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Dedicated to Reinhard Lang on the occasion of his 60 th birthday.

A THEOREM OF MICHAEL MÜRMANN REVISITED

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Аннотация. A fundamental theorem of Mürmann [2] characterizing equilibrium distributions of physical clusters is reconsidered. We recover this result by means of the integration by parts formula approach to Gibbs processes due to Nguyen Xuan Xanh and Hans Zessin [4].

Gibbs processes; equilibrium distributions; physical clusters.

1. THE PROBLEM

Let E denote the $d-$ dimensional Euclidean space $\mathbb{R}^d$ and $\rho = z\lambda^d \, (z > 0)$ the Lebesgue measure on it. We consider simple point processes P on the space $\mathcal{M}^{\cdot}(E)$ of locally finite subsets η of E , for which we write $P \in \mathcal{PM}^{\cdot}(E)$ for short. Often we do not write the underlying space E, and as usual η is often considered as a simple point measure on E. We'll write $\mathcal{M}^{\cdot}_f$ for finite configurations in $\mathcal{M}^{\cdot}$.

We are given a parameter $R > 0$, which enables us to define the notions of an $R-$cluster or an $R-$cluster property as follows.

$x \in \mathcal{M}^{\cdot}$ is called an $R-$cluster (Rcl) iff x is void, or a singleton, or if the graph obtained by joining interacting points ('particles') in x is connected. We then write $x \in \mathcal{R}^{\cdot}$. Observe that $\mathcal{R}^{\cdot}$ is a measurable subset of $\mathcal{M}^{\cdot}$ containing 0 and all singletons. We write $\mathcal{R}^{\cdot}_f$ for the finite elements in $\mathcal{R}^{\cdot}$. The cluster property defined by the parameter R is given by the following measurable subset of $\mathcal{M}^{\cdot} \times \mathcal{M}^{\cdot}$:

$$((x,\eta) \in D := D_R \textit{ iff } x \in \mathcal{R}^{\cdot} \textit{ with } \eta(\dot{B}_R(x)) = 0). \tag{1.1}$$

Here $\dot{B}_R(x) = B_R(x) \setminus x$, where $B_R(x) := \cup_{a \in x} B_R(a)$ with $B_R(a)$ denoting the open ball in E centered in a with radius R. $\dot{B}_R(0) = \emptyset$. We define also the subset D^* in $\mathcal{M}^{\cdot} \times \mathcal{M}^{\cdot}$ by $((x,\eta) \in D^* \textit{ iff } \eta(\dot{B}_R(x)) = 0)$. Thus $\mathbf{1}_D(x,\eta) = \mathbf{1}_{D^*}(x,\eta) \cdot \mathbf{1}_{\mathcal{R}^{\cdot}}$.

Observe that 0 is an Rcl for any $\eta \in \mathcal{M}^{\cdot}$. Furthermore, D is translation invariant. We remark that D induces in each configuration $\eta \in \mathcal{M}^{\cdot} \setminus \{0\}$ an equivalence relation by means of

$$(a \sim b \textit{ iff there exists } x \subseteq \eta \textit{ with } a, b \in x \textit{ and } (x, \eta) \in D) \tag{1.2}$$

Thus two elements a, b of η are equivalent iff there exists an '$R-$path' in η from a to b. The equivalence classes are the $r-$clusters in η.

Consider then the random element on $(\mathcal{M}^{\cdot}, P)$ defined by

$$\gamma : \eta \longrightarrow \Sigma_{x \subseteq \eta} \mathbf{1}_D(x, \eta) \cdot \delta_x =: \mu. \tag{1.3}$$

(If $\eta = 0$ then $\mu = \delta_0$.) It is obvious that γ is a point process on $\mathcal{R}^{\cdot}$ with distribution Q concentrated on

$$\mathcal{M}^{(\cdot)}(\mathcal{R}^{\cdot}) = \{\mu \mid x, y \in \mu, x \neq y \Rightarrow B_{R/2}(x) \cap B_{R/2}(y) = \emptyset\}. \tag{1.4}$$

γ resp. Q, the image of P under γ, is called the *cluster process belonging to* P.Note that γ is a Borel isomorphism between $\mathcal{M}^{\cdot}(E)$ and $\mathcal{M}^{(\cdot)}(\mathcal{R}^{\cdot})$ with inverse

$$\chi_{(0)} : \mu \longrightarrow \mu_{(0)} := \Sigma_{x \in \mu} x, \tag{1.5}$$

and the relation $Q = \gamma P$, the image of P under γ, establishes a 1-1 correspondence between the simple point processes P in E and the cluster processes Q in $\mathcal{R}^{\cdot}$, i.e. the point processes $Q \varepsilon \mathcal{M}^{(\cdot)}(\mathcal{R}^{\cdot})$.

The *aim of this paper* is the description of cluster processes belonging to Gibbs processes in E interacting by finite range pair potentials. Such processes will be introduced now.

