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SOME PROPERTIES OF THE ε_∞-PRODUCT OF QUOTIENT BORNOLOGICAL SPACES

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Аннотация. Some properties of the ε_∞-product defined in [4] are obtained by a study of a kind of isomorphism between the computation of this ε_∞-product and the ordinary ε-product of L. Schwartz [9]. The paper contains several corollaries.

1. INTRODUCTION

The ε-product in the category of locally convex spaces was defined by L. Schwartz [9]. Later, L. Waelbroeck [10] gave a simple definition of the ε-product in the category of Banach spaces, while in [1] we defined the ε_c-product in the category of quotient bornological spaces.

For a nuclear b-space N, we showed in [2] that if Ω is a finite or a σ-finite measure space and $1 \le p \le \infty$, then the functors $L^p_{loc}(\Omega, N\varepsilon.)$ and $N\varepsilon L^p(\Omega, .)$ are isomorphic on the category of b-spaces of L. Waelbroeck [1]. Next, we established in [3] that for a nuclear b-space N and a b-space E, if X is a compact space (resp. locally compact space that is countable at infinity) then the exact functors $C(X, N\varepsilon.E)$ and $N\varepsilon C(X, .)$ are isomorphic on the category of b-spaces.

In a recent paper [4], we defined the ε_∞-product of a b-space by a quotient bornological space and we proved that if G is an εb-space and $E \mid F$ is a quotient bornological space, then $(G\varepsilon E) \mid (G\varepsilon F)$ is isomorphic to $G\varepsilon_\infty(E \mid F)$.

In the present paper, we are going to prove that if N is a nuclear b-space and G is a b-space, the quotient bornological spaces $G\varepsilon_\infty(N\varepsilon(E \mid F))$ and $N\varepsilon(G\varepsilon_\infty(E \mid F))$ are isomorphic for each quotient bornological space $E \mid F$ where ε_∞ is the ε_∞-product defined in [4] and we will give some interesting consequences.

First we need to fix the notation and recall some definitions. Let **EV** be the category of vector spaces and linear mappings over the scalar field $\mathbb{R}$ or $\mathbb{C}$, and **Ban** be the category of Banach spaces and bounded linear mappings. We denote by $\mathbf{Ban}(E_1, E_2)$ the Banach space of all bounded linear mappings $E_1 \longrightarrow E_2$, where E_1 and E_2 are Banach spaces.

1- Let $(E, \|\cdot\|_E)$ be a Banach space. A Banach subspace F of E is a vector subspace endowed with a Banach norm $\|\cdot\|_F$ such that the inclusion map $(F, \|\cdot\|_F) \longrightarrow (E, \|\cdot\|_E)$ is bounded. Observe that the norm $\|\cdot\|_F$ of F is not necessarily the same as the norm induced by $\|.\|_E$ on F, and the Banach subspace F is not necessarily closed in E. A quotient Banach space $E \mid F$ is a vector space E/F, where E is a Banach space and F a Banach subspace. It is clear that $E \mid F$ is not necessarily an object of the category **Ban**, but it is if F is closed in E. If $E \mid F$ and $E_1 \mid F_1$ are two quotient Banach spaces, a strict morphism $u : E \mid F \longrightarrow E_1 \mid F_1$ is a linear mapping $u : x + F \longmapsto u_1(x) + F_1$, where $u_1 : E \longrightarrow E_1$ is a bounded linear mapping such that $u_1(F) \subseteq F_1$. We say that u_1 induces u. Two bounded linear mappings u_1, $u_2 : E \longrightarrow E_1$ both inducing a strict morphism, induce the same strict morphism iff the linear mapping $u_1 - u_2 : E \longrightarrow F_1$ is bounded. Let $E \mid F$ be a quotient Banach space and E_0 a Banach subspace of E such that F is a Banach subspace of E_0. Then the natural injection $E_0 \rightarrow E$ induces a strict morphism $E_0 \mid F \longrightarrow E \mid F$, and the identity mapping $Id_E : E \rightarrow E$ induces a strict morphism $E \mid F \longrightarrow E \mid E_0$.

