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BLOW-UP THE SYMMETRY ANALYSIS FOR THE HIROTA-SATSUMA EQUATIONS

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Аннотация. The paper investigates the invariance and integrability properties of Hirota-Satsuma equations. Painleve analysis for the general similarity reduced ordinary differential equation is performed. Using Rung-Kutta-Merson method in shooting and matching technique, the nonlinear ordinary differential equations are solved that were numerically converted from similarity reduction.

1. INTRODUCTION

In 1981, R. Hirota and J. Satsuma first proposed the well-known Hirota – Satsuma KDV equation [1]. This equation describes an interaction of two long waves with different dispersion relations.

It is well known that the nonlinear partial differential equations are widely used to describe many important phenomena in physics, biology, chemistry, etc. This equations play a crucial rule in applied mathematics and physics and have many applications in physics and Engineering.

For the past two decades the Lie group method has been applied to solve a wide range of problems and to explore many physically interesting solutions of nonlinear phenomena [2]–[5]. In recent years several extensions and modifications of the classical Lie algorithm have been proposed in order to arrive at new solutions of PDE [6].

The present paper gives a systematic investigation of the invariance and integrability properties of Hirota-Satsuma coupled KDV equation. This enables us to obtain similarity reductions and allows us to derive a great variety of particular solutions which have not been reported for Hirota – Satsuma KDV equation.

An ordinary differential equation (ODE) is said to be Painleve type or to have the Painleve property if all its solutions are free from movable critical points. A critical point is a branching point or a singularity in the solution of the ODE. It is movable if its location depends on the initial values. The most well known ode of Painleve type are the so-called Painleve equations, PI-PVI [7].

The connection between complete integrability and Painleve property was first noticed by Ablowitz – Segur [8], who observed that the similarity reductions of nonlinear PDE solved by inverse scattering transform give rise to nonlinear ODEs.

First, we introduce a generalized nonlinear Hirota-Satsuma KDV equation in the following form:

$$u_t = \frac{1}{2}u_{xxx} - 3uu_x + 3(vw)_x, \tag{1.1}$$

$$v_t = -v_{xxx} + 3uv_x, \tag{1.2}$$

$$w_t = -w_{xxx} + 3uw_x. \tag{1.3}$$

This paper is arranged as follows. In Section 2, we briefly describe the invariance analysis and obtain the reduction system for equations (1.1)–(1.3). In Section 3, we describe the improved Painleve analysis and obtained new solutions of equations' (1.1)–(1.3). In Section 4, Shooting Method is used to study the reduction similarity system of Hirota-Satsuma KDV equation.

2. INVARIANCE ANALYSIS

Let us consider one-parameter Lie group of infinitesimal transformations of the form:

$$x \longrightarrow X = x + \epsilon\xi_1(x,t,u,v,w) + O(\epsilon^2),$$

$$t \longrightarrow T = t + \epsilon\xi_2(x,t,u,v,w) + O(\epsilon^2),$$

$$u \longrightarrow U = u + \epsilon\phi_1(x,t,u,v,w) + O(\epsilon^2),$$

$$v \longrightarrow V = v + \epsilon\phi_2(x,t,u,v,w) + O(\epsilon^2),$$

$$w \longrightarrow W = w + \epsilon\phi_3(x,t,u,v,w) + O(\epsilon^2), \quad \epsilon << 1, \tag{2.1}$$

depending one infinitesimal parameter ϵ. This Lie-Group based similarity method has already been applied successfully to construct and classify all possible classes of similarity solutions.

Applying the infinitesimal Lie-group technique [5], a straightforward calculation yields the following generators of the ϵ Lie group:

$$\phi_1 = -\frac{2}{3}k_4 u,$$
$$\phi_2 = k_1 v,$$
$$\phi_3 = -\frac{1}{3}(3k_1 + 4k_4)w,$$
$$\xi_1 = k_2 + \frac{1}{3}k_4 x,$$
$$\xi_2 = k_3 + k_4 t, \tag{2.2}$$

where k_1, k_2, k_3 and k_4 are arbitrary constants.