2. THE GIBBSIAN FRAMEWORK

We are given an even and stable pairpotential ϕ on E with finite range $R > 0$. The associated Boltzmann factor is

$$f(a, \eta) = exp(-E(a, \eta)), a \in E, \eta \in \mathcal{M}^{\cdot}(E). \tag{2.1}$$

Here

$$E(a, \eta) = \sum_{a \neq b \in \eta} \phi(b - a), a \in E, \eta \in \mathcal{M}^{\cdot}(E), \tag{2.2}$$

is the *energy of a in η*.

Recall that a simple point process P in E is called a *Gibbs process for* (ϕ, ρ) iff P solves the following *integration by parts formula*

$$C_P(h) = \int_{\mathcal{M}^{\cdot}(E)} \int_E h(a, \eta + \delta_a) \cdot f(a, \eta)\, \varrho(da) P(d\eta), h \in F_+(E \times \mathcal{M}^{\cdot}(E)). \tag{Σ_ϱ^ϕ}$$

Here C_P denotes the Campbell measure of P and $F_+(X)$ the collection of non-negative, measurable functions on the measurable space X. We then write $P \in \mathcal{G}(\phi, \varrho)$. In the sequel we'll write always F_+ instead of $F_+(X)$.

The above definition of a Gibbs process is justified by a theorem of Nguyen X.X., H. Zessin [4], saying that Gibbs processes in the DLR (=Dobrushin/Lanford/Ruelle) sense are equivalently defined by (Σ_ϱ^ϕ). Iterating the integration by parts formula implies immediately the important

Lemma 1. *If* $P \in \mathcal{G}(\phi, \varrho)$, *then* P *satisfies*

$$\int\limits_{\mathcal{M}^\cdot}\int\limits_{E} \Sigma_{x \subset \eta, x \in \mathcal{M}_f^\cdot}\, h(x,\eta) P(d\eta)$$

$$= \int\limits_{\mathcal{M}^\cdot}\int\limits_{\mathcal{M}_f^\cdot} h(x, \eta + x) \cdot \exp(-E(x,\eta)) W_\varrho(dx) P(d\eta), h \in F_+.$$

Here $E(x,\eta)$ *is the energy of* x *given* η*, defined by*

$$E(\delta_{a_1} + ... + \delta_{a_n}, \eta) = E(a_1, \eta) + E(a_2, \eta + \delta_{a_1}) + ... + E(a_n, \eta + \delta_{a_1} + ... + \delta_{a_{n-1}}).$$

Moreover

$$W_\varrho(\varphi) = \sum_{n \geq 0} \frac{1}{n!} \int\limits_{E^n} \varphi(\delta_{a_1} + ... + \delta_{a_n})\, \varrho(da_1) ... \varrho(da_n),\ \varphi \in F_+.$$

We'll not use the DLR-approach in the sequel. This has been done by Mürmann in [2]. Instead we use systematically the integration by parts approach accompanied by the following equation due to Ruelle which is equivalent to (Σ_ϱ^ϕ) if the underlying point process P is of first order. The simplicity and beauty of this approach to Gibbs processes will be visible in the sequel. Ruelle's equation is

$$(\mathcal{R})\ P(\varphi) = \int\limits_{\mathcal{M}_{\Lambda^c}^\cdot}\int\limits_{\mathcal{M}_\Lambda^\cdot} \varphi(\xi + \eta) \cdot \exp(-E_\Lambda(\xi + \eta)) W_\varrho(d\xi) P(d\eta), \varphi \varepsilon F_+, \Lambda \varepsilon \mathcal{B}_0(E).$$

Here $\mathcal{B}_0(E)$ denotes the collection of all bounded Borel sets in E, and $\exp(-E_\Lambda(\xi + \eta)) = E(\xi, \eta)$ if $\xi \subseteq \Lambda, \eta \subseteq \Lambda^c$. (For a proof of the equivalence of $(\mathcal{R})$ and (Σ_ϱ^ϕ) see [4].)

3. MUERMANN'S FIRST THEOREM

If $(x, \eta) \in D$ we say that x is a cluster *for* η; if in addition $x \subseteq \eta$ then x is called a cluster *in* η. We consider the random variable

$$cd_D : \mathcal{M}^{\cdot}(E) \longrightarrow \mathbb{N}_0 \cup \{\infty\}, \eta \longmapsto \sum_{x \subseteq \eta} \mathbf{1}_D(x, \eta), \tag{3.1}$$

which counts the $R-$ clusters in η. We use also the notations $D_\infty = \{(x, \eta) \in D \mid |x| = +\infty\}$, $D_f = \{(x, \eta) \in D \mid |x| < +\infty\}$; cd_{D_f} as well as cd_{D_∞} are defined similarly.

Given (ϕ, ρ) as above, let P be from now on a Gibbs process for (ϕ, ρ) , being concentrated on $\mathcal{M}^{\cdot}_\infty$, the set of infinite configurations, which satisfies also the condition

$$P\{cd_{D_\infty} = 0\} = 1. \tag{3.2}$$

As a consequence P is concentrated on

$$\mathcal{M}^{\cdot}_D = \{cd_{D_f} = +\infty, cd_{D_\infty} = 0\}. \tag{3.3}$$

In this situation γ resp. Q is called the *Mürmann cluster process for* P. The Mürmann cluster process is a point process in $\mathcal{R}^{\cdot}_f$ which is concentrated on $\mathcal{M}^{(\cdot)}(\mathcal{R}^{\cdot}_f)$.

