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ASYMPTOTICS OF BROWNIAN INTEGRALS AND PRESSURE: BOSE-EINSTEIN STATISTICS

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Аннотация. The paper studies the asymptotics of the Brownian integrals with paths restricted to a bounded domain of $\mathbb{R}^\nu$, when the domain is dilated to infinity. The framework is that of the Bose-Einstein statistics with paths observed within random time intervals which are integer multiplies of some fixed $\beta > 0$. The three first terms of the asymptotics are found explicitly via the functional integrals. In the case of a gas of interacting Brownian loops an expression for the volume term of the asymptotics of the log-partition function is found and the correction term is proved to by order be the boundary area of the domain.

1. INTRODUCTION

In [1] the large volume asymptotics of the Brownian integrals with paths observed in a fixed time interval β restricted to a bounded domain of $\mathbb{R}^\nu$ was studied.

In the present paper we consider similar problem for the Brownian integrals with random time intervals which are integer multiplies of β. This problem can be considered as a natural generalization of the famous Kac problem [2] on the asymptotics of the function $\sum\limits_{i=0}^{\infty} e^{-\beta\lambda_i}$ as β goes to zero, where λ_i are the eigenvalues of the Laplacian $-\Delta$ in a bounded domain Λ.

In the special case where the integrand is one, the Brownian integrals are nothing else but the logarithms of the grand canonical partition functions of the ideal quantum gases in their functional integral representations [3]. The functional integration method allows one to replace the quantum mechanical problem by a corresponding classical problem for a system of interacting Brownian trajectories. This method with application of Feynman-Kac formula was used first by Ginibre in [4]. The systems of interacting Brownian trajectories we call Ginibre gases (see [5] and [6]).

The case considered in [1] corresponds to the Ginibre gas with Maxwell-Boltzmann statistics while the present paper considers the case of Ginibre gas with Bose-Einstein statistics. The class of admissible domains Λ consists of bounded convex domains with convex holes possessing smooth boundaries of the class C^3.

We obtain the three first terms of the asymptotics for the case of small activity (Theorem 1 and 2). The first two terms are proportional respectively to the volume and to the area of the boundary of Λ. We prove that in two dimensional case the third term is proportional to the Euler-Poincare characteristic of the domain. In this part our analysis relies on the modified techniques from [1] and involves some specific properties of the Brownian bridge process outlined in the Appendix.

We consider also the Ginibre gas with repulsive two-body interaction at low activity. Applying the previous results together with the results on the decay of correlations from [6] we find an explicit expression for the pressure in terms of functional integrals and prove that the correction term is of order of the area of the boundary of Λ (Theorem 3). The proof is based on the cluster expansion method.

Similar result for the case of Maxwell-Boltzmann statistics was obtained in [7].

2. GINIBRE GAS WITH BOSE–EINSTEIN STATISTICS

For $\beta > 0$ fixed and $j = 1, 2, \dots$ let $\mathcal{X}_{j\beta}$ be the space of Brownian loops of time interval $j\beta$ in $\mathbb{R}^\nu$, $\nu \geq 1$, defined by

$$\mathcal{X}_{j\beta} \equiv \{\mathbf{X} \in C\left([0, j\beta], \mathbb{R}^\nu\right) \ | \mathbf{X}(0) = \mathbf{X}(j\beta)\}$$

In the topology of uniform convergence $\mathcal{X}_{j\beta}$ is a Polish space with Borel σ-algebra $\mathfrak{B}_{j\beta}$. Let $\mathcal{X}^u_{j\beta}$ be the set of loops $\mathbf{X}$ which start and end at the point $u \in \mathbb{R}^\nu$. In $(\mathcal{X}_{j\beta}, \mathfrak{B}_{j\beta})$ we consider a non-normalized Brownian bridge measure $P^u_{j\beta}\ :\ P^u_{j\beta}\left(\mathcal{X}_{j\beta}\right) = (\pi j\beta)^{\nu/2}$. (see the details in [6])

The underlying one particle space $\mathcal{X}$ is defined as a topological sum of the spaces $\mathcal{X}_{j\beta}$:

$$\mathcal{X} = \bigcup_{j=1}^{\infty} \mathcal{X}_{j\beta}.$$

The natural σ-algebra in $\mathcal{X}$ generated by the σ-algebras $\mathfrak{B}_{j\beta}$ we denote by $\mathfrak{B}(\mathcal{X})$. The elements of $\mathcal{X}$ we call composite loops and put $|\mathbf{X}| = j$ if $\mathbf{X} \in \mathcal{X}_{j\beta}$.