The extremal Lie group of transformations, admitted by (2.2), is thus seen to depend on four arbitrary group constants (k_1, k_2, k_3, k_4). Consequently, the infinitesimal generators are:

$$\chi_1 = v\frac{\partial}{\partial v} - w\frac{\partial}{\partial w},$$
$$\chi_2 = \frac{\partial}{\partial x}, \tag{2.3}$$
$$\chi_3 = \frac{\partial}{\partial t}, \tag{2.4}$$
$$\chi_4 = \frac{1}{2}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{2}{3}u\frac{\partial}{\partial u} - \frac{4}{3}w\frac{\partial}{\partial w}. \tag{2.5}$$

The commutation relation between these generators are given in the following table:

	χ_1	χ_2	χ_3	χ_4
χ_1	0	0	0	0
χ_2	0	0	0	$\frac{1}{3}\chi_2$
χ_3	0	0	0	χ_3
χ_4	0	$-\frac{1}{3}\chi_2$	$-\chi_3$	0

Group-invariant solutions can be found by solving the characteristic equation:

$$\frac{dt}{\xi_2} = \frac{dx}{\xi_1} = \frac{du}{\varphi_1} = \frac{dv}{\varphi_2} = \frac{dw}{\varphi_3} \tag{2.6}$$

After solving the characteristic equation (9) associated with the infinitesimal symmetry (2.2), one obtains:

$$z = \frac{3k_2 + k_4 x}{(k_3 + k_4 t)^{\frac{1}{3}}}, \tag{2.7}$$

and

$$\omega_1(z) = (k_3 + k_4 t)^{\frac{2}{3}} u$$
$$\omega_2(z) = (k_3 + k_4 t)^{-\frac{k_1}{k_4}} v,$$

$$\omega_3(z) = (k_3 + k_4 t)^{(\frac{4}{3}+\frac{k_1}{k_4})} w, \tag{2.8}$$

where z is the similarity variable and $w_1(z), w_2(z), w_3(z)$ represent the similarity functions.

Substituting (2.7) and (2.8) into equations (1.1)–(1.3), we obtain an ordinary differential equations of the form:

$$\begin{aligned} &3k_1^2 w_1'''(z) + 2zw_1'(z) + 4w_1(z) \\ &\quad - 18w_1(z)w_1'(z) + 18w_3(z)w_2'(z) + 18w_2(z)w_3'(z) = 0, \end{aligned} \tag{2.9}$$

$$3k_4^2 \omega_2'''(z) - zw_2'(z) + \frac{3k_1}{k_4} w_2(z) - 9w_1(z)w_2'(z) = 0, \tag{2.10}$$

$$3k_4^2 \omega_3'''(z) - zw_3'(z) - \frac{3k_1}{k_4} w_3(z) - 9w_1(z)w_3'(z) - 4w_3(z) = 0, \tag{2.11}$$

where

$$w_i' = \frac{dw_i}{dz}, \quad w_i'' = \frac{d^2 w_i}{dz^2} \quad \text{and} \quad w_i''' = \frac{d^3 w_i}{dz^3} \quad (i = 1, 2, 3).$$

In the following sections we solve the ordinary differential equations (2.9)–(2.11) by two different methods.

3. PAINLEVÉ ANALYSIS

Following [9], we outline the WTC algorithm for testing ODEs for the Painleve property. Each of the three main steps of the algorithm is illustrated by the system (2.9)–(2.11).

We assume a Laurent series solution

$$w_i(z) = g^{\alpha_i}(z) \sum_{k=0}^{\infty} w_{i,k}(z) g^k(z), \quad i = 1, 2, 3, \tag{3.1}$$

where the coefficients $w_{i,k}(z)$ are analytic functions of z with $w_{i,0}(z) \neq 0$ in a neighborhood of the manifold.

3.1. **Step 1 (Determination of the Dominant Behavior).** To investigate the singularity structure analysis (2.9)–(2.11), we apply a local Laurent expansion in a neighborhood of a noncharacteristic singular $g(z) = 0$.

