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Dedicated to Reinhard Lang on the occasion of his 60 th birthday.

POINT PROCESSES IN GENERAL POSITION

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Аннотация. The aim of this note is to introduce for point processes in $\mathbb{R}^d$ the notions *general position* and *reinforced general position*, and to characterize these processes. As a consequence we show that Poisson processes P_ρ with an infinite intensity measures ρ are in general position iff ρ is diffuse in the sense that any affine subspace of dimension $d-1$ is a ρ-nullset. Moreover, P_ρ is in reinforced general position iff in addition any $(d-1)$-sphere is a ρ-nullset.

Point processes; general positions; physical clusters.

1. INTRODUCTION

The starting point of this note is Krickeberg's characterization of simple point processes within the class of point processes in terms of their moment measures. (See [1], theorem 3, corollary 2): If P is a point process in a basic phase space X of second order, then P is simple iff

$$\left(1_{D_2} \cdot \nu_P^2\right) \circ \pi^{-1} = \nu_P^1.$$

Here $\nu_P^k, k = 1, 2$, denote the first and second moment measures of P, D_2 the diagonal of X^2 and $\pi : D_2 \to X, (x, x) \mapsto x$, the projection. If P is a Poisson point process P_ρ with intensity measure ρ then P_ρ is simple iff ρ is diffuse.

The aim here is to strengthen this result. We are looking for characterizations of point processes which are in general or even reinforced general position. These notions will be made precise below.

We analyse this problem in full generality as Krickeberg did it in the special case. But instead using moment measures we shall work with reduced moment measures.

The motivation and point of departure is given by the observation that in the case of Poisson processes these results are always used implicitly in the domain of stochastic geometry, in particular for the construction of random tesselations, without mathematical justification. Exceptions are Krickeberg's work [1] as well as Møller's lectures [3] where already valuable indications with respect to this problem can be found.

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2. BASIC CONCEPTS

The starting point is a measurable space $(X, \mathcal{B})$ in which configurations of points will be realized, which are locally finite in the sense that only a finite number of points hit each member of a class $\mathcal{B}_0$ of "bounded"sets. Following Ripley [5] we assume that $\mathcal{B}$ contains all singleton subsets and that $\mathcal{B}_0$ is a nonempty subset of $\mathcal{B}$ which is hereditary, i.e.

$$\big(B \in \mathcal{B}_0, C \in B \cap \mathcal{B} \Rightarrow C \in \mathcal{B}_0\big),$$

closed under finite unions and σ-bounded. The latter means that there exists an increasing sequence $X_1, X_2, \dots$ in $\mathcal{B}_0$ covering X. We assume also that $(X, \mathcal{B}, \mathcal{B}_0)$ is countably separated, i. e. there exists a countable π-system $\tilde{\mathcal{B}}_0$ in $\mathcal{B}_0$ separating the points of X. Such spaces $(X, \mathcal{B}, \mathcal{B}_0)$ are called phase spaces here. Examples of phase spaces are all separable metric spaces where $\mathcal{B}$ in the Borel σ-field and $\mathcal{B}_0$ the collection of all metrically bounded Borel sets.

Given a phase space $(X, \mathcal{B}, \mathcal{B}_0)$, we then consider random locally finite measures μ on X, which are defined as follows: Let $\mathcal{M} = \mathcal{M}(X)$ denote the set of all measures μ on $(X, \mathcal{B})$ which are locally finite, i. e. finite on $\mathcal{B}_0$. Note that such measures are σ-finite because X is σ-bounded. $\mathcal{M}$ is endowed with the σ-field $\mathcal{F}$, generated by all variables

$$\zeta_B : \mu \mapsto \mu(B), B \in \mathcal{B}_0.$$

Important measurable subsets are

$$\begin{aligned} \mathcal{M}^{\circ} &= \{\mu \in \mathcal{M} \big| \ \mu \text{ diffuse}\}, \\ \mathcal{M}^{\cdot\cdot} &= \{\mu \in \mathcal{M} \big| \ \mu(B) \in \mathbb{N}_0 \ \forall B \in \mathcal{B}_0\} \text{ (point measures)}, \\ \mathcal{M}^{\cdot} &= \{\mu \in \mathcal{M}^{\cdot\cdot} \big| \ \mu(\{x\}) \leq 1 \ \forall x \in X\} \text{ (simple point measures)}. \end{aligned}$$

