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A NOTE ON 4-DIMENSIONAL LOCALLY CONFORMALLY FLAT WALKER MANIFOLDS

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Abstract. A 4-dimensional Walker metric on a semi Riemanian manifold M, for the canonical metric with $c = 0$, have been investigated by M. Chaichi, E. García–Río and Y. Matsushita. The paper generalizes these notions to the case of constant $c \neq 0$. Specially the form of defining functions of this metric in locally conformally flat 4-dimensional Walker manifolds is found.

§1. INTRODUCTION

A Walker n-manifold, admits a field of parallel null r-planes, with $r \leq \frac{n}{2}$. The canonical forms of the metrics have been investigated by Walker [4]. The canonical form of this metric contain three functions $a(x,y,z,t)$, $b(x,y,z,t)$ and $c(x,y,z,t)$. Our interest is in 4-dimensional Walker manifolds with parallel null 2-planes. In [2], Einstein, Osserman and locally conformally flat Walker manifolds were investigated in the restricted form of metric when $c(x,y,z,t) = 0$. Following [2], in this paper we focus on the case $c(x,y,z,t) = c$ and investigate locally conformally flat 4-dimensional Walker manifolds admitting parallel null 2-plane.

Definition 1. A Walker manifold is a triple (M, g, D) consisting of an n-dimensional manifold M, an indefinite metric g and an r-dimensional parallel distribution D. If $\dim M = 4$ and $\dim D = 2$, g has signature $(--++)$ and in suitable coordinates, g can be given by

$$g(x,y,z,t) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a(x,y,z,t) & c(x,y,z,t) \\ 0 & 1 & c(x,y,z,t) & b(x,y,z,t) \end{pmatrix} \tag{1.1}$$

for some functions $a(x,y,z,t)$, $b(x,y,z,t)$, $c(x,y,z,t)$ and $D = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle$ (ref. [4], [5]). The special case where $c(x,y,z,t) \equiv 0$ was studied in [2], where the components of Levi–Civita connection, and the curvature tensor has been calculated and the conditions that creates locally conformally flat are given. We are interested in the case of the Walker metric g as in (0.1) with constant c.

§2. CURVATURE TENSOR OF 4-DIMENSIONAL NEUTRAL METRIC ADDMITTING A PARALLEL NULL 2-PLANE

The canonical form of Walker metric has been obtained by A. G. Walker in [4], [5], showing existence of suitable coordinates where the metric expresses as (0.1).
Now the Levi–Civita connection of a Walker metric (0.1) is given by

$$\nabla_{\partial_x}\partial_z = \frac{1}{2}a_x\partial_x + \frac{1}{2}c_x\partial_y, \qquad \nabla_{\partial_x}\partial_z = \frac{1}{2}a_x\partial_x + \frac{1}{2}c_x\partial_y,$$

$$\nabla_{\partial_x}\partial_t = \frac{1}{2}c_x\partial_x + \frac{1}{2}b_x\partial_y, \qquad \nabla_{\partial_y}\partial_z = \frac{1}{2}a_y\partial_x + \frac{1}{2}c_y\partial_y,$$

$$\nabla_{\partial_y}\partial_t = \frac{1}{2}c_y\partial_x + \frac{1}{2}b_y\partial_y,$$

$$\nabla_{\partial_z}\partial_z = \frac{1}{2}(a_z + ca_y + aa_x)\partial_x + \frac{1}{2}(2c_z - a_t + ba_y + ca_x)\partial_y - \frac{1}{2}a_x\partial_z - \frac{1}{2}a_y\partial_t,$$

$$\nabla_{\partial_z}\partial_t = \frac{1}{2}(a_t + cc_y + ac_x)\partial_x + \frac{1}{2}(bc_y + b_z + cc_x)\partial_y - \frac{1}{2}c_x\partial_z - \frac{1}{2}c_y\partial_t,$$