The category of quotient Banach spaces and strict morphisms we called $\tilde{\mathbf{q}}$**Ban**, it is a subcategory of vector spaces **EV** and contains the category **Ban** (any Banach space E will be identified with the quotient Banach space $E \mid \{0\}$, moreover if $u_1 : E \rightarrow E_1$ is a bounded linear mapping, then u_1 induces a strict morphism $E \mid \{0\} \rightarrow E_1 \mid \{0\}$ and every strict morphism $E \mid \{0\} \rightarrow E_1 \mid \{0\}$ is induced by a unique bounded linear mapping $u_1 : E \rightarrow E_1$).

The category $\tilde{\mathbf{q}}$**Ban** is not Abelian. If E is a Banach space and F a closed subspace of E, it would be very nice if the quotient Banach space $E \mid F$ were isomorphic to the quotient $(E/F) \mid \{0\}$. This is not the case in $\tilde{\mathbf{q}}$**Ban** unless F is complemented in E.

L. Waelbroeck [12] introduced an Abelian category **qBan** generated by $\tilde{\mathbf{q}}$**Ban** and inverses of pseudo-isomorphims, i.e. has the same objects as $\tilde{\mathbf{q}}$**Ban** and every morphism u of **qBan** can be expressed as $u = v \circ s^{-1}$, where s is a pseudo-isomorphism and v is a strict morphism. For more information about quotient Banach spaces we refer the reader to [12].

2- Let E be a real or complex vector space, and B be an absolutely convex set in E. Let E_B be the vector space generated by B i.e. $E_B = \cup_{\lambda > 0} \lambda B$. The Minkowski functional of B is a semi-norm on E_B. It is a norm, if and only if B does not contain any nonzero subspace of E. The set B is completant if its Minkowski functional is a Banach norm.

A bounded structure β on a vector space E is defined by a set of "bounded" subsets of E with the following properties:

(1) Every finite subset of E is bounded.
(2) Every union of two bounded subsets is bounded.
(3) Every subset of a bounded subset is bounded.
(4) A set homothetic to a bounded subset is bounded.
(5) Each bounded subset is contained in a completant bounded subset.

A b-space (E, β) is a vector space E with a boundedness β. A subspace F of a b-space E is bornologically closed if the subspace $F \cap E_B$ is closed in E_B for every completant bounded subset B of E.

Given two b-spaces (E, β_E) and (F, β_F), a linear mapping $u : E \longrightarrow F$ is bounded, if it maps bounded sets of E into bounded sets of F. The mapping u is bornologically surjective if for every $B' \in \beta_F$, there exists $B \in \beta_E$ such that $u(B) = B'$. A Schwartz b-space G is a b-space satisfying the following condition: for each completant bounded disk A of G there exists a completant bounded disk B of G such that the inclusion mapping $i_{B'B} : G_A \to G_B$ is compact.

We denote by **b** the category of b-spaces and bounded linear mappings. For more information about b-spaces we refer the reader to [5, 6] and [11].

Let (E, β_E) be a b-space. A b-subspace of E is a subspace F with a boundedness β_F such that (F, β_F) is a b-space and $\beta_F \subseteq \beta_E$. We note that the boundedness β_F of F is not necessary the same as the boundedness induced by β_E on F, and then the b-subspace F is not necessary bornologically closed in E. A quotient bornological space $E \mid F$ is a vector space E/F, where E is a b-space and F a b-subspace of E. Observe that $E \mid F$ is not necessarily an object of the category of b-spaces **b**, but it is if F is bornologically closed in E. If $E \mid F$ and $E_1 \mid F_1$ are quotient bornological spaces, a strict morphism $u : E \mid F \longrightarrow E_1 \mid F_1$ is induced by a bounded linear mapping $u_1 : E \longrightarrow E_1$ whose restriction to F is a bounded linear mapping $F \longrightarrow F_1$. Two bounded linear mappings $u_1, v_1 : E \longrightarrow E_1$, both inducing a strict morphism, induce the same strict morphism $E \mid F \longrightarrow E_1 \mid F_1$ iff the linear mapping $u_1 - v_1 : E \longrightarrow F_1$ is bounded.