If one defines *percolation* in this model as the occurrence of an infinite cluster with positive probability, we are working in a situation where percolation does not occur.

We now calculate the Campbell measure of γ resp.Q : Let h be non negative and measurable. By definition

$$C_Q(h) = \int \sum_{x \subseteq \eta, x \in \mathcal{R}^{\cdot}_f} \mathbf{1}_{D^*}(x, \eta) \cdot h(x, \gamma(\eta)) P(d\eta).$$

Since P is a Gibbs process for (ϕ, ρ) and a.s. all clusters are finite, this equals

$$\int \int \mathbf{1}_{D^*}(x, \eta + x) \cdot h(x, \gamma(\eta + x)) \cdot w(x, \eta) \cdot u(x) \cdot \mathbf{1}_{\mathcal{R}^{\cdot}_f}(x) W_\varrho(dx) P(d\eta).$$

Here $u(x)$ denotes the part of the Boltzmann factor which depends only on x and $w(x, \eta)$ the one taking into account only the interaction between x and η. Now observe that $(x, \eta + x) \in D^*$ iff $(x, \eta) \in D^*$. Obviously one then has $\gamma(\eta + x) = \gamma(\eta) + \delta_x$ and $w \equiv 1$. As a consequence we obtain

$$C_Q(h) = \int \int h(x, \gamma(\eta) + \delta_x) \cdot \mathbf{1}_{D^*}(x, \gamma(\eta)_{(0)}) \cdot u(x) \cdot \mathbf{1}_{\mathcal{R}^{\cdot}_f}(x) W_\varrho(dx) P(d\eta).$$

We call the measure

$$M_\varrho = u \cdot \mathbf{1}_{\mathcal{R}^{\cdot}_f} W_\varrho$$

appearing here the *Mürmann measure* on $\mathcal{R}^{\cdot}_f$. Now all ingredients are developped for the following first basic result of Mürmann ([2]).

Theorem 1. *If ρ and ϕ are given as above and $P \in \mathcal{G}(\phi, \varrho)$ is concentrated on $\mathcal{M}^{\cdot}_{\infty}$ and satisfies condition (3.2), then Q, the associated Mürmann cluster process, is a* **Poisson exclusion process** *for (D^*, M_ρ). I.e. Q is a simple point process in $\mathcal{R}^{\cdot}_f$, solution of the following integration by parts formula*

$$(\Sigma_\varrho^{D^*}) \quad C_Q(h) = \int\limits_{\mathcal{M}^{\cdot}(\mathcal{R}^{\cdot}_f)} \int\limits_{\mathcal{R}^{\cdot}_f} h(x, \mu + \delta_x) \cdot \mathbf{1}_{D^*}(x, \mu_{(0)})\, M_\varrho(dx) Q(d\mu), h \in F_+.$$

We see that Q, the Mürmann cluster process, is again a Gibbs process, but now in $\mathcal{R}^{\cdot}_f$. On the other hand its interaction structure is much simpler: the clusters are hard cores but do not interact otherwise.

The theorem implies immediately that the intensity of Q is given by

$$\nu_Q(h) = \int\limits_{\mathcal{R}^{\cdot}_f} h(x) \cdot P(D_x^*) M_\varrho(dx), h \varepsilon F_+.$$

The intensity measure $\nu_Q(h)$ is *Radon* (i.e. locally finite) if this is the case for the Mürmann measure, i.e. if M_ρ satisfies Mürmann's condition

$$(\mathcal{M}) \quad M_\rho \quad \textit{is finite on} \quad \mathcal{B}_0(\mathcal{R}^{\cdot}_f).$$

The meaning of this condition is the following: The collection $\mathcal{B}_0(\mathcal{R}^{\cdot}_f)$ of bounded Borel sets in $\mathcal{R}^{\cdot}_f$ is generated by the collection $\mathcal{F}_\Lambda, \Lambda \varepsilon \mathcal{B}_0(E)$, of events meeting Λ. (For a formal definition see (5.1).) Moreover, γ then is diffuse $P-a.s.$ since ρ and thereby M_ρ is diffuse. The main significance of condition Mürmann's condition $(\mathcal{M})$ is however that it implies also condition (3.2) if the underlying P is of first order. A sufficient condition for Mürmann's condition is that z is small enough. This aspect will be developped below.

4. THE CONVERSE OF MUERMANN'S FIRST THEOREM

Let$Q$ be a Poisson exclusion process for (D_R^*, M_ρ^ϕ) on $\mathcal{R}^{\cdot}_f$. Then Q is concentrated on $\mathcal{M}^{(\cdot)}(\mathcal{R}^{\cdot}_f)$. Consider the image P of Q under the measurable transformation $\chi_{(0)}$. This mapping dissolves the clusters of μ into its particles.