$$\nabla_{\partial_t}\partial_t = \frac{1}{2}(2c_t - b_z + cb_y + ab_x)\partial_x + \frac{1}{2}(b_t - bb_y + cb_x)\partial_y - \frac{1}{2}b_x\partial_z - \frac{1}{2}b_y\partial_t,$$

where $a_x = \frac{\partial}{\partial x}a(x,y,z,t)$ means partial derivative and ∂_k denotes the coordinate vector field $\frac{\partial}{\partial k}$, $k = x,y,z,t$. For the special case of $c(x,y,z,t) \equiv c$ in (1.1), we get

$$g(x,y,z,t) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a(x,y,z,t) & c \\ 0 & 1 & c & b(x,y,z,t) \end{pmatrix}. \tag{2.1}$$

Let R denotes the curvature tensor taken with the convention $R(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$. Then, the nonzero components of the curvature tensor are computed as :

$$R_{\partial_z\partial_x}\partial_x = \frac{1}{2}a_{xx}\partial_x, \qquad R_{\partial_z\partial_x}\partial_y = \frac{1}{2}a_{yx}\partial_x,$$

$$R_{\partial_z\partial_x}\partial_z = \frac{1}{2}(ca_{yx} + aa_{xx}) + \frac{1}{4}(a_yb_x + 2ba_{yx} + 2ca_{tx} - 2a_{tx})\partial_y - \frac{1}{2}a_{xx}\partial_z - \frac{1}{2}a_{yx}\partial_t,$$

$$R_{\partial_z\partial_x}\partial_t = \frac{1}{4}(2a_{tx} - b_xa_y)\partial_x, \qquad R_{\partial_t}\partial_x\partial_x = \frac{1}{2}b_{xx}\partial_y,$$

$$R_{\partial_t\partial_x}\partial_y = \frac{1}{2}b_{yx}\partial_y, \qquad R_{\partial_t\partial_x}\partial_z = \frac{1}{4}(2b_{zx} - a_x b_x)\partial_y,$$

$$R_{\partial_t}\partial_x\partial_t = \frac{1}{4}(-2b_{zx} + 2cb_{yx} + a_x b_x + 2ab_{xx})\partial_x +$$

$$+\frac{1}{2}(bb_{yx} - cb_{xx})\partial_y - \frac{1}{2}b_{xx}\partial_z - \frac{1}{2}b_{yx}\partial_t,$$

$$R_{\partial_z}\partial_y\partial_x = \frac{1}{2}a_{xy}\partial_x, \qquad R_{\partial_z\partial_y}\partial_y = \frac{1}{2}a_{yy}\partial_x,$$

$$R_{\partial_z\partial_y}\partial_z = \frac{1}{2}(ca_{yy} + aa_{yx}b_x)\partial_x + \frac{1}{4}(-2a_{ty}+$$

$$+a_y b_y + 2ba_{yy} + 2ca_{xy})\partial_y - \frac{1}{2}a_{xy}\partial_z - \frac{1}{2}a_{yy}\partial_t,$$

$$R_{\partial_z}\partial_y\partial_t = \frac{1}{4}(2a_{ty} - b_y a_y)\partial_x, \qquad R_{\partial_t}\partial_y\partial_x = \frac{1}{2}b_{xy}\partial_y,$$

$$R_{\partial_t\partial_y}\partial_y = \frac{1}{2}b_{yy}\partial_y, \qquad R_{\partial_t\partial_y}\partial_z = \frac{1}{4}(2b_{zy} - a_y b_x)\partial_y,$$

$$R_{\partial_t}\partial_y\partial_t = \frac{1}{4}(-2b_{zy} + 2cb_{yy} + a_y b_x + 2ab_{xy})\partial_x$$

$$+\frac{1}{2}(bb_{yy} + cb_{xy})\partial_y - \frac{1}{2}b_{xy}\partial_z - \frac{1}{2}b_{yy}\partial_t,$$

$$R_{\partial_t}\partial_z\partial_x = \frac{1}{4}(2b_z a_y - 2a_{tx})\partial_x + \frac{1}{4}(2b_{xz} - a_x b_x)\frac{1}{4}\partial_y,$$