We call $\tilde{\mathbf{q}}$ the category of quotient bornological spaces and strict morphisms. A pseudo-isomorphism $u : E \mid F \longrightarrow E_1 \mid F_1$ is a strict morphism induced by a bounded linear mapping $u_1 : E \longrightarrow E_1$ which is bornologically surjective and such that $u_1^{-1}(F_1) = F$ i.e. $B \in \beta_F$ if $B \in \beta_E$ and $u_1(B) \in \beta_{F_1}$.

The category $\tilde{\mathbf{q}}$ is not Abelian because it contains the category $\tilde{\mathbf{q}}\mathbf{Ban}$. In [13], Waelbroeck introduced an Abelian category $\mathbf{q}$ generated by $\tilde{\mathbf{q}}$ and the inverses of pseudo-isomorphims i.e. has the same objects as $\tilde{\mathbf{q}}$ and every morphism u of $\mathbf{q}$ can be expressed as $u = v \circ s^{-1}$, where s is a pseudo-isomorphism and v is a strict morphism.

2. MAIN RESULTS

To show our main results concerning the ε_∞-product defined in [4], recall that the usual ε-product of two Banach spaces E and F is the Banach space $E\varepsilon F$ of linear mappings $E' \longrightarrow F$ whose restrictions to the closed unit ball $B_{E'}$ of E' are $\sigma(E', E)$-continuous where E' is the topological dual of E. The ε-product is symmetric i.e. the Banach spaces $E\varepsilon F$ and $F\varepsilon E$ are isometrically isomorphic (Proposition 2 of [10]). If E_i and F_i are Banach spaces and $u_i : E_i \longrightarrow F_i$ are bounded linear mappings, $i = 1, 2$, the ε-product of u_1 and u_2 is the bounded linear mapping $u_1\varepsilon u_2 : E_1\varepsilon E_2 \longrightarrow F_1\varepsilon F_2$, $f \longmapsto u_2 \circ f \circ u_1'$, where u_1' is the dual mapping of u_1. It is clear that $u_1\varepsilon u_2$ is injective whenever u_1 and u_2 are injective.

If G is a Banach space and F is a Banach subspace of a Banach space E, then $G\varepsilon F$ is a Banach subspace of $G\varepsilon E$. For more information about the ε-product see [10].

The ε-product of two b-spaces G and E is the b-space $G\varepsilon E = \cup_{B,C} (G_B\varepsilon E_C)$, where B and C respectively range over the bounded completant subsets of the b-spaces G and E respectively. It is clear that if F is a b-subspace of G, then the space $F\varepsilon E$ is a b-subspace of $G\varepsilon E$.

A Banach space E is an $\mathcal{L}_{\infty,\lambda}$-space, $\lambda \geq 1$, if and only if every finite-dimensional subspace F of E is contained in a finite-dimensional subspace F_1 of E such that

$d(F_1, l_n^\infty) \leq \lambda$, where $n = \dim F_1$, and l_n^∞ is $\mathbf{K}^n$ ($\mathbf{K} = \mathbb{R}$ or $\mathbb{C}$) with the norm $\sup_{1\leq i\leq n} |x_i|$, where $d(X,Y) = \inf\{\|T\| \left\|T^{-1}\right\|, T : X \longrightarrow Y$ isomorphism is the Banach-Mazur distance of the Banach spaces X and Y. A Banach space E is an $\mathcal{L}_\infty$-space if it is an $\mathcal{L}_{\infty,\lambda}$-space for some $\lambda \geq 1$. Any complemented subspace of an $\mathcal{L}_\infty$-space is an $\mathcal{L}_\infty$-space. For more information about $\mathcal{L}_\infty$-spaces the reader is refered to [8].