The traces of $\mathcal{F}$ in these spaces are denoted by $\mathcal{F}^{\circ}, \mathcal{F}^{\cdot\cdot}, \mathcal{F}^{\cdot}$. Elements $\mu \in \mathcal{M}^{\cdot}$ are often identified with locally finite subsets of X.

A random measure in X is a probability measure P on $(\mathcal{M}, \mathcal{F})$, write $P \in \mathcal{PM}$ for short. Probabilities P on $(\mathcal{M}^{\cdot\cdot}, \mathcal{F}^{\cdot\cdot})$ resp. $(\mathcal{M}^{\cdot}, \mathcal{F}^{\cdot})$ are called point processes resp. simple point processes in X, and we write $P \in \mathcal{PM}^{\cdot\cdot}$ resp. $P \in \mathcal{PM}^{\cdot}$.

Given $\mu \in \mathcal{M}$, $k \geq 1$, μ^k denotes the k-th power of μ. If $\mu \in \mathcal{M}^{\cdot\cdot}$, the reduced k-th power of μ is the measure on X^k defined by

$$\dot{\mu}^k(f) = \int_{X^k} \mu(dx_1)(\mu - \delta_{x_1})(dx_2)\dots(\mu - \delta_{x_1} - \dots - \delta_{x_{k-1}})(dx_k)f(x_1,\dots,x_k)$$

Here $f \in \mathcal{H}_+(X^k)$, the space of all non negative measurable functions on X^k.

Note that $\mu \mapsto \mu^k(f)$ and $\mu \mapsto \dot{\mu}^k(f)$ are measurable. If $\mu \in \mathcal{M}^{\cdot}$ then $k!\dot{\mu}^k$ counts the subsets of μ of cardinality k. Given $P \in \mathcal{PM}^{\cdot\cdot}$ the following versions of the Campbell measure of order k are well defined:

$$C_P^k(h) = \int_{\mathcal{M}^{\cdot\cdot}} \int_{X^k} h(x_1,\dots,x_k;\mu)\ \mu^k(dx_1,\dots,dx_k)\ P(d\mu),$$

$$\dot{C}_P^k(h) = \int_{\mathcal{M}^{\cdot\cdot}} \int_{X^k} h(x_1,\dots,x_k;\mu)\ \dot{\mu}^k(dx_1,\dots,dx_k)\ P(d\mu).$$

Here $h \in \mathcal{H}_+\big(X^k \times \mathcal{M}^{\cdot\cdot}\big)$. If $f \in \mathcal{H}_+(X^k)$ we write

$$\nu_P^k\ (f) = C_P^k\ (f \otimes 1) \quad \text{and} \quad \dot{\nu}_P\ (f) = \dot{C}_P^k\ (f \otimes 1).$$

ν_P^k is called the k-th moment measure and $\dot{\nu}_P^k$ the reduced k-th moment measure.

3. KRICKEBERG'S CRITERIUM

To prepare later refinements we first derive Krickeberg's characterization of simple point processes in X by an argument which we shall use later.