$$R_{\partial_t\partial_z}\partial_y = \frac{1}{4}(b_y a_y - 2a_{yt})\partial_x + \frac{1}{4}(2b_{yz} - a_y b_x)\partial_y,$$

$$\begin{aligned}
R_{\partial_t\partial_z}\partial_z &= \frac{1}{4}(-2ca_{yt} - 2aa_{xt} + ca_y b_y + aa_y b_x)\partial_x + \frac{1}{4}(2b_{zz} + 2a_{tt} - a_y b_t - \\
&- 2a_{yt}b - 2ca_{xt} - ca_x b_y - aa_x b_x + a_t b_y + b_z a_x - a_z b_x)\partial_y + \\
&+ \frac{1}{4}(2a_{xt} - a_y b_x)\partial_z + \frac{1}{4}(2a_{yt} - a_y b_y)\partial_t, \\
R_{\partial_t\partial_z}\partial_t &= \frac{1}{4}(-2b_{zz} + a_z b_x + 2ab_{xz} - 2a_{tt} - b_z a_x + 2cb_{yz} + b_t a_y - b_y a_t + \\
&+ bb_y a_y + cb_y a_x)\partial_x + \frac{1}{4}(2bb_{yz} + 2cb_{xz} - b_x ba_y - cb_x a_x)\partial_y + \\
&+ \frac{1}{4}(b_x a_x - 2b_{xz})\partial_z + \frac{1}{4}(b_x a_y - 2b_{yz})\partial_t.
\end{aligned} \tag{2.2}$$

§3. LOCALLY CONFORMALLY FLAT 4-DIMENSIONAL NEUTRAL METRICS ADMITTING A PARALLEL NULL 2-PLANE FIELDS

A semi-Riemannian manifold is locally conformally flat if and only if its Weyl tensor, given by

$$W(X, Y, Z, V) = R(X, Y, Z, V)$$

$$-\frac{Sc}{(n-1)(n-2)}\left(g(X,Z)g(Y,V)-g(X,V)g(Y,Z)\right)$$
$$-\frac{1}{(n-2)}\big(Ric(X,Z)g(Y,V)-Ric(Y,Z)g(X,V)$$
$$+Ric(Y,V)g(X,Z)g(X,V)g(Y,Z)\big)$$

vanishes. The nonzero components of Weyl tensor of a special Walker metric (2.1) are given by

