvature.*

Hence we come to semi-Riemannian manifolds with harmonic curvature (cf. [7]).

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Поступила 11 января 2008