Recall from [2], that a b-space G is an εb-space if the bounded linear mapping

$$Id_G\varepsilon u : G\varepsilon E \longrightarrow G\varepsilon F, \quad f \longmapsto u \circ f$$

is bornologically surjective whenever $u : E \longrightarrow F$ is a surjective bounded linear mapping between Banach spaces.

As in Proposition 6.2 of [1], it is easy to show that a b-space G is an εb-space if and only if for every bounded linear mapping $u : X \longrightarrow Y$ which is bornologically surjective, the bounded linear mapping $Id_G\varepsilon u : G\varepsilon X \longrightarrow G\varepsilon Y$ is bornologically surjective, where X and Y are b-spaces. As a consequence, if $E \mid F$ is a quotient bornological space, then it defines the exact sequence

$$0 \longrightarrow F \longrightarrow E \longrightarrow E \mid F \longrightarrow 0.$$

Its image by the exact functor $G\varepsilon : \mathbf{q} \longrightarrow \mathbf{q}$ is the exact sequence

$$0 \longrightarrow G\varepsilon F \longrightarrow G\varepsilon E \longrightarrow G\varepsilon \left(E \mid F\right) \longrightarrow 0$$

in the category $\mathbf{q}$. Finally, we obtain $G\varepsilon \left(E \mid F\right) = \left(G\varepsilon E\right) \mid \left(G\varepsilon F\right)$.

We start with the following elementary Lemma:

Lemma 1. *Let G be an εb-space and $(E_i \mid F_i)_{i\in I}$ an inductive system in the category $\mathbf{q}$. Then $G\varepsilon(\cup_i (E_i \mid F_i)) \simeq \cup_i (G\varepsilon (E_i \mid F_i))$ where $\cup_i$ denotes the inductive limit in $\mathbf{q}$.*

Proof: Recall that in [6], Houzel proved that the inductive limit is an exact functor on the category $\mathbf{b}$. It follows from Theorem 4.1 of [13] that this functor admits an exact extension to the category of quotient bornological spaces.

If we apply the exact functor $\cup_i(\cdot)$ to the following exact complex in $\mathbf{q}$:

$$0 \longrightarrow F_i \longrightarrow E_i \longrightarrow E_i \mid F_i \longrightarrow 0 \tag{2.1}$$

we obtain

$$0 \longrightarrow \cup_i F_i \longrightarrow \cup_i E_i \longrightarrow \cup_i \left(E_i \mid F_i\right) \longrightarrow 0$$

and hence

$$\cup_i \left(E_i \mid F_i\right)) \simeq co\ker\left(\cup_i F_i \longrightarrow \cup_i E_i\right) \simeq \left(\cup_i E_i\right) \mid \left(\cup_i F_i\right).$$

Now, if we apply successively the exact functors $\cup_i(\cdot)$ and $G\varepsilon.$ to the exact complex (2.1), we obtain the following exact complex:

$$0 \longrightarrow G\varepsilon(\cup_i F_i) \longrightarrow G\varepsilon(\cup_i E_i) \longrightarrow G\varepsilon(\cup_i \left(E_i \mid F_i\right)) \longrightarrow 0$$

and then

$$G\varepsilon(\cup_i \left(E_i \mid F_i\right)) \simeq \left(G\varepsilon(\cup_i E_i)\right) \mid \left(G\varepsilon(\cup_i F_i)\right)).$$

Now, from the definition of the ε-product of two b-spaces it follows that

$$\left(G\varepsilon(\cup_i E_i)\right) \mid \left(G\varepsilon(\cup_i F_i)\right) \simeq \cup_i \left(G\varepsilon E_i\right) \mid \cup_i \left(G\varepsilon F_i\right)$$

$$\simeq \cup_i (G\varepsilon E_i \mid G\varepsilon F_i) \simeq \cup_i (G\varepsilon (E_i \mid F_i)).$$

This proves the Lemma.

Recall that a b-space G is nuclear if each bounded completant subset B of G is included in a bounded completant A of G such that the inclusion $i_{AB} : G_B \longrightarrow G_A$ is a nuclear mapping. For more information about nuclear b-spaces we refer the reader to [5].