Observe first that simple point measures are characterized within the class of point measures as follows: $\mu \in \mathcal{M}^{\cdot\cdot}$ is simple, i. e. $\mu \in \mathcal{M}^{\cdot}$ iff $\dot{\mu}^2\ (D_2) = 0$. Here $D_2 = \{(x,x) \in X^2 \big| x \in X\}$ denotes the diagonal in X^2. Since $D_2 \in \mathcal{B}(X) \otimes\ \mathcal{B}(X)$ and $\mu \mapsto \dot{\mu}^2\ (D_2)$ is measurable, so is $\mathcal{M}^{\cdot}$. One immediately obtains

Theorem 1. *Let $P \in \mathcal{PM}^{\cdot\cdot}$. Then the following assertions are equivalent:*

(3.1) $$P \in \mathcal{PM}^{\cdot},\ \textit{i. e. } P \textit{ is simple};$$

(3.2) $$\dot{\nu}_P^2\ (D_2) = 0.$$

This result contains Krickeberg's criterium in [1] stated at the beginning.

Corollary 1. Let $\rho \in \mathcal{M}$. Then the following assertions are equivalent:

(3.3) $$P_\rho \in \mathcal{PM}^{\cdot};$$

(3.4) $$\rho^2\ (D_2) = 0;$$

(3.5) $$\rho \in \mathcal{M}^{\circ}.$$

Here the equivalence (3.3) $\iff$ (3.4) is due to the fact that $\dot{\nu}^2_{P_\rho} = \rho^2$, which in turn follows immediately from Mecke's characterization of the Poisson process by means of its Campbell measure. (See [2].) The equivalence (3.4) $\iff$ (3.5) follows directly from

$$\rho^2\ (D_2) = \int \rho\ (dx)\ \rho\ (\{x\}).$$

If $X = \mathbb{R}^d$ and ρ the Lebesque measure on X then $P_\rho \in \mathcal{PM}^{\cdot}$. Recall that in general $P_\rho \in \mathcal{PM}^{\cdot\cdot}$.

4. POINT PROCESSES IN GENERAL POSITION

We now assume that $(X, \mathcal{B}, \mathcal{B}_0)$ is a linear phase space. This means that X has the structure of a vector space of dimension d such that its scalar multiplication and addition are measurable operations. Our main example is $X = \mathbb{R}^d$.

The aim is to characterize (within $\mathcal{M}^{\cdot\cdot}_\infty = \mathcal{M}^{\cdot\cdot} \cap \{\zeta_X = +\infty\}$) those configurations which are in general position. Here $\mu \in \mathcal{M}^{\cdot\cdot}$ is **in general position**, we then write $\mu \in \mathcal{M}^{\cdot}_{gp}$, iff

(4.1) $(\nu \leq \mu, 2 \leq |\nu| \leq d+1 \Rightarrow \nu$ affinely independent)

Here $|\nu| = \nu(X)$; $\nu \leq \mu$ means that ν is a non-void subconfiguration of μ. Finally, $\nu \in \mathcal{M}^{\cdot\cdot}_f$ is called **affinely independent**, write $\nu\,$**ai**, iff $\nu \in \mathcal{M}^{\cdot}$, and if any element $x \in \nu$ is not contained in aff $(\nu - \delta_x)$, the linear variety in X generated by $\nu - \delta_x$. By definition $\mathcal{M}^{\cdot}_{gp}$ is a subset of $\mathcal{M}^{\cdot}$

ν is called **affinely dependent** (**ad**) otherwise. Any singleton is affinely independent; and $\delta_x + \delta_y$ is affinely dependent iff $x = y$. If ν is **ad** then so is $\nu + \delta_x$ for any $x \in X$; thus any non-void subconfiguration of an **ai** ν is **ai**.

We now give an equivalent description of configurations in general position: Let

$$D_k = \{(x_1, \ldots, x_k) \in X^k \big| \delta_{x_1} + \ldots + \delta_{x_k} \mathbf{ad}\}\ 2 \leq k \leq d+1.$$

D_k is a measurable subset of X^k. The following result is the main lemma of this note.