We calculate the Campbell measure of P. Using that Q is a Poisson exclusion process one has for any given $h \varepsilon F_+$,

$$\begin{aligned} C_P(h) &= \int \int h(a, \mu_{(0)}) \mu_{(0)}(da) Q(d\mu) \\ &= \int \int \int h(a, \mu_{(0)}) x(da) \mu(dx) Q(d\mu) \\ &= \int \int \int h(a, (\mu + \delta_x)_{(0)}) \cdot \mathbf{1}_{D^*}(x, \mu_{(0)}) \cdot \mathbf{1}_{\mathcal{R}^{\cdot}_f}(x) \cdot u(x) x(da) W_\varrho(dx) Q(d\mu). \end{aligned}$$

We then study the inner double integral of the last triple integral, which we denote by I_μ, using partial integration with respect to $W_\varrho(dx)$. As a result we obtain

$$I_\mu = \int\int h(a,\mu_{(0)}+x+\delta_a)\cdot \mathbf{1}_{D^*}(x+\delta_a,\mu_{(0)})\cdot \mathbf{1}_{\mathcal{R}^\cdot_f}(x+\delta_a)\cdot u(x+\delta_a)W_\varrho(dx)\varrho(da).$$

The next step is the analysis of the double integral

$$I_a = \int\int h(a,\mu_{(0)}+x+\delta_a)\cdot \mathbf{1}_{D^*}(x+\delta_a,\mu_{(0)})\cdot \mathbf{1}_{\mathcal{R}^\cdot_f}(x+\delta_a)\cdot u(x+\delta_a)W_\varrho(dx)Q(d\mu).$$

Here we subdivide the variable x into $R-$clusters in such a way that they form, together with a , an $R-$cluster. To be more precise, the following representation is true:

$$\begin{aligned}
&\mathbf{1}_{D^*}(x+\delta_a,\mu_{(0)})\cdot \mathbf{1}_{\mathcal{R}^\cdot_f}(x+\delta_a)\cdot u(x+\delta_a)\\
&= \Sigma_{k\geq 0}\tfrac{1}{k!}\Sigma_{z_1\subseteq x}\Sigma_{z_2\subseteq x-z_1}\cdots\Sigma_{z_{k-1}\subseteq x-(z_1+\cdots+z_{k-2})}\mathbf{1}_{D^*}(a,\mu_{(0)})\\
&\quad\times\Pi_{j=1}^{k-1}\mathbf{1}_{D^*}(z_j,z_1+\cdots+z_{j-1}+\mu_{(0)}))\cdot\\
&\quad\cdot\mathbf{1}_{D^*}(x-(z_1+\cdots+z_{k-1}),z_1+\cdots+z_{k-1}+\mu_{(0)}))\\
&\times\Pi_{j=1}^{k-1}\mathbf{1}_{\mathcal{R}^\cdot_f}(z_j)\cdot\mathbf{1}_{\dot B(z_j)}(a)\cdot u(z_j)\cdot\exp(-E(a,z_j))\cdot\mathbf{1}_{\mathcal{R}^\cdot_f}(x-(z_1+\cdots+z_{k-1}))\\
&\quad\times\mathbf{1}_{\dot B(x-(z_1+\cdots+z_{k-1}))}(a)\cdot u(x-(z_1+\cdots+z_{k-1}))\cdot\\
&\quad\cdot\exp(-E(a,x-(z_1+\cdots+z_{k-1}))).
\end{aligned}$$

Using again partial integration with respect to W_ϱ, the inner integral of I_a equals

$$\begin{aligned}
&\Sigma_{k\geq 0}\frac{1}{k!}\int\cdots^k\int h(a,\mu_{(0)}+x+(z_1+\cdots+z_{k-1})+\delta_a)\cdot\mathbf{1}_{D^*}(a,\mu_{(0)})\\
&\times\Pi_{j=1}^{k-1}\,\mathbf{1}_{D^*}(z_j,z_1+\cdots+z_{j-1}+\mu_{(0)}))\cdot\mathbf{1}_{D^*}(x,z_1+\cdots+z_{k-1}+\mu_{(0)}))\\
&\times\Pi_{j=1}^{k-1}\mathbf{1}_{\mathcal{R}^\cdot_f}(z_j)\cdot\mathbf{1}_{\dot B(z_j)}(a)\cdot u(z_j)\cdot\exp(-E(a,z_j)))\\
&\times\mathbf{1}_{\mathcal{R}^\cdot_f}(x)\cdot\mathbf{1}_{\dot B(x)}(a)\cdot u(x)\cdot\exp(-E(a,x))\\
&\times W_\varrho(dz_1)\cdots W_\varrho(dz_{k-1})W_\varrho(dx).
\end{aligned}$$