Assume that the leading orders of the solutions of system (2.9)–(2.11) have the form

$$w_i(z) = \chi_i \ g^{\alpha_i}(z), \quad i = 1, 2, 3, \tag{3.2}$$

or

$$w_1(z) = \chi_1 \quad g^{\alpha_1}(z),$$
$$w_2(z) = \chi_2 \quad g^{\alpha_2}(z),$$
$$w_3(z) = \chi_3 \quad g^{\alpha_3}(z), \tag{3.3}$$

where χ_1, χ_2 and χ_3 are constants.

We substitute (3.3) into the equations (2.9)–(2.11) to determine the leading exponents α_i. Balancing between the highest order term and the non linear terms, we get:

$$\alpha_1 = \alpha_2 = \alpha_3 = -2.$$

The traditional Painleve test requires that all α_1, α_2 and α_3 be integers and at least one of them be negative.

If one or more exponents α_1, α_2 and α_3 remain undetermined, we assign integer values to the free $\alpha_i (i = 1, 2, 3)$ so that every equation in (2.9)–(2.11) has at least two different terms with equal lowest exponents.

For each solution α_i, we substitute

$$w_i(z) = w_{i,0}(z) g^{\alpha_i}(z), \quad i = 1, 2, 3,$$

i.e.

$$w_1(z) = a_0(z) g^{-2}(z),$$
$$w_2(z) = b_0(z) g^{-2}(z),$$
$$w_3(z) = c_0(z) g^{-2}(z) \tag{3.4}$$

(where $a_0(z), b_0(z)$ and $c_0(z)$ do not vanish) into (2.9)–(2.11). We then solve the nonlinear equation for $w_{i,0}$, found by balancing the leading terms with the lowest exponent of $g(z)$.

If any of the solutions contradicts the assumption $w_{i,0}(z) \neq 0$, then that branch of the algorithm fails the Painleve test [10].

If an α_i is non-integer, all the α_i are positive, or the assumption $w_{i,0}(z) \neq 0$ fails, then that branch of the algorithm terminates and does not pass the Painleve test.

We substitute (3.4) into (2.9)–(2.11). Requiring that the leading terms be $g^{-5}(z)$, and balancing at $g^{-5}(z)$ we obtain

$$a_0(z) = 4k_4^2 g'^2(z), \quad b_0(z) = \frac{4k_4^4}{c_0(z)} g'^4(z), \tag{3.5}$$

where $c_0(z)$ is arbitrary function.

3.2. **Step 2 (Determination of the resonance).** For each α_i and $w_{i,0}(z)$, we calculate the $r_1 \leq \ldots \leq r_m$ for which $w_{i,k}(z)$ are arbitrary functions (3.1). We substitute:

$$w_i(z) = w_{i,0}(z)g^{\alpha_i}(z) + w_{i,r}(z)g^{\alpha_i+r}(z). \tag{3.6}$$

or

$$\begin{aligned} w_1(z) &= 4k_4^2 g'^2(z)g^{-2}(z) + a_r(z)g^{r-2}(z), \\ w_2(z) &= \frac{4k_4^4}{c_0(z)} g'^4(z)g^{-2}(z) + b_r(z)g^{r-2}(z), \\ w_1(z) &= c_0(z)g^{-2}(z) + c_r(z)g^{r-2}(z). \end{aligned} \tag{3.7}$$

into (2.9)–(2.11) and equate the coefficients of the dominant (with $g^{r-5}(z)$). We get the resonances values are:

$$r_1 = -2, r_2 = -1, r_3 = 0, r_4 = 2, r_5 = 3, r_6 = 4, r_7 = 6, r_8 = 7, r_9 = 8.$$

3.3. **Step 3 (Finding the Constants of Integration and Checking Compatibility Conditions).** By convention, the resonance $r_1 = -2$ is ignored since it violates the hypothesis that $g^{-2}(x,t)$ is the dominant term in the expansion near $g(z) = 0$.

Furthermore, this is not a principal branch since the series has only eight arbitrary functions instead of the required nine (as the term corresponding to the resonance $r_1 = -2$ does not contribute to the expansion). Thus, this leads to a particular solution, while the general solution may still be multivalued.