Theorem 1. *Let I be a set, G an $\mathcal{L}_\infty$-space and $E \mid F$ a quotient bornological space. Then $l^\infty(I)\,\varepsilon\,(G\varepsilon(E \mid F)) \simeq G\varepsilon\,(l^\infty(I)\,\varepsilon\,(E \mid F))$.*

Proof: Since the Banach spaces $l^\infty(I)$ and G are $\mathcal{L}_\infty$-spaces, their ε-product $l^\infty(I)\,\varepsilon G$ is an $\mathcal{L}_\infty$-space. Hence a Theorem of Kaballo [7] implies that the functor $(l^\infty(I)\,\varepsilon G)\,\varepsilon.$ is exact on the category **Ban**. It follows from Theorem 4.1 of [13] that this functor admits an unique and exact extension to the category of quotient Banach spaces **qBan**. Then, for each quotient Banach space $E \mid F$, we have

$$(l^\infty(I)\,\varepsilon G)\,\varepsilon(E \mid F) = ((l^\infty(I)\,\varepsilon G)\,\varepsilon E) \mid ((l^\infty(I)\,\varepsilon G)\,\varepsilon F).$$

On the other hand,

$$((l^\infty(I)\,\varepsilon G)\,\varepsilon E) \mid ((l^\infty(I)\,\varepsilon G)\,\varepsilon F) = (l^\infty(I)\,\varepsilon\,(G\varepsilon E)) \mid l^\infty(I)\,\varepsilon\,(G\varepsilon F)$$
$$= l^\infty(I)\,\varepsilon\,((G\varepsilon E) \mid (G\varepsilon F)) = l^\infty(I)\,\varepsilon\,(G\varepsilon\,(E \mid F))$$

Now, given quotient bornological space $E \mid F$, let (B, C) be a couple of bounded completant subsets, B is bounded in E, C is bounded in F and $C \subset B$. This set of couples is ordered by the relation $(B, C) \prec (B_1, C_1)$ if and only if $B \subset B_1$ and $C \subset C_1$. For this order, the set of couples (B, C) is a net and the family $(E_B \mid F_C)_{(B,C)}$ is an inductive system in **q** and we can write $E \mid F \simeq \cup_{(B,C)}(E_B \mid F_C)$. This proves that each quotient bornological space $E \mid F$ is an inductive limit of an inductive system of quotient Banach spaces $E_B \mid F_C$. It follows from the equality

$$l^\infty(I)\,\varepsilon\,(G\varepsilon(E_B \mid F_C)) \simeq G\varepsilon\,(l^\infty(I)\,\varepsilon\,(E_B \mid F_C))$$

and the exactness of the inductive limit that

$$\cup_{(B,C)}\,(l^\infty(I)\,\varepsilon\,(G\varepsilon(E_B \mid F_C))) \simeq \cup_{(B,C)}\,(G\varepsilon\,(l^\infty(I)\,\varepsilon\,(E_B \mid F_C))).$$

Finally, a double application of Lemma 1, gives the following result:

$$\cup_{(B,C)}\,(l^\infty(I)\,\varepsilon\,(G\varepsilon(E_B \mid F_C))) = \left(l^\infty(I)\,\varepsilon \cup_{(B,C)}\,(G\varepsilon(E_B \mid F_C))\right)$$
$$= \left(l^\infty(I)\,\varepsilon\,\left(G\varepsilon \cup_{(B,C)}\,(E_B \mid F_C)\right)\right) = l^\infty(I)\,\varepsilon\,(G\varepsilon(E \mid F))$$

and

$$\cup_{(B,C)}\,(G\varepsilon\,(l^\infty(I)\,\varepsilon\,(E_B \mid F_C))) = G\varepsilon \cup_{(B,C)}\,(l^\infty(I)\,\varepsilon\,(E_B \mid F_C))$$
$$= G\varepsilon\left(l^\infty(I)\,\varepsilon \cup_{(B,C)}\,(E_B \mid F_C)\right) = G\varepsilon\,(l^\infty(I)\,\varepsilon\,(E \mid F)).$$