Lemma 1. *If $\mu \in \mathcal{M}^{\cdot\cdot}_\infty$, the following assertions are equivalent:*

(4.1) $\quad \mu \in \mathcal{M}^{\cdot}_{gp}$, *i.e. μ is in general position;*

(4.2) $\quad \dot{\mu}^k(D_k) = 0$ *for* $2 \leq k \leq d+1$;

(4.3) $\quad \dot{\mu}^{d+1}\ (D_{d+1}) = 0.$

Proof The implications (4.1) $\Rightarrow$ (4.2) $\Rightarrow$ (4.3) are obvious.

ad (4.3) $\Rightarrow$ (4.2): Observe first that

$(\alpha)\ \dot{\mu}^{d+1}\ (D_{d+1}) = \int\limits_{X^d} \dot{\mu}^d(dx_1, \ldots, dx_d) \int\limits_X (\mu - \delta_{x_1} - \ldots - \delta_{x_d})(dx) \mathbf{1}_{D_{d+1}(x_1,\ldots,x_d)}\ (x),$
where $D_{d+1}(x_1, \ldots, x_d) = \{x \in X | \delta_{x_1} + \ldots + \delta_{x_d} + \delta_x \ \mathbf{ad}\}$.
But

$$D_{d+1}\ (x_1, \ldots, x_d) = \begin{cases} \text{aff}\ (\delta_{x_1} + \ldots + \delta_{x_d}) & ,\ \text{if}\ (x_1, \ldots, x_d) \notin D_d \\ X & ,\ \text{else.} \end{cases}$$

Therefore under assumption (4.3) one obtains
$(\beta)\ \int\limits_{D_d} \dot{\mu}^d(d_{x_1}, \ldots, d_{x_d})\ (\mu - \delta_{x_1} - \ldots - \delta_{x_d})\ (X) = 0.$

Since the integrand is $\equiv +\infty$, one has $\dot{\mu}^d(D_d) = 0$. Iterating this argument yields (4.2).

ad (4.2) $\Rightarrow$ (4.1): Under (4.2) we have $\dot{\mu}^2(D_2) = 0$, which is equivalent to μ being simple. Then it is obvious that (4.2) implies (4.1).

A direct consequence is the measurability of $\mathcal{M}^{\cdot}_{gp}$ because all D_k are measurable and also $\mu \mapsto \dot{\mu}^k(D_k)$. Moreover one immediately has the

Theorem 2. *If $P \in P\mathcal{M}^{\cdot\cdot}_{\infty}$ then the following assertions are equivalent:*

(4.4) $\quad P \in P\mathcal{M}^{\cdot}_{gp}$, *i.e. P is in general position*;

(4.5) $\quad \dot{\nu}^k_P(D_k) = 0$ *for* $2 \leq k \leq d+1$;

(4.6) $\quad \dot{\nu}^{d+1}_P\ (D_{d+1}) = 0$.

We remark here that a necessary condition for a point process to be concentrated on $\mathcal{M}^{\cdot\cdot}_{\infty}$ is that ν^1_P is an infinite measure. If $\rho \in \mathcal{M}$ then $P_\rho \in \mathcal{PM}^{\cdot\cdot}_{\infty}$ iff ρ is infinite.

Corollary 2. Let $\rho \in \mathcal{M}$ be infinite. Then the following assertions are equivalent:

(4.7) $\quad P_\rho \in \mathcal{PM}^{\cdot}_{gp}$, *i.e. P_ρ is in general position*;

(4.8) $\quad \rho^k\ (D_k) = 0$ *for all* $2 \leq k \leq d+1$;

(4.9) $\quad \rho^{d+1}\ (D_{d+1}) = 0$;

(4.10) $\quad$ *any affine subspace of X of dimension $d-1$ is a ρ-null set.*

Here (4.7) $\Leftrightarrow$ (4.8) $\Leftrightarrow$ (4.9) follows from the theorem because $\dot{\nu}^k_P = \rho^k$ by Mecke's theorem.

ad (4.10) $\Rightarrow$ (4.8): Observe first that for $2 \leq k \leq d+1$

$$(\alpha) \quad \rho^{k+1}\ (D_{k+1}) = \int\limits_{D_k} \rho^k(dx_1, \ldots, dx_k)\rho(X) +$$

$$+ \int\limits_{D_k^c} \rho^k(dx_1, \ldots, dx_k)\rho(\text{aff}(\ \delta_{x_1} + \ldots + \delta_{x_k}))$$

where $D_1 = \emptyset$.