The constants of integration at level k are found by substituting the system of ordinary differential equations possessing the Painleve property. The arbitrariness of $w_{i,r}(z)$ must be verified up to the resonance level. This is done by substituting (3.5) into (3.1). We get

$$w_i(z) = g^{\alpha_i}(z) \sum_{k=0}^{r_m} w_{i,k}(z)g^k(z), \quad i = 1, 2, 3. \tag{3.8}$$

Therefore

$$\begin{aligned} w_1(z) &= \sum_{k=0}^{8} a_k(z)g^{k-2}(z), \\ w_2(z) &= \sum_{k=0}^{8} b_k(z)g^{k-2}(z), \\ w_3(z) &= \sum_{k=0}^{8} c_k(z)g^{k-2}(z). \end{aligned} \tag{3.9}$$

From equation (3.5) and (3.9) we obtain:

$$w_1(z) = 4k_4^2 g'^2(z)g^{-2}(z) + a_1(z)g^{-1}(z) + a_2(z) + a_3(z)g(z) + a_4(z)g^2(z)$$
$$+a_5(z)g^3(z) + a_6(z)g^4(z) + a_7(z)g^5(z) + a_8(z)g^6(z),$$
$$w_2(z) = \frac{4k_4^4}{c_0(z)} g'^4(z)g^{-2}(z) + b_1(z)g^{-1}(z) + b_2(z) + b_3(z)g(z) + b_4(z)g^2(z)$$
$$+b_5(z)g^3(z) + b_6(z)g^4(z) + b_7(z)g^5(z) + b_8(z)g^6(z),$$
$$w_3(z) = c_0(z)g^{-2}(z) + c_1(z)g^{-1}(z) + c_2(z) + c_3(z)g(z) + c_4(z)g^2(z)$$
$$+c_5(z)g^3(z) + c_6(z)g^4(z) + c_7(z)g^5(z) + a_8(z)g^6(z). \tag{3.10}$$

We now substitute (3.10) into equations (2.9)–(2.11) and group the terms in the same powers of $g(z)$. So, we get the coefficients of $g^{k-2}(z)$ at level k.

To find the functions $a_1(z), b_1(z)$ and $c_1(z)$, we equate the coefficients by $g^{-4}(z)$ to zero at level $k = 1$. By solving the equations, we obtain:

$$a_1(z) = -4k_4^2 g''(z),$$
$$b1(z) = -\frac{4(-k_4^4 c_0'(z)g'^3(z) + 3c_0(z)k_4^4 g'^2(z)g''(z))}{c_0^2(z)},$$
$$c_1(z) = -\frac{c_0'(z)g'(z) - c_0(z)g''(z)}{g'^2(z)}. \tag{3.11}$$

To find the functions $a_2(z), b_2(z)$ and $c_2(z)$ we equate the coefficients by $g^{-3}(z)$ to zero at level $k = r_4 = 2$. By solving the equations, we obtain:

$$a_2(z) = -\frac{zg'^2(z) + 9k_4^2 g''^2(z) - 12k_4^2 g'(z)g'''(z)}{9g'^2(z)}.$$
$$c_2(z) = -\frac{b_2(z)c_0^2(z)}{4k_4^4 g'^4(z)} - \frac{1}{18c_0(z)k_4^2 g'^4(z)}(4zc_0^2(z)g'^2(z) - 18k_4^2 c_0'^2(z)$$
$$+g'^2(z) + 72c_0(z)k_4^2 c_0'(z)g'(z)g''(z) - 63c_0^2(z)k_4^2 g''^2(z)$$
$$-12c_0^2(z)k_4^2 g'(z)g'''(z)), \tag{3.12}$$

where the function $b_2(z)$ is arbitrary.

To find the functions $a_3(z), b_3(z)$ and $c_3(z)$, we equate the coefficients of $g^{-2}(z)$ to zero at level $k = r_5 = 3$. Solving the equations, we obtain:

$$b_3(z) = \frac{1}{3c_0^4(z)g'^2(z)}(-3c_0^4(z)b_2'(z)g'(z) + c_0^3(z)k_1k_4g'^3(z)$$
$$-14k_4^4 c_0'^3(z)g'^3(z) + 6a_3(z)c_0^3(z)k_4^2 g'4(z) + 24c_0(z)k_4^4 c_0'(z)$$
$$+g'^3(z)c_0''(z) + 60c_0(z)k_4^4 c_0'^2(z)g'^2(z)g''(z) - 30c_0^2(z)k_4^4 g'^2(z)$$
$$+c_0''(z)g''(z) - 42c_0^2(z)k_4^4 c_0'(z)g'(z)g''^2(z) + 9c_0^3(z)k_4^4 g''^3(z)$$