Now, a b-space is a b$\mathcal{L}_\infty$-space if it is a bornological inductive limit of $\mathcal{L}_\infty$-spaces. Since the inductive limit functor is exact on the category of b-spaces [6], it is clear that any b$\mathcal{L}_\infty$-space is an εb-space. Another concrete example is given in ([2], Example 2.4). In fact, for $r \in \mathbb{N}^*$ and X a compact manifold, we defined the space $C^{r+0}(X) = \cup_\varepsilon\, C^{r+\varepsilon}(X)$, where $0 < \varepsilon < 1$ and $C^{r+\varepsilon}(X)$ is the Banach space of functions of class C^r on X such that: for all $k \in \mathbb{N}^n$, $|k| \leq r$, $D^k f$ is continuously o-Hölderian of exposant ε, on which we placeed the following boundedness of b-space: $B \subset C^{r+0}(X)$

is bounded if there is $\varepsilon > 0$ such that B is bounded in $C^{r+\varepsilon}(X)$. The space $C^{r+0}(X)$ is a bornological inductive limit of the inductive system $(C^{r+\varepsilon}(X))_{\varepsilon>0}$, where each $C^{r+\varepsilon}(X)$ is an $\mathcal{L}_\infty$-space, and hence it is an $\mathcal{L}_\infty$-space.

As a consequence of Theorem 1 and the exactness of the bornological inductive limit, we obtain

Corollary 1. *Let I be a set, G be a b$\mathcal{L}_\infty$-space and $E \mid F$ be a quotient bornological space. Then $l^\infty(I)\,\varepsilon\,(G\varepsilon(E \mid F)) \simeq G\varepsilon\,(l^\infty(I)\,\varepsilon\,(E \mid F))$.*

Another example of b$\mathcal{L}_\infty$-spaces: let N be a nuclear b-space, then there exists a net $(I, \leq)$ and a base $(B_{0,i})_{i\in I}$ of the bornology of N such that the Banach space $N_{B_{0,i}}$ is isometrically isomorphic to c_0 and $N = \cup_{i\in I} N_{B_{0,i}}$ as b-spaces [5]. Since c_0 is an $\mathcal{L}_\infty$-space [9], it is clear that every nuclear b-space is a b$\mathcal{L}_\infty$-space.

Corollary 2. *Let I be a set, N a nuclear b-space and $E \mid F$ a quotient bornological space. Then $l^\infty(I)\,\varepsilon\,(N\varepsilon(E \mid F)) \simeq N\varepsilon\,(l^\infty(I)\,\varepsilon\,(E \mid F))$.*

Since the Banach space $l^\infty(I)$ is an εb-space and its ε-product by the εb-space N is an εb-space. Hence the functor $(l^\infty(I)\,\varepsilon G)\,\varepsilon$ is exact on the category **Ban**. A version of the proof of Theorem 1, produces the following result:

Theorem 2. *Let I be a set, G an εb-space and $E \mid F$ a quotient bornological space. Then $l^\infty(I)\,\varepsilon\,(G\varepsilon(E \mid F)) \simeq G\varepsilon\,(l^\infty(I)\,\varepsilon\,(E \mid F))$.*

Now, we prove the same property for the ε_∞-product of a b-space by a quotient bornological space defined in [4]. First, we recall the definition of the ε_∞-product.

In the category **Ban**, a left exact l^∞-resolution of a Banach space G is a strongly left exact complex

$$0 \longrightarrow G \xrightarrow{u} l^\infty(I) \xrightarrow{v} l^\infty(J)$$

i.e. a complex such that $Ker(v) = Im(u)$ and the image of v is closed in $l^\infty(J)$.