$$\begin{aligned}
W(\partial_x,\partial_y,\partial_z,\partial_t) &= -\frac{1}{12}a_{xx}-\frac{1}{12}b_{yy},\\
W(\partial_x,\partial_z,\partial_x,\partial_t) &= \frac{1}{4}b_{xy},\\
W(\partial_x,\partial_z,\partial_x,\partial_z) &= -\frac{1}{6}a_{xx}-\frac{1}{6}b_{yy},\\
W(\partial_x,\partial_z,\partial_y,\partial_z) &= -\frac{1}{4}a_{xy},\\
W(\partial_x,\partial_z,\partial_z,\partial_t) &= \frac{1}{4}a_{xt}-\frac{1}{4}b_{zy}-\frac{1}{12}ca_{xx}+\frac{1}{4}ab_{xy}+\frac{1}{6}cb_{yy},\\
W(\partial_x,\partial_t,\partial_x,\partial_t) &= -\frac{1}{2}b_{xx},\\
W(\partial_x,\partial_t,\partial_y,\partial_z) &= \frac{1}{12}b_{yy}+\frac{1}{12}a_{xx},\\
W(\partial_x,\partial_t,\partial_y,\partial_t) &= -\frac{1}{4}b_{xy},\\
W(\partial_x,\partial_t,\partial_z,\partial_t) &= -\frac{1}{4}ab_{xx}-\frac{1}{4}cb_{xy}-\frac{1}{12}bb_{yy}-\frac{1}{12}ba_{xx},\\
W(\partial_y,\partial_z,\partial_y,\partial_z) &= -\frac{1}{2}a_{yy},\\
W(\partial_y,\partial_z,\partial_y,\partial_t) &= \frac{1}{4}a_{xy},\\
W(\partial_y,\partial_z,\partial_z,\partial_t) &= \frac{1}{12}aa_{xx}+\frac{1}{4}ca_{xy}+\frac{1}{4}ba_{yy}+\frac{1}{12}ab_{yy},\\
W(\partial_y,\partial_t,\partial_y,\partial_t) &= -\frac{1}{6}a_{xx}-\frac{1}{6}b_{yy},\\
W(\partial_y,\partial_t,\partial_z,\partial_t) &= -\frac{1}{4}b_{zy}+\frac{1}{4}a_{xt}-\frac{1}{4}ba_{xy}+\frac{1}{12}cb_{yy}-\frac{1}{6}ca_{xx},\\
W(\partial_z,\partial_t,\partial_z,\partial_t) &= -\frac{1}{2}a_{tt}-\frac{1}{2}b_{zz}-\frac{1}{4}a_xb_z+\frac{1}{4}a_yb_t+\frac{1}{4}b_xa_z-\frac{1}{4}b_ya_t\\
&+\frac{1}{4}cb_ya_x-\frac{1}{4}cb_xa_y+\frac{1}{2}ab_{zx}+\frac{1}{2}ca_{xt}+\frac{1}{2}cb_{zy}\\
&-\frac{1}{4}b^2a_{yy}+\frac{1}{2}ba_{yt}-\frac{1}{6}c^2a_{xx}-\frac{1}{6}c^2b_{yy}-\frac{1}{2}cab_{xy}\\
&-\frac{1}{12}abb_{yy}-\frac{1}{12}baa_{xx}-\frac{1}{2}cba_{xy}-\frac{1}{4}a^2b_{xx}.
\end{aligned}\tag{3.1}$$

Now it is possible to obtain the form of a locally conformally flat Walker metric as follows :

Theorem 1. *A Walker metric (2.1) is locally conformally flat if and only if the functions $a(x,y,z,t)$ and $b(x,y,z,t)$ satisfy*

$$\begin{aligned} a(x,y,z,t) &= \kappa x^2 + xR(z,t) + yS(z,t) + \xi(z,t), \\ b(x,y,z,t) &= -\kappa y^2 + yP(z,t) + xQ(z,t) + \eta(z,t), \end{aligned} \tag{3.2}$$

where the functions $R(z,t)$, $S(z,t)$, $P(z,t)$, $Q(z,t)$, $\xi(z,t)$, $\eta(z,t)$ and c satisfy the following relations

$$\begin{gathered} R_t - P_z = 2c\kappa, \\ -2R_{tt} - 2Q_{zz} + RQ_z - 2\kappa\eta_z + SQ_t + R_zQ - R_tP + 2QS_t + 2\kappa cP = 0, \\ -2S_{tt} - 2P_{zz} + 2SQ_z - RP_z + SP_t + QS_z + 2\kappa\xi_t + PS_t - 2\kappa cR = 0, \\ -2\xi_{tt} - 2\eta_{zz} + 2\xi Q_z + 2cR_t + 2cP_z + 2S_t\eta - R\eta_z + S\eta_t + Q\xi_z - P\xi_t + cRP - cQS = 0, \end{gathered} \tag{3.3}$$

$\kappa \in \mathbb{R}$ is any constant.

Since a four-dimensional manifold is locally conformally flat if and only if the Weyl tensor vanishes, we consider (2.1) to be a system of PDE's. We shall prove the theorem in four steps.