$$-4c_0^2(z)k_4^4g'^3(z)c_0''''(z) - 32c_0^2(z)kc_0'(z)g'^2(z)'''g(z)$$

$$+38c_0^3(z)k_4^4g'(z)g''(z)g'''(z) + 13c_0^3(z)k_4^4g'^2(z)g'''(z),$$

$$\begin{aligned}
c_3(z) = {} & \frac{1}{36c_0^2(z)k_4^4g'^6(z)}(9c_0^4(z)b_2'(z)g'(z) + 18b_2(z)c_0^3(z)c_0'(z)g'(z) \\
& - 3c_0^3(z)k_1^4g'^3(z) + 4c_0^3(z)k_4^2g'^3(z) + 8zc_0^2(z)k_4^2c_0'(z)g'^3(z) \\
& + 36k_4^4c_0'^3(z)g'^3(z) + 18a_3(z)c_0^3(z)k_4^2g'^4(z) + 72c_0(z)k_4^4 \\
& + c_0'(z)g'^3(z)c_0''(z) - 36b_2(z)c_0^4(z)g''(z) - 16zc_0^3(z)k_4^2g'^2(z)g''(z) \\
& + 72c_0(z)k_4^4c_0'^2(z)g'^2(z)g''(z) + 90c_0^2(z)k_4^4g'^2(z)c_0''(z)g''(z) \\
& - 432c_0^2(z)k_4^4c_0'(z)g'(z)g''^2(z) + 387c_0^3(z)k_4^4g''^3(z) \\
& + 12c_0^2(z)k_4^4g'^3(z)c_0''(z) + 72c_0^2(z)k_4^4c_0'(z)g'^2(z)g'''(z) \\
& - 90c_0^3(z)k_4^4g'(z)g''(z)g'''(z) - 33c_0^3(z)k_4^4g'^2(z)g''''(z)),
\end{aligned} \tag{3.13}$$

where the function $a_3(z)$ is arbitrary.

To find the functions $a_4(z), b_4(z)$ and $c_4(z)$, we equate the coefficients of $g^{-1}(z)$ to zero at level $k = r_6 = 4$ and solve the equations.

To find the functions $a_5(z), b_5(z)$ and $c_5(z)$, we equate the coefficients of $g^0(z)$ to zero at level $r_6 = 5$ and solve the equations.

To find the functions $a_6(z), b_6(z)$ and $c_6(z)$, we equate the coefficients of $g(z)$ to zero at level $k = r_7 = 6$ and solve the equations.

To find the functions $a_7(z), b_7(z)$ and $c_7(z)$, we equate the coefficients of $g^2(z)$ to zero at level $k = r_8 = 7$ and solve the equations.

To find the functions $a_8(z), b_8(z)$ and $c_8(z)$, we equate the coefficients of $g^3(z)$ to zero at level $k = r_9 = 4$ and solve the equations.

Substitute from equations (3.11)–(3.13) into (3.10), we obtain w_1, w_2, w_3. Consequently, (2.8) yields u, v, w.

4. SECOND METHOD: SHOOTING METHOD

We try to solve equations (2.9)–(2.11) by using shooting method. The boundary and matching conditions of the problem can be written as:

$$w_1 = -\frac{1}{3}, \quad w_1' = \frac{1}{3}, \quad w_2 = -1, \quad w_2' = 1, \quad w_2'' = 0, \quad \text{at} \quad z = -1,$$

$$w_1 = \frac{1}{3}, \quad w_1' = \frac{1}{3}, \quad w_2 = 1, \quad w_3 = 1 \quad \text{at} \quad z = 1. \tag{4.1}$$