Since $l^\infty(K)$ is an $\mathcal{L}_\infty$-space, it follows from [1], that for each quotient bornological space $E \mid F$, we have

$$l^\infty(K)\varepsilon(E \mid F) = (l^\infty(K)\varepsilon E) \mid (l^\infty(K)\varepsilon F) \quad \text{for} \quad K = I, J.$$

The bounded linear mapping $v\varepsilon Id_E : l^\infty(I)\varepsilon E \longrightarrow l^\infty(J)\varepsilon E$ induces a strict morphism

$$v\varepsilon Id_{E|F} : (l^\infty(I)\varepsilon E) \mid (l^\infty(I)\varepsilon F) \longrightarrow (l^\infty(J)\varepsilon E) \mid (l^\infty(J)\varepsilon F)$$

and as the category **q** is Abelian, the object $Ker(v\varepsilon Id_{E|F})$ exists, and we obtain the following left exact sequence in **q**:

$$0 \longrightarrow Ker(v\varepsilon Id_{E|F}) \overset{u\varepsilon Id_{E|F}}{\longrightarrow} (l^\infty(I)\varepsilon E) \mid (l^\infty(I)\varepsilon F)$$
$$\overset{v\varepsilon Id_{E|F}}{\longrightarrow} (l^\infty(J)\varepsilon E) \mid (l^\infty(J)\varepsilon F),$$

where

$$Ker(v\varepsilon Id_{E|F}) = (v\varepsilon Id_E)^{-1}(l^\infty(J)\varepsilon F) \mid (l^\infty(I)\varepsilon F).$$

and $(v\varepsilon Id_E)^{-1}(l^\infty(J)\varepsilon F)$ is a b-subspace of the b-space $l^\infty(I)\varepsilon E$ for the following boudedness: a subset B of $(v\varepsilon Id_E)^{-1}(l^\infty(J)\varepsilon F)$ is bounded if it is bounded in $l^\infty(I)\varepsilon E$ and its image $(v\varepsilon Id_E)(B)$ is bounded in $l^\infty(J)\varepsilon F$.

We let $G\varepsilon_{Res}(E \mid F) = Ker(v\varepsilon Id_{E|F})$. This defines a functor

$$G\varepsilon_{Res}. : \mathbf{q} \longrightarrow \mathbf{q}, \quad E \mid F \longrightarrow G\varepsilon_{Res}(E \mid F).$$

The Banach space G has several left exact l^∞-resolutions, and we proved in [4] that the object $G\varepsilon_{Res}(E \mid F)$ does not depend on the left exact l^∞-resolutions of G (Theorem 2.4 of [4]). Then, we defined the ε_∞-product of a b-space G by a quotient bornological space $E \mid F$ as the quotient bornological space $G\varepsilon_\infty(E \mid F) = G\varepsilon_{Res}(E \mid F)$.

Now we are in a position to prove the principal result which concerns the ε_∞-product defined in [4].

Theorem 3. *Let N be an $\mathcal{L}_\infty$-space, G a Banach space and $E \mid F$ a quotient bornological space. Then $G\varepsilon_\infty(N\varepsilon(E \mid F)) \simeq N\varepsilon(G\varepsilon_\infty(E \mid F))$.*

Proof: Let

$$0 \longrightarrow G \xrightarrow{\Phi} l^\infty(I) \xrightarrow{\Psi} l^\infty(J)$$

be a left exact l^∞-resolution of G. As N is an $\mathcal{L}_\infty$-space, we have

$$N\varepsilon(E \mid F) \simeq (N\varepsilon E) \mid (N\varepsilon F).$$

Then the bounded linear mapping

$$\Psi\varepsilon Id_{(N\varepsilon E)} : l^\infty(I)\varepsilon(N\varepsilon E) \longrightarrow l^\infty(J)\varepsilon(N\varepsilon E)$$

induces a strict morphism

$$\Psi\varepsilon Id_{N\varepsilon(E|F)} : l^\infty(I)\varepsilon(N\varepsilon(E \mid F)) \longrightarrow l^\infty(J)\varepsilon(N\varepsilon(E \mid F))$$

which has a kernel in the category $\mathbf{q}$. We obtain then the following left exact complex in $\mathbf{q}$:

$$0 \longrightarrow G\varepsilon_\infty N\varepsilon(E \mid F) \xrightarrow{\Phi\varepsilon Id_{N\varepsilon(E|F)}} l^\infty(I)\varepsilon(N\varepsilon(E \mid F)) \xrightarrow{\Psi\varepsilon Id_{N\varepsilon(E|F)}} l^\infty(J)\varepsilon(N\varepsilon(E \mid F)).$$