First step :

$$\begin{aligned} W(\partial_x,\partial_z,\partial_x,\partial_t) &= \frac{1}{4}b_{xy} = 0, \\ W(\partial_x,\partial_z,\partial_y,\partial_z) &= -\frac{1}{4}a_{xy} = 0, \\ W(\partial_x,\partial_t,\partial_x,\partial_t) &= -\frac{1}{2}b_{xx} = 0, \\ W(\partial_x,\partial_t,\partial_y,\partial_t) &= -\frac{1}{4}b_{xy} = 0, \\ W(\partial_y,\partial_z,\partial_y,\partial_z) &= -\frac{1}{2}a_{yy} = 0, \\ W(\partial_y,\partial_z,\partial_y,\partial_t) &= \frac{1}{4}a_{xy} = 0. \end{aligned} \tag{3.4}$$

From the equation (2.2) in the Weyl condition (3.4), we see that a and b become

$$\begin{aligned} a &= a(x,y,z,t) = \overline{a}(x,z,t) + \widehat{a}(y,z,t), \\ b &= b(x,y,z,t) = \overline{b}(x,z,t) + \widehat{b}(y,z,t). \end{aligned} \tag{3.5}$$

Then we have $a_{yy}(y,z,t) = 0$, so $\widehat{a}_{yy}(y,z,t) = 0$, and $\widehat{a}(y,z,t) = yS(z,t) + \xi(z,t)$. Similarly, $b_{xx}(x,z,t) = 0$, so $\overline{b}_{xx}(x,z,t) = 0$, and $\overline{b}(x,z,t) = xQ(z,t) + \eta(z,t)$.

Second step : Considering the result of step 1, the Weyl equations (3.1) reduce to

$$W(\partial_x,\partial_y,\partial_z,\partial_t)=-\frac{1}{12}a_{xx}-\frac{1}{12}b_{yy}=0, \tag{3.6}$$

$$\begin{aligned}
W(\partial_x,\partial_z,\partial_x,\partial_z)&=-\frac{1}{6}a_{xx}-\frac{1}{6}b_{yy}=0\\
W(\partial_x,\partial_z,\partial_z,\partial_t)&=\frac{1}{4}a_{xt}-\frac{1}{4}b_{zy}-\frac{1}{12}ca_{xx}+\frac{1}{4}ab_{xy}+\frac{1}{6}cb_{yy}=0,\\
W(\partial_x,\partial_t,\partial_y,\partial_z)&=\frac{1}{12}b_{yy}+\frac{1}{12}a_{xx}=0,\\
W(\partial_x,\partial_t,\partial_z,\partial_t)&=-\frac{1}{4}ab_{xx}-\frac{1}{4}cb_{xy}-\frac{1}{12}bb_{yy}-\frac{1}{12}ba_{xx}=0,\\
W(\partial_y,\partial_z,\partial_z,\partial_t)&=\frac{1}{12}aa_{xx}+\frac{1}{4}ca_{xy}+\frac{1}{4}ba_{yy}+\frac{1}{12}ab_{yy}=0,\\
W(\partial_y,\partial_t,\partial_y,\partial_t)&=-\frac{1}{6}a_{xx}-\frac{1}{6}b_{yy}=0,\\
W(\partial_y,\partial_t,\partial_z,\partial_t)&=-\frac{1}{4}b_{zy}+\frac{1}{4}a_{xt}-\frac{1}{4}ba_{xy}+\frac{1}{12}cb_{yy}-\frac{1}{6}ca_{xx}=0,\\
W(\partial_z,\partial_t,\partial_z,\partial_t)&=-\frac{1}{2}a_{tt}-\frac{1}{2}b_{zz}-\frac{1}{4}a_xb_z+\frac{1}{4}a_yb_t+\frac{1}{4}b_xa_z-\frac{1}{4}b_ya_t+\\
&+\frac{1}{4}cb_ya_x-\frac{1}{4}cb_xa_y+\frac{1}{2}ab_{zx}+\frac{1}{2}ca_{xt}+\frac{1}{2}cb_{zy}-\\
&-\frac{1}{4}b^2a_{yy}+\frac{1}{2}ba_{yt}-\frac{1}{6}c^2a_{xx}-\frac{1}{6}c^2b_{yy}-\frac{1}{2}cab_{xy}-\\
&-\frac{1}{12}abb_{yy}-\frac{1}{12}baa_{xx}-\frac{1}{2}cba_{xy}-\frac{1}{4}a^2b_{xx}=0.
\end{aligned} \tag{3.7}$$