Then integrating this with respect to $Q(d\mu)$ and using again that Q is a Poisson exclusion process yields

$$\begin{aligned}
I_a \quad = \quad &\Sigma_{k\geq 0}\frac{1}{k!}\int\int\cdots^k\int h(a,\mu_{(0)}+\delta_a)\cdot\mathbf{1}_{D^*}(a,\mu_{(0)}-(x+z_1+\cdots+z_{k-1}))\\
&\times\Pi_{j=1}^{k-1}\,\mathbf{1}_{D^*}(z_j,\mu_{(0)}-(x+z_j+\cdots+z_{k-1}))\cdot\mathbf{1}_{D^*}(x,\mu_{(0)}-x)\\
&\times\Pi_{j=1}^{k-1}\mathbf{1}_{\dot B(z_j)}(a)\cdot\mathbf{1}_{\dot B(x)}(a)\cdot\Pi_{j=1}^{k-1}\exp(-E(a,z_j))\cdot\exp(-E(a,x))\\
&\times\mu(dz_1)\cdots\mu(dz_{k-1})\mu(dx)Q(d\mu).
\end{aligned}$$

Observe finally that in I_a the sum

$$\begin{aligned}&\Sigma_{k\geq 0}\frac{1}{k!}\int\cdots^{k}\int \mathbf{1}_{D^*}(a,\mu_{(0)}-(x+z_1+\cdots+z_{k-1}))\\&\times\Pi_{j=1}^{k-1}\mathbf{1}_{D^*}(z_j,\mu_{(0)}-(x+z_j+\cdots+z_{k-1}))\cdot\mathbf{1}_{D^*}(x,\mu_{(0)}-x)\\&\times\Pi_{j=1}^{k-1}\mathbf{1}_{\dot{B}(z_j)}(a)\cdot\mathbf{1}_{\dot{B}(x)}(a)\cdot\Pi_{j=1}^{k-1}\exp(-E(a,z_j))\cdot\exp(-E(a,x))\\&\times\mu(dz_1)\cdots\mu(dz_{k-1})\mu(dx)\end{aligned}$$

equals $\exp(-E(a,\mu_{(0)}))$.

To summarize, we have shown Mürmann's second theorem.

Theorem 2. *Let Q be a Poisson exclusion process for (D_R^*, M_ρ^ϕ) on $\mathcal{R}^{\cdot}_f$. Then the image P of Q under the measurable transformation $\chi_{(0)}$ is a Gibbs process in E for (ϕ,ϱ).*

Summarizing both theorems, we see that the relation $Q=\gamma P$ establishes a 1-1 correpondence between Gibbs processes P in E for (ϕ,ϱ) concentrated on $\mathcal{M}^{\cdot}_\infty$ and satisfying condition (3.2) , and Poisson exclusion processes for (D_R^*, M_ϱ^ϕ) in $\mathcal{R}^{\cdot}_f$. Here ϕ is the given even, stable, pair potential of finite range R.

This result is very valuable. It reduces the study of Gibbs processes to the one of Poisson exclusion processes. Moreover, since there is a one-to-one correspondence between these processes, existence, extremality or uniqueness resp. phase transition occurs in one class if and only if this phenomenon occurs in the other one.

5. ABSENCE OF PERCOLATION AND MUERMANN'S CONDITION

Given (ϕ,ϱ) as above, we fix some Gibbs process P in E for ϕ (of finite range R) and ϱ. We assume in addition that P is of first order. In this situation P is then equivalently described by Ruelle's equation. To pose *the problem* we introduce some notations first: For an open $\Lambda\in\mathcal{B}_0(E)$ let $\Lambda(r)$ denote the $r-$ dilatation $r\cdot\Lambda, r\geq 1$. Moreover

$$\mathcal{F}_\Lambda=\{x\in\mathcal{R}^{\cdot}\mid x\cap\Lambda\neq\emptyset\},\tag{5.1}$$

$$\mathcal{F}_{\Lambda,\Lambda(r)^c}=\{x\in\mathcal{F}_\Lambda\mid x\cap\Lambda(r)^c\neq\emptyset\},\tag{5.2}$$

$$\mathcal{G}_{\mathcal{F}_{\Lambda,\Lambda(r)^c}}=\{\mu\in\mathcal{M}^{\cdot}(\mathcal{R}^{\cdot})\mid \mu(\mathcal{F}_{\Lambda,\Lambda(r)^c})\geq 1\}.\tag{5.3}$$

If Λ' is a bounded Borel set containing $\Lambda(r)$ such that $d(\Lambda(r),(\Lambda')^c)>R$, then it is evident that

$$
\begin{aligned}
&P\{\text{ there is a particle in } \Lambda \text{ clustering with infinitely many others }\}\\
&\quad \le P\{\text{ there is a p. in } \Lambda \text{ which clusters with some p. in } \Lambda' \setminus \Lambda(r),\\
&\qquad\qquad \text{regardless eventual clustering with some p. outside } \Lambda'\}
\end{aligned}
$$

More formally: If $\pi_{\Lambda'} : \eta \longrightarrow \mathbf{1}_{\Lambda'} \cdot \eta$ and $\gamma_{\Lambda'} = \gamma \circ \pi_{\Lambda'}$, then

$$
P\{\exists x \in \gamma : |x| = +\infty, x \in \mathcal{F}_\Lambda\} \le P\{\gamma_{\Lambda'} \in \mathcal{G}_{\mathcal{F}_{\Lambda,\Lambda(r)^c}}\}. \tag{5.4}
$$

The problem now is to estimate from above the probability on the right hand side (and thereby on the left).