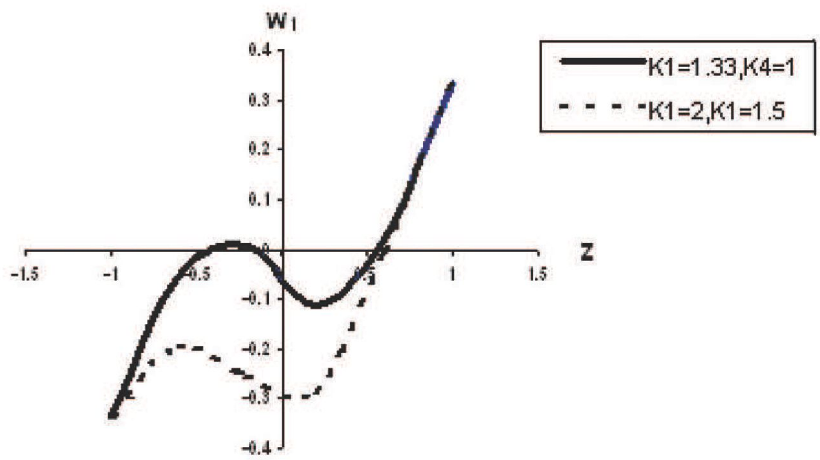

Fig. 1. The relation between z, w_1 in two cases of constants.

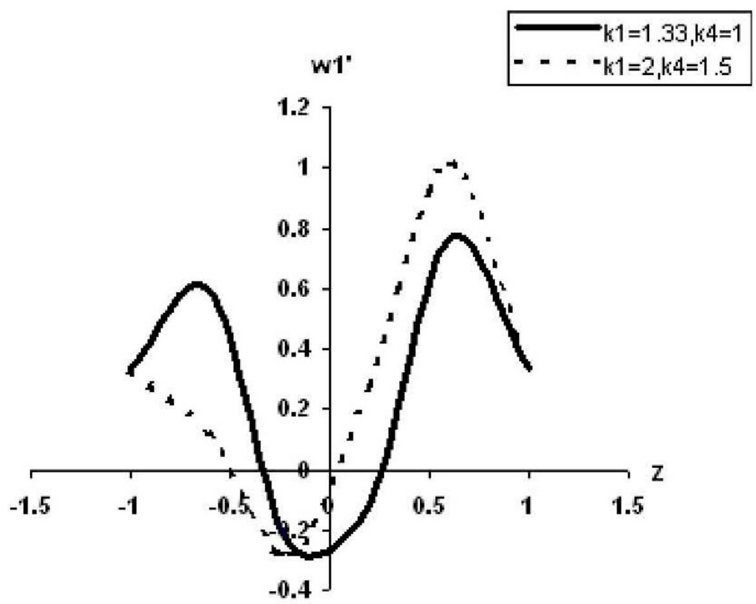

Fig. 2. The relation between z, w_1' in two cases of constants.

4.1. **Numerical Solution.** Equations (2.9)–(2.11) represent the governing equations of the problem under consideration. These equations are nonlinear and therefore, must be solved numerically by Runge-Kutta-Merson method within the shooting and matching technique [11]–[14].

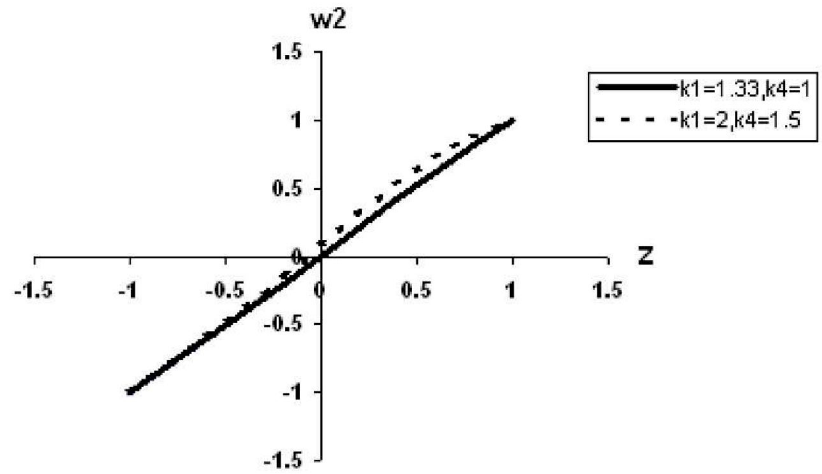

Fig. 3. The relation between z, w_2 in two cases of constants.