Under assumption (4.10) the second integral on the right hand side always vanishes. On the other hand $\rho(X) = +\infty$, and an induction starting with $k = 1$ yields (4.8).

ad (4.8) $\Rightarrow$ (4.10): Assumption (4.10) combined with (α) yields for $k = d$

$$(\beta) \qquad \rho^{d+1}(D_{d+1}) = \int\limits_{D_d^c} \rho^d(dx_1, \ldots, dx_d)\ \rho\ (\text{aff}(\delta_{x_1} + \ldots + \delta_{x_d})),$$

because D_d is a ρ^d- null set.

This implies (4.10) by the following argument: Suppose that A is an affine subspace of X of dimension $d-1$. Then (β) yields

$$0 = \rho^{d+1}(D_{d+1}) \geq \int\limits_{A^d \cap D_d^c} \rho^d(dx_1, \ldots, dx_d)\ \rho\ (\text{aff}(\ \delta_{x_1} + \ldots + \delta_{x_d})).$$

But $((x_1, \ldots, x_d) \in A^d \cap D_d^c \Rightarrow \text{aff}(\ \delta_{x_1} + \ldots + \delta_{x_d}) = A)$, so that

$$0 = \rho^{d+1}(D_{d+1}) \geq \rho(A)^{d+1}.$$

This implies $0 = \rho(A)$.

Condition (4.10) is a strong version of ρ being diffuse. As a consequence, the Poisson process P_ρ in $\mathbb{R}^d$ with intensity measure $\rho = z \cdot \lambda^d, z > 0$, is in general position.

5. POINT PROCESSES IN REINFORCED GENERAL POSITION

The aim now is to describe within the class of point processes in general position those whose configurations are non circular in the sense that subconfigurations of cardinality $n+1$ between 1 and $d+2$ are not situated on an $(n-1)$-sphere.

We therefore assume now that X is equipped with an inner product. However there is no essential loss of generality in working only with $X = \mathbb{R}^d$, which we shall do now. A configuration $\mu \in \mathcal{M}^{\cdot}_{gp}$ is called **in reinforced general position**, write then $\mu \in \mathcal{M}^{\cdot}_{rgp}$, iff

(5.1.) $(\nu \subseteq \mu, 3 \leq |\nu| \leq d+2 \Rightarrow \nu$ non circular$)$.

Here $\nu \in \mathcal{M}^{\cdot\cdot}_f$ is called **non circular (nc)** iff $\nu \in \mathcal{M}^{\cdot}_{gp}$, and if any element $x \in \nu$ is not contained in $sph\ (\nu - \delta_x)$, the circum sphere generated by the **ai** subconfiguration $\nu - \delta_x$ in aff $(\nu - \delta_x)$. ν is called **circular (c)** otherwise.

Any singleton is **nc**; any $\delta_x + \delta_y, x \neq y$, also, and any $\delta_x + \delta_y + \delta_z$ **ai** is **nc**. All $\nu \in \mathcal{M}^{\cdot\cdot}_f$ which are not in general position are **c**. If ν is in general position, then ν is circular if there exists $x \in \nu$ such that $x \in\ sph\ (\nu - x)$.

We now proceed in complete analogy to the case of processes in general position. Let

$$S_k = \{(x_1, \ldots, x_k) \in X^k \big| \delta_{x_1} + \ldots + \delta_{x_k} \text{ is } \mathbf{c}\},\ 3 \leq k \leq d+2.$$

This is a measurable subset of X^k.