Write $\mathcal{E}$ for the event $\mathcal{G}_{\mathcal{F}_{\Lambda,\Lambda(r)^c}}$. We first use Ruelle's equation to get

$$
P\{\gamma_{\Lambda'} \in \mathcal{E}\} = \int\limits_{\mathcal{M}^{\cdot}_{(\Lambda')^c}} \int\limits_{\mathcal{M}^{\cdot}_{\Lambda'}} \mathbf{1}_{\mathcal{E}}(\gamma(\xi)) \cdot \exp(-E_{\Lambda'}(\xi + \eta)) W_\varrho(d\xi) P(d\eta).
$$

We then analyze the inner integral $I_\eta(\mathcal{E})$ for *any* measurable event $\mathcal{E}$ in $\mathcal{M}_f^{(\cdot)}(\mathcal{R}^{\cdot}_f)$ by means of the following cluster representation:

Lemma 2. *For any measurable $\mathcal{E}$ in $\mathcal{M}_f^{(\cdot)}(\mathcal{R}^{\cdot}_f)$ and any $\eta \in \mathcal{M}^{\cdot}_{(\Lambda')^c}$*

$$
I_\eta(\mathcal{E}) = \int\limits_{\mathcal{M}_f^{(\cdot)}(\mathcal{R}^{\cdot}_f)} \mathbf{1}_{\mathcal{E}}(\mu) \cdot \exp(-\mathcal{W}(\mu \mid \eta))\, W_{M_{\varrho_{\Lambda'}}}(d\mu). \tag{5.5}
$$

Here for $\mu = \delta_{z_1} + \cdots + \delta_{z_k}$

$$
\begin{aligned}
\mathcal{W}(\mu \mid \eta)) &= \sum_{j=1}^{k} \mathcal{W}(z_j \mid \eta), \text{ where}\\
\mathcal{W}(z_j \mid \eta) &= \sum_{a \in z_j} E(a, \eta).
\end{aligned}
$$

Proof. Expressing as above the integrand of $I_\eta(\mathcal{E})$ by means of the cluster decomposition of ξ yields

$$
\begin{aligned}
&\mathbf{1}_{\mathcal{E}}(\gamma(\xi)) \cdot \exp(-E_{\Lambda'}(\xi + \eta)) =\\
&= \Sigma_{k \ge 0} \tfrac{1}{k!} \Sigma_{z_1 \subseteq \xi} \Sigma_{z_2 \subseteq \xi - z_1} \cdots \Sigma_{z_{k-1} \subseteq \xi - (z_1 + \cdots + z_{k-2})} \mathbf{1}_{\mathcal{E} \cap \mathcal{M}_f^{(\cdot)}(\mathcal{R}^{\cdot}_f)}(\delta_{\xi - (z_1 + \cdots + z_{k-1})} +\\
&\quad + \delta_{z_1} + \cdots + \delta_{z_{k-1}}) \times \Pi_{j=1}^{k-1} u(z_j) \cdot \exp(-\mathcal{W}(z_j \mid \eta)))\\
&\quad \times u(\xi - (z_1 + \cdots + z_{k-1})) \cdot \exp(-\mathcal{W}(\delta_{\xi - (z_1 + \cdots + z_{k-1})} \mid \eta))).
\end{aligned}
$$

Applying again partial integration with respect to $W_{\varrho_{\Lambda'}}$ yields the assertion of the lemma.

Before starting the estimation we separate a cluster inside $\mathcal{F}_{\Lambda,\Lambda(r)^c}$ in $I_\eta(\mathcal{E})$, where $\mathcal{E}$ is from now on given by $\mathcal{G}_{\mathcal{F}_{\Lambda,\Lambda(r)^c}}$, and obtain

$$I_\eta(\mathcal{E}) = \int\limits_{\mathcal{M}_f^{(\cdot)}(\mathcal{R}_f^{\cdot})} \frac{1}{\mu(\mathcal{F}_{\Lambda,\Lambda(r)^c})} \cdot \int\limits_{\mathcal{F}_{\Lambda,\Lambda(r)^c}} \mathbf{1}_{\mathcal{E}}(\mu) \cdot \exp(-\mathcal{W}(\mu \mid \eta))\mu(dx) W_{M_{\varrho_{\Lambda'}}}(d\mu).$$

Again partially integrating yields

$$I_\eta(\mathcal{E}) = \int\limits_{\mathcal{M}_f^{\cdot}(\mathcal{R}_{\{}^{\cdot})} \frac{1}{\mu(\mathcal{F}_{\Lambda,\Lambda(r)^c})+1} \times \int\limits_{\mathcal{F}_{\Lambda,\Lambda(r)^c}} \mathbf{1}_{\mathcal{E}}(\mu+\delta_x) \cdot \mathbf{1}_{\mathcal{M}_f^{(\cdot)}(\mathcal{R}_f^{\cdot})}(\mu+\delta_x) \cdot$$
$$\cdot \exp(-\mathcal{W}(\mu+\delta_x \mid \eta)) M_{\varrho_{\Lambda'}}(dx) W_{M_{\varrho_{\Lambda'}}}(d\mu).$$