Equation (3.6) gives $a_{xx}=-b_{yy}$ and thus (3.5) shows that $\overline{a}_{xx}(x,z,t)=-\widehat{b}_{yy}(y,z,t)$ and moreover we see that the both sides of equation are independent of x and y, and in fact they depend only on z and t. Thus, equation (3.6) reduces to the following.

$$\overline{a}_{xx}(x,z,t)=\kappa(z,t),\qquad \widehat{b}_{yy}(y,z,t)=-\kappa(z,t).$$

Then $\overline{a}$ $(resp,\widehat{b})$ is a quadratic function of $x(resp,y)$, with z and t parameters. Therefore we have

$$\begin{aligned}
a&=a(x,y,z,t)=\overline{a}(x,z,t)+\widehat{a}(y,z,y)=x^2\kappa(z,t)+xR(z,t)+\xi(z,t)+\widehat{a}(y,z,t),\\
b&=b(x,y,z,t)=\overline{b}(x,z,t)+\widehat{b}(y,z,y)=\overline{b}(x,z,t)-y^2\kappa(z,t)+yP(z,t)+\eta(z,t),
\end{aligned}$$

for some function $\kappa(z,t)$, $P(z,t)$, $R(z,t)$, $\xi(z,t)$ and $\eta(z,t)$.

Third step : Considering the result of step one, equations (3.7), we conclude that $a_{xt} - b_{zy} - ca_{xx} \equiv 0$. Now from the above we have $a_{xt} = b_{zy} + ca_{xx}$ and thus

$$2x\kappa_t(z,t) + R_t(z,t) = -2y\kappa_z(z,t) + P_z(z,t) + 2c\kappa(z,t).$$

Then $\kappa_t = 0$, $\kappa_z = 0$, so $\kappa(z,t) = \kappa$, $P_z = R_t - 2c\kappa$. This shows the first equation at (3.3) and the metric expression as in (3.2)

$$\begin{aligned} a &= a(x,y,z,t) = \kappa x^2 + xR(z,t) + yS(z,t) + \xi(z,t), \\ b &= b(x,y,z,t)) = -\kappa y^2 + yP(z,t) + xQ(z,t) + \eta(z,t). \end{aligned}$$

Fourth step : It follows from the previous steps that the last of equation (3.1) reduces to the equation

$$\begin{aligned} W(\partial_z,\partial_t,\partial_z,\partial_t) = &-\frac{1}{2}a_{tt} - \frac{1}{2}b_{zz} - \frac{1}{4}a_x b_z + \frac{1}{4}a_y b_t + \frac{1}{4}a_z b_x - \frac{1}{4}a_t b_y + \\ &+ \frac{1}{4}ca_x b_y - \frac{1}{4}cb_x a_y + \frac{1}{2}b_{zx}a + \frac{1}{2}ca_{xt} + \frac{1}{2}cb_{zy} + \frac{1}{2}ba_{yt} = 0. \end{aligned}$$

Then the special Walker metric defined as (3.2) is locally conformally flat if and only if the component of the Weyl tensor $W(\partial_z,\partial_t,\partial_z,\partial_t)$ vanishes. Moreover, since

$$\begin{aligned} &2a_{tt} + 2b_{zz} + a_x b_z - a_y b_t - a_z b_x + a_t b_y - ca_x b_y + \\ &\quad + cb_x a_y - 2b_{zx}a - 2ca_{xt} - 2cb_{zy} - 2ba_{yt} = 0. \end{aligned}$$