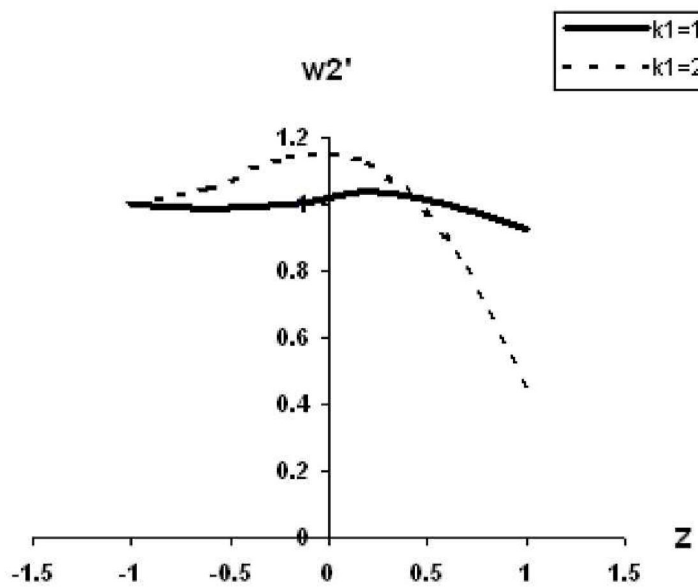

Fig. 4. The relation between z, w_2' in two cases of constants.

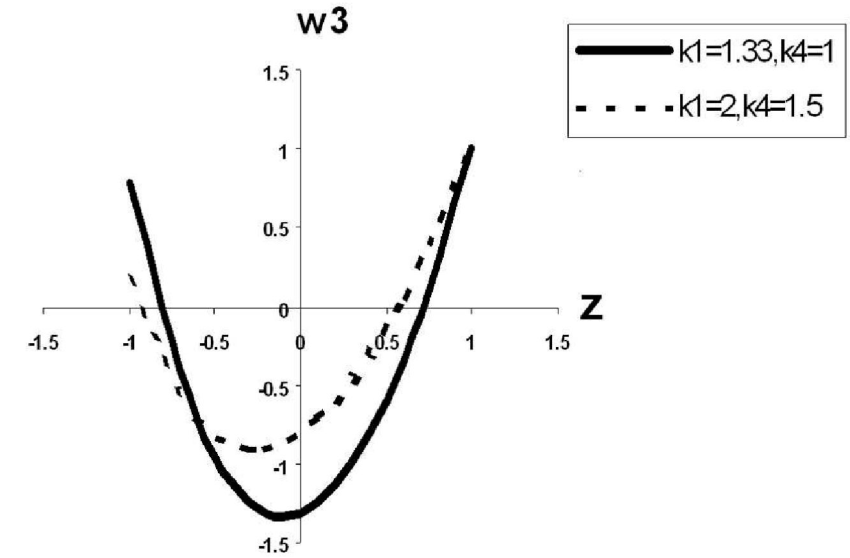

Fig. 5. The relation between z, w_3 in two cases of constants.

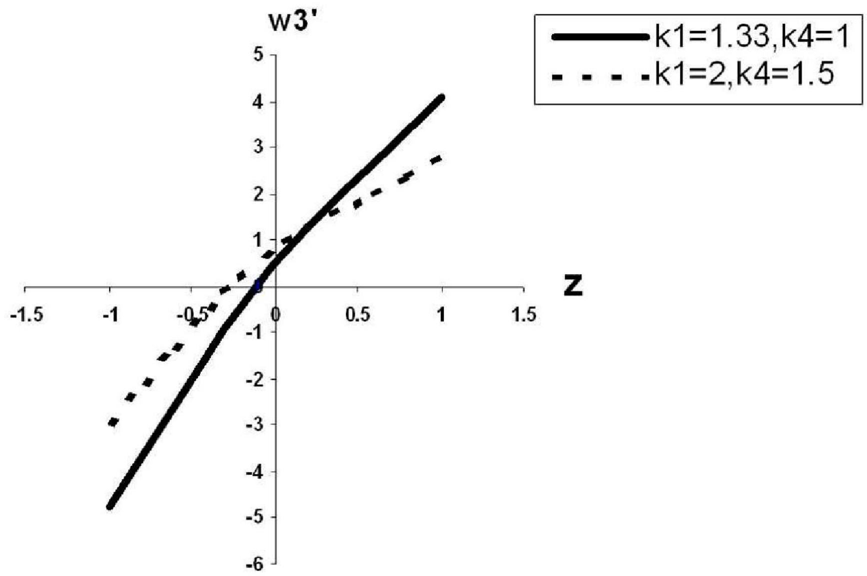

Fig. 6. The relation between z, w_3' in two cases of constants.