We now *start estimating from above* and use that $(\mu+\delta_x)\varepsilon\mathcal{M}_f^{(\cdot)}(\mathcal{R}_f^{\cdot})$ implies $\mu\varepsilon\mathcal{M}_f^{(\cdot)}(\mathcal{R}_f^{\cdot})$. Moreover

$$\mathcal{W}(\mu+\delta_x \mid \eta) = \mathcal{W}(\mu \mid \eta) + \mathcal{W}(x \mid \eta),$$
$$\textit{where } \mathcal{W}(x \mid \eta) \geq -B \cdot |x| \textit{ by stability of } \phi.$$

As a consequence one obtains

$$I_\eta(\mathcal{E}) \leq M_{e^B \cdot \varrho_{\Lambda'}}(\mathcal{F}_{\Lambda,\Lambda(r)^c}) \cdot \int\limits_{\mathcal{M}_f^{(\cdot)}(\mathcal{R}_f^{\cdot})} \exp(-\mathcal{W}(\mu \mid \eta))\, W_{M_{\varrho_{\Lambda'}}}(d\mu).$$

Then dissolving the clusters of μ into its particles, by the lemma above the integral on the right hand side equals

$$\int\limits_{\mathcal{M}_{\Lambda'}^{\cdot}} \exp(-E_{\Lambda'}(\xi+\eta)) W_\varrho(d\xi).$$

Now integrating the last inequality with respect to $\mathbf{1}_{\mathcal{M}_{(\Lambda')^c}^{\cdot}} \cdot P$, and using Ruelle's equation implies

$$P\{\gamma_{\Lambda'} \in \mathcal{G}_{\mathcal{F}_{\Lambda,\Lambda(r)^c}}\} \leq M_{e^B \cdot \varrho_{\Lambda'}}(\mathcal{F}_{\Lambda,\Lambda(r)^c}),$$

so that finally we otain for any $P \in \mathcal{G}(\phi, \varrho)$ of first order Mürmann's

Main estimate. If Λ' is a bounded Borel set containing $\Lambda(r)$ such that $d(\Lambda(r), (\Lambda')^c) > R$, then

$$P\{ \textit{ a p. in } \Lambda \textit{ clusters with infinitely many other p. } \} \leq M_{e^B \cdot \varrho_{\Lambda'}}(\mathcal{F}_{\Lambda,\Lambda(r)^c}).$$

This estimate implies then Mürmann's

Main Lemma. If $M_{e^B \cdot \varrho}$ is a Radon measure then

$$P\{cd_{D_\infty} \geq 1\} = 0.$$

The proof immediately follows from the following decomposition of $\mathcal{F}_\Lambda$:

$$\mathcal{F}_\Lambda = \mathcal{F}_\Lambda^{\Lambda^c} \cup \mathcal{F}_{\Lambda,\Lambda^c}^{\Lambda(2)^c} \cup \mathcal{F}_{\Lambda,\Lambda(2)^c}^{\Lambda(3)^c} \cup \cdots .$$

Here $\mathcal{F}_{\Lambda,\Lambda'}^{\Lambda''}$ is the event meeting Λ and Λ', but not Λ''. If $\mathcal{F}_\Lambda$ has finite measure with respect to $M_{e^B \cdot \varrho}$ then for any positive ϵ there exists N such that $M_{e^B \cdot \varrho}(\mathcal{F}_{\Lambda,\Lambda(N)^c}) < \epsilon$, and thus for any Λ' containing $\Lambda(N)$,which itself contains Λ, one has

$$M_{e^B \cdot \varrho_{\Lambda'}}(\mathcal{F}_{\Lambda,\Lambda(N)^c}) < \epsilon.$$

Combining this with Mürmann's estimate proves the main lemma.

6. CONCLUDING REMARKS

The question remains, under which conditions on (ϕ, ϱ) the Mürmann measure is Radon. The well known answer is that this is the case if z is small enough. To be more complete we give some indications to this question. We use the following simple observation by separating a particle inside Λ: For any bounded Borel set Λ of positive $\varrho-$measure

$$\int\limits_{\mathcal{M}_f^{\cdot}(E)} \frac{1}{|\eta| + 1} \cdot \mathbf{1}_{\mathcal{R}_f^{\cdot}}(\eta + \delta_0) \cdot u(\eta + \delta_0) W_\varrho(d\eta) \tag{6.1}$$

$$\leq \frac{1}{\varrho(\Lambda)} \cdot M_\varrho(\mathcal{F}_\Lambda) \tag{6.2}$$

$$\leq \int\limits_{\mathcal{M}_f^{\cdot}(E)} \mathbf{1}_{\mathcal{R}_f^{\cdot}}(\eta + \delta_0) \cdot u(\eta + \delta_0) W_\varrho(d\eta). \tag{6.3}$$

The upper and lower bound in (6.3) and (6.1) are power series in z with the same radius of convergence. As a consequence the radius of convergence of $M_\varrho(\mathcal{F}_\Lambda)$, if considered as a power series, does not depend on Λ and coincides with the one of $W_\varrho \star \delta_0(u \cdot \mathbf{1}_{\mathcal{R}_f^{\cdot}})$. Here $\star$ denotes convolution.