On the other hand, the image of the left exact complex

$$0 \longrightarrow G\varepsilon_\infty(E \mid F) \xrightarrow{\Phi\varepsilon Id_{N\varepsilon(E|F)}} l^\infty(I)\varepsilon(E \mid F) \xrightarrow{\Psi\varepsilon Id_{N\varepsilon(E|F)}} l^\infty(J)\varepsilon(E \mid F)$$

by the exact functor $N\varepsilon. : \mathbf{q} \longrightarrow \mathbf{q}$ is the following left exact complex:

$$0 \longrightarrow N\varepsilon(G\varepsilon_\infty(E \mid F)) \longrightarrow N\varepsilon(l^\infty(I)\varepsilon(E \mid F)) \longrightarrow N\varepsilon(l^\infty(J)\varepsilon(E \mid F)).$$

Now, by Theorem 1, we have

$$l^\infty(K)\varepsilon(N\varepsilon(E \mid F)) = N\varepsilon(l^\infty(K)\varepsilon(E \mid F)) \quad \text{for} \quad K = I, J$$

therefore

$$Ker\left(\Psi\varepsilon Id_{N\varepsilon(E|F)}\right) = Ker\left(Id_N\varepsilon\left(\Psi\varepsilon Id_{E|F}\right)\right),$$

implying

$$G\varepsilon_\infty(N\varepsilon(E \mid F)) \simeq N\varepsilon(G\varepsilon_\infty(E \mid F)).$$

As a consequence of the Theorem 2 and Lemma 1, we obtain:

Corollary 3. Let N be an $\mathcal{L}_\infty$-space, G a b-space and $E \mid F$ a quotient bornological space. Then $G\varepsilon_\infty(N\varepsilon(E \mid F)) \simeq N\varepsilon(G\varepsilon_\infty(E \mid F))$.

Proof: In fact, since $G = \cup_B G_B$ where B ranges over bounded completant subsets of G, and since

$$G_B\varepsilon_\infty(N\varepsilon(E \mid F)) \simeq N\varepsilon(G_B\varepsilon_\infty(E \mid F)),$$

we have

$$\cup_B(G_B\varepsilon_\infty(N\varepsilon(E \mid F))) \simeq \cup_B(N\varepsilon(G_B\varepsilon_\infty(E \mid F))).$$

It follows from Lemma 1, that

$$\cup_B(N\varepsilon(G_B\varepsilon_\infty(E \mid F))) = N\varepsilon(\cup_B(G_B\varepsilon_\infty(E \mid F))) = N\varepsilon(G\varepsilon_\infty(E \mid F)),$$

therefore

$$G\varepsilon_\infty(N\varepsilon(E \mid F)) \simeq N\varepsilon(G\varepsilon_\infty(E \mid F)).$$

This ends the proof.

Other consequences of Theorem 1 and Corollary 1 (resp. Corollaries 2, 3 and Theorem 2) are as follows:

Corollary 4. *Let N be a $b\mathcal{L}_\infty$-space, G a b-space and $E \mid F$ a quotient bornological space. Then $G\varepsilon_\infty(N\varepsilon(E \mid F)) \simeq N\varepsilon(G\varepsilon_\infty(E \mid F))$.*

Corollary 5. *Let N be a nuclear b-space, G a b-space and $E \mid F$ a quotient bornological space. Then $G\varepsilon_\infty(N\varepsilon(E \mid F)) \simeq N\varepsilon(G\varepsilon_\infty(E \mid F))$.*

Corollary 6. *Let N be an εb-space, G a b-space and $E \mid F$ a quotient bornological space. Then $G\varepsilon_\infty(N\varepsilon(E \mid F)) \simeq N\varepsilon(G\varepsilon_\infty(E \mid F))$.*

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