The system of nonlinear ordinary differential equations (2.9)–(2.11) can be written as follows:

$$\omega_1'''(z) = \frac{1}{3k_4^2}(-2zw_1'(z) - 4w_1(z) + 18w_1(z)w_1'(z) - 18w_3(z)w_2'(z) - 18w_2(z)w_3'(z)), \tag{4.2}$$

$$\omega_2'''(z) = \frac{1}{3k_4^2}(zw_2'(z) - \frac{3k_1}{k_4}w_2(z) + 9w_1(z)w_2'(z)), \tag{4.3}$$

$$\omega_3'''(z) = \frac{1}{3k_4^2}(zw_3'(z) + \frac{3k_1}{k_4}w_2(z) + 9w_1(z)w_2'(z) + 4w_3(z)). \tag{4.4}$$

We take,

$$w_1 = Y_1, \quad w_2 = Y_4, \quad w_3 = Y_7,$$

and hence equations (4.2)–(4.4) can be written as

$$Y_1' = Y_2, \quad Y_2' = Y_3,$$

$$Y_3' = \frac{1}{3k_4^2}(-2zY_2 - 4Y_1 + 18Y_1Y_2 - 18Y_7Y_5 - 18Y_4Y_8),$$

$$Y_4' = Y_5, \quad Y_5' = Y_6,$$

$$Y_6' = \frac{1}{3k_4^2}(zY_5 - \frac{3k_1}{k_4}Y_4 + 9Y_1Y_5),$$

$$Y_7' = Y_8, \quad Y_8' = Y_9,$$

$$Y_9' = \frac{1}{3k_4^2}(zY_8 + \frac{3k_1}{k_4}Y_7 + 9Y_1Y_8 + 4Y_7), \tag{4.5}$$

subject to the boundary conditions

$$Y_1(-1) = -\frac{1}{3}, \;\; Y_2(-1) = \frac{1}{3}, \quad Y_4(-1) = -1, \quad Y_5(-1) = 1, \quad Y_6(-1) = 0,$$

$$Y_1(1) = \frac{1}{3}, \quad Y_2(1) = \frac{1}{3}, \quad Y_4(1) = 1, \quad Y_7(1) = 1.$$

To apply the shooting method we use the subroutine D02HAF from the NAG Fortran library which requires the supply of starting values of the missing initial and terminal conditions. We take two special cases for k_1, k_4:

1) $k_1 = \frac{4}{3}$, $k_4 = 1$. The supplied values are

$$\begin{aligned} &Y_3(-1) = 5.36, \quad Y_7(-1) = .786, \quad Y_8(-1) = -4.75, \quad Y_9(-1) = 4.37,\\ &Y_3(1) = -5.3, \quad Y_5(1) = 0.924, \quad Y_6(1) = -0.183,\\ &Y_8(1) = 4.11, \quad Y_9(1) = 4.58. \end{aligned} \tag{4.6}$$

2) $k_1 = 2$, $k_4 = 1.5$. The supplied values are

$$Y_3(-1) = 1.86, \quad Y_7(-1) = 0.151, \quad Y_8(-1) = -2.95, \quad Y_9(-1) = 3.67,$$
$$Y_3(1) = -4.74, \quad Y_5(1) = 0.46, \quad Y_6(1) = -1.35,$$
$$Y_8(1) = 2.78, \quad Y_9(1) = 2.92. \tag{4.7}$$

The subroutine uses Runge-Kutta-Merson method with variable step size in order to control the local truncation error, then it applies modified Newton-Raphson technique to make successive corrections to the estimated boundary values.

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