Thus it remains to investigate the power series $W_\varrho \star \delta_0(u \cdot \mathbf{1}_{\mathcal{R}_f^{\cdot}})$. Here we use the observation in [2] that $\mathbf{1}_{\mathcal{R}_f^{\cdot}}(\eta + \delta_0) \leq |g|(\eta + \delta_0)$, where g denotes the *Ursell* function belonging to the hard-core potential defined by R. If combined with the stability of the potential one obtains

$$W_\varrho \star \delta_0(u \cdot \mathbf{1}_{\mathcal{R}_f^{\cdot}}) \leq e^B \cdot W_{e^B \varrho} \star \delta_0(|g|).$$

On the other hand one can find in Ruelle's book [5] that the integral on the right hand side of this inequality is finite if z is small enough. This is one of Ruelle's important contributions and follows by means of the method of strong cluster estimates.

7. APPENDIX: THE POISSON CASE

We consider now the case when the underlying process is a Poisson process with an intensity of the following form: $\sigma = \rho \otimes \tau$, where ρ is given as above by the Lebesgue measure $z\lambda^d(z > 0)$ and τ is a probability on $]0, +\infty[$. The corresponding Poisson process in $X = E\times]0, +\infty[$ is then concentrated on the following collection of configurations:

$$\mathcal{M}^{\cdot}_{\succ} := \{\nu = \Sigma_{a\epsilon\eta}\delta_{(a,r_a)} \mid \eta\epsilon\mathcal{M}^{\cdot}(E), r_a > 0\}. \tag{7.1}$$

The definition of the corresponding *clusters* and *cluster property* is as follows: We call the elements $x = (a, r)$ particles and represent them geometrically as open balls $b(x) = B_r(a)$. Two particles x, x' are sayed to interact if $b(x) \cap b(x') \neq \emptyset$. Then $z\epsilon\mathcal{M}^{\cdot}_{\succ}$ is called a *cluster* iff the graph obtained by joining interacting particles is connected. Denote by $\mathcal{C}^{\cdot}$ the collection of all such clusters and $\mathcal{C}^{\cdot}_f$ the one of finite clusters. $\mathcal{C}^{\cdot}$ and $\mathcal{C}^{\cdot}_f$ are measurable subsets of $\mathcal{M}^{\cdot}_{\succ}$.

The *cluster property* defined by R is the measurable subset $D = D_R$ in $\mathcal{M}^{\cdot}_{\succ} \times \mathcal{M}^{\cdot}_{\succ}$ defined by

$(z, \nu)\epsilon D$ *iff*

$z\epsilon\mathcal{C}^{\cdot}$ *such that the particles in* ν *with centers outside the support of* z

do not interact with particles from z.

Thus one cannot add a particle to z from η to enlarge the cluster z. Consider now the point process on $(\mathcal{M}^{\cdot}(E), P_\sigma)$, defined by

$$\gamma : \nu \longmapsto \kappa := \sum_{z \subseteq \nu} \mathbf{1}_D(z, \nu) \cdot \delta_z.$$

It has its values in

$$\mathcal{M}^{(\cdot)}(\mathcal{C}^{\cdot}) := \{\kappa\epsilon\mathcal{M}^{\cdot}(\mathcal{C}^{\cdot}) \mid z, z'\epsilon\kappa, z \neq z' \Rightarrow B(z) \cap B(z') = \emptyset\}.$$

Here $B(z) = \cup_{x\epsilon z} b(x)$. We denote its distribution by Q_σ. As above γ is a Borel isomorphism between $\mathcal{M}^{\cdot}_{\succ}$ and $\mathcal{M}^{(\cdot)}(\mathcal{C}^{\cdot})$ with inverse

$$\chi_{(0)} : \kappa \longmapsto \kappa_{(0)} := \Sigma_{z\epsilon\kappa} z;$$

and the relation $Q = \gamma P$. establishes a 1-1 correspondence between $\mathcal{P}\mathcal{M}^{\cdot}_{\succ}$ and $\mathcal{P}\mathcal{M}^{(\cdot)}(\mathcal{C}^{\cdot})$.

The problem is to describe Q_σ. The following result is also due to Mürmann [3]. Exactly the same reasoning as in the Gibbsian case yields the

Theorem 3. *Given (σ, R) as above, the relation $Q = \gamma P$ establishes a 1-1 correspondence between P_σ and the uniquely determined Poisson exclusion process Q_σ on $C_f^{\cdot}$ for (D^*, M_σ) if P_σ has only finite clusters a.s. . Here M_σ is the measure $\mathbf{1}_{C_f^{\cdot}} \cdot W_\sigma$.*

This result is also important for the following reasons: It implies the existence and uniqueness of Poisson cluster exclusion processes in the situation considered here if there is no percolation.

The question of absence of percolation has been studied in a recent paper of Gouéré [1]. The author proves that finiteness of $\int_0^\infty r^d \nu(dr)$ is equivalent to the existence of some strictly positive z_0 such that for all z strictly smaller than z_0 percolation does not take place.

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