Then

$$\begin{aligned} W(\partial_z,\partial_t,\partial_z,\partial_t) = &\frac{1}{4}x(-2R_{tt} - 2Q_{zz} + RQ_z - 2\kappa\eta_z + SQ_t + R_zQ - R_tP + 2QS_t + 2\kappa cP) + \\ &+\frac{1}{4}y(-2S_{tt} - 2P_{zz} + 2SQ_z - RP_z + SP_t + QS_z + 2\kappa\xi_t + PS_t - 2\kappa cR) + \\ &+\frac{1}{2}xy(-\kappa P_z + 2\kappa R_t - \kappa^2 c) + \frac{1}{4}\big(-2\xi_{tt} - 2\eta_{zz} + 2\xi Q_z + 2cR_t + \\ &+2cP_z + 2S_t\eta - R\eta_z + S\eta_t + Q\xi_z - P\xi_t + cRP - cQS\big) = 0, \end{aligned}$$

and (3.3) follows.

Remark 1. Putting $c = 0$ we get Theorem 13 in [2].

Definition 2. Let Ric and Sc denote the Ricci tensor and the scalar curvature of (M,g), defined by $Ric = trace\,\{Z \to R(X,Z)Y\}$ and $Sc = traceRic$, respectively.

Lemma 1. *The scalar curvature of a Walker metric (2.1) is given by* $Sc = b_{yy} + a_{xx}$.

It follows from the equation expressing the components of the curvature tensor that the Ricci curvature is

$$Ric = \begin{pmatrix} 0 & 0 & \frac{1}{2}a_{xx} & \frac{1}{2}b_{xy} \\ 0 & 0 & \frac{1}{2}a_{xy} & \frac{1}{2}b_{yy} \\ \frac{1}{2}a_{xx} & \frac{1}{2}a_{xy} & \frac{1}{2}A_{33} & \frac{1}{2}(a_{tx} - a_y b_x + b_{yz}) \\ \frac{1}{2}b_{xy} & \frac{1}{2}b_{yy} & \frac{1}{2}(a_{tx} - a_y b_x + b_{yz}) & \frac{1}{2}A_{44} \end{pmatrix},$$

where $A_{33} = 2ca_{xy} + aa_{xx} + ba_{yy} + a_y b_y - 2a_{yt}$, $A_{44} = -2b_{zx} + a_x b_x + 2cb_{xy} + ab_{xx} + bb_{yy}$ and the Ricci operator $\langle \widehat{R}ic(X), Y\rangle = Ric(X, Y)$ satisfies

$$\widehat{R}ic = \begin{pmatrix} \frac{1}{2}a_{xx} & \frac{1}{2}a_{xy} & \frac{1}{2}B_{13} & \frac{1}{2}(a_{tx} - a_y b_x + b_{yz} - cb_{yy} - ab_{xy}) \\ \frac{1}{2}b_{xy} & \frac{1}{2}b_{yy} & \frac{1}{2}B_{23} & \frac{1}{2}(ab_{xx} + cb_{yx} + a_x b_x - 2b_{zx}) \\ 0 & 0 & \frac{1}{2}a_{xx} & \frac{1}{2}b_{xy} \\ 0 & 0 & \frac{1}{2}a_{xy} & \frac{1}{2}b_{yy} \end{pmatrix},$$

where $B_{13} = ca_{xy} + ba_{yy} + a_y b_y - 2a_{yt}$ and $B_{23} = a_{tx} - a_y b_x + b_{yz} - ca_{xx} - ba_{xy}$. Now, the result follows by a straightforward calculation.

Remark 2. Considering (2.14) in Lemma 1 and (2.3) we see that any locally conformally flat Walker metric (2.2) has vanishing scaler curvature, hence they necessarily are semi-Riemannian manifolds with harmonic curvature (ref. [7]).

Резюме. 4-мерная метрика Уолкера на полу-римановом многообразии M, для канонической метрики с $c = 0$, было исследовано М. Чаичи, Э. Гарсия–Рио и Й. Мацушита. В настоящей статье эти понятия обобщаются на случай постоянного $c \neq 0$. В частности, найден вид определяющих функций этой метрики в локально-конформно плоских 4-мерных многообразиях Уолкера.

R E F E R E N C E S

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