Lemma 2. *Let $\mu \in \mathcal{P}M^{\cdot}_{gp} \cap \mathcal{M}^{\cdot}_{\infty}$. Then the following assertions are equivalent:*

(5.1) μ *is in reinforced general position, i.e.* $\mu \in \mathcal{M}^{\cdot}_{rgp}$;

(5.2) $\dot{\mu}^k(S_k) = 0$ *for any* $3 \leq k \leq d+2$;

(5.3) $\dot{\mu}^{d+2}(S_{d+2}) = 0.$

The proof uses the same ideas as the one of lemma 1 (the main lemma) and will not be repeated here. Again lemma 2 immediately implies measurability of $\mathcal{M}^{\cdot}_{rgp}$, and one has

Theorem 3. *If $P \in \mathcal{P}M^{\cdot}_{gp} \cap \mathcal{M}^{\cdot}_{\infty}$, then the following assertions are equivalent:*

(5.4) P *is in reinforced general position, i.e.* $P \in \mathcal{P}\mathcal{M}^{\cdot}_{rgp}$;

(5.5) $\dot{\nu}^k_P\ (S_k) = 0$ *for all* $3 \leq k \leq d+2$;

(5.6) $\dot{\nu}^{d+2}_P\ (S_{d+2}) = 0.$

If specialized to Poisson processes we get in the same way as above

Corollary 3. Let $\rho \in \mathcal{M}$ be infinite such that any affine subspace of dimension $d-1$ is a ρ-null set. Then P_ρ is in reinforced general position iff any $(d-1)$-sphere in X is a ρ- null set.

If ρ is the Lebesque measure on X, then P_ρ is in reinforced general position.

6. APPLICATIONS TO GIBBSIAN POINT PROCESSES

Given $\rho \in \mathcal{M} \setminus \{0\}$, P_ρ is the unique solution of the equation

$$\textbf{(Mecke)} \qquad C_P^{'} = \rho \otimes P, \qquad P \in \mathcal{M}^{\cdot\cdot}(X).$$

Here $C_P^{'}$ denotes the reduced Campbell measure.
We call a point process P Gibbs for ρ if and only if

$$(\mathbf{\Sigma}_\rho) \qquad C_P^{'} \ll \rho \otimes P$$

If in $(\mathbf{\Sigma}_\rho)$ the density V is given in addition to ρ then P is called a Gibbs process for ρ and V, and we write $P \in \mathcal{G}(\rho, V)$. V is called the Boltzmannfactor. If V has the form $exp(-\beta \cdot E^{\phi})$ for some β strictly positive and some potential ϕ then P satisfies

($\mathbf{\Sigma}_\rho$) iff P is a Gibbs state for (ϕ, ρ) in the sense of Dobrushin/Lanford/Ruelle. Here $E^\phi(x, \mu)$ denotes the socalled energy of x in μ. It is then easy to see that

$$\dot{\nu}_P^k(f) = \int\limits_{X^k} f(x) \int\limits_{\mathcal{M}^{\cdot\cdot}} exp(-\beta \cdot E^\phi(x, \mu)) P(d\mu) \rho^k(dx),$$

$f \geq 0$ and measurable. For the details we refer to [4]. Thus we get the following

Corollary 4. *If ρ is infinite such that any affine subspace of dimension $d-1$ is a ρ-nullset and any $(d-1)-sphere$ is a ρ-nullset then any $P \in \mathcal{G}(\rho, V)$ is in reinforced general position.*

Список литературы

[1] K. Krickeberg, "Moments of point processes," E. F. Harding and D. G. Kendall, *Stochastic Geometry* (Wiley, 1974).

[2] J. Mecke, "Stationäre Maße auf lokalkompakten Abelschen Gruppen," Z. W-Theorie verw. Geb. **9**, 36-58 (1967).

[3] J. Møller, "Lectures on random Voronoi tesselations," Lecture notes in statistics **87**, (Springer, 1994).

[4] X. X. Nguyen and H. Zessin, "Integral and differential characterizations of the Gibbs process," Math. Nachr. **88**, 105-115 (1979).

[5] B. D. Ripley, "Locally finite random sets: foundations for point process theory," The Annals of Probability **4**, 983-994 (1976).

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