Let $0 < z \leq 1$ be a parameter called activity or fugacity. We define a measure P^u_z on $\mathcal{X}^u = \bigcup\limits_{j=1}^{\infty} \mathcal{X}^u_{j\beta}$ by the formula

$$P^u_z = \sum_{j=1}^{\infty} \frac{z^j}{j} P^u_{j\beta}.$$

Evidently P^u_z is a finite measure for all z, $0 < z \leq 1$. Using a natural bijection $\tau : \mathcal{X}^0 \times \mathbb{R}^\nu \to \mathcal{X}$ defined by $\tau\left(\mathbf{X}^0, u\right) = \mathbf{X}^0 + u$, $\mathbf{X}^0 \in \mathcal{X}^0$, $u \in \mathbb{R}^\nu$, we define a σ-finite measure ρ_z on $\mathcal{X}$ by

$$\rho_z = \left(P^0_z \times \lambda\right) \circ \tau^{-1},$$

where λ is the Lebesgue measure on $\mathbb{R}^\nu$. The triple $(\mathcal{X}, \mathfrak{B}(\mathcal{X}), \rho_z)$ is the one particle space of our system.

The configuration space of our system is

$$\mathcal{M}(\mathcal{X}) = \{\omega \subset \mathcal{X} \,||\omega| < \infty\}$$

where $|\cdot|$ stands for the number of elements in a finite set.

An element $\omega \in \mathcal{M}(\mathcal{X})$ is a finite configuration of composite loops in $\mathbb{R}^\nu$ of random time intervals multiple to β. We denote by $\mathcal{F}(\mathcal{X})$ the canonical σ-algebra in $\mathcal{M}(\mathcal{X})$.

On $\mathcal{M}(\mathcal{X})$ we consider the following σ-finite measure W_{ρ_z} given by the formula

$$W_{\rho_z} = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathcal{X}^n} \varphi(\mathbf{X}_1, \ldots, \mathbf{X}_n)\, \rho_z(d\mathbf{X}_1) \cdots \rho_z(d\mathbf{X}_n) \tag{2.1}$$

(For the details, see [6]).

For any domain $\Lambda \in \mathbb{R}^\nu$ let $\mathcal{X}(\Lambda)$ be the set of all composite loops "living" in Λ:

$$\mathcal{X}(\Lambda) = \{\mathbf{X} \in \mathcal{X} \mid \mathbf{X}(t) \in \Lambda,\ \forall t \in [0, |\mathbf{X}|\beta]\}.$$

In the same way, let

$$\mathcal{M}(\Lambda) = \{\omega \subset \mathcal{X}(\Lambda) \mid |\omega| < \infty\}$$

be the set of finite configurations of composite loops in Λ. The restriction of the measure ρ_z on $\mathcal{X}(\Lambda)$ (respectively of W_{ρ_z} on $\mathcal{M}(\Lambda)$) we denote by $\rho_{z,\Lambda}$ (respectively by $W_{z,\Lambda}$). The triple $(\mathcal{M}(\Lambda), \mathfrak{F}(\Lambda), W_{z,\Lambda})$ we call the ideal Ginibre gas in Λ with Bose-Einstein statistics and activity z.

Note that for Λ bounded both measures $\rho_{z,\Lambda}$ and $W_{z,\Lambda}$ are finite. Moreover

$$W_{z,\Lambda}(\mathcal{M}(\Lambda)) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathcal{X}^u(\Lambda)} \rho_{z,\Lambda}(d\mathbf{X}_1) \cdots \rho_{z,\Lambda}(d\mathbf{X}_n) = \exp\{\rho_{z,\Lambda}(\mathcal{X}(\Lambda))\} \tag{2.2}$$

is the grand partition function $\Xi_{id}(\Lambda, z)$ of the ideal Ginibre gas in Λ.

To define the energy of configuration $\omega \in \mathcal{M}(\Lambda)$ we consider the space $C([0,\beta], \mathbb{R}^\nu)$ of all continuous trajectories of time intervals β in $\mathbb{R}^\nu$ which we call elementary trajectories. We will say that an elementary trajectory x is an elementary constituent of a composite loop $\mathbf{X} \in \mathcal{X}$, and we will write $x \in \mathbf{X}$, if for some i, $i = 0, 1, \ldots, |\mathbf{X}| - 1$, $x(t) = \mathbf{X}(i\beta + t)$ for all $t \in [0, \beta]$.

Let

$$\tilde{\Phi}(x) = \int_0^\beta \Phi(x(t))\, dt, \qquad x \in C([0,\beta], \mathbb{R}^\nu), \tag{2.3}$$

where $\Phi : \mathbb{R}^\nu \to \mathbb{R}$ is a continuous function (see below for the conditions on Φ). The energy $U(\omega)$ of a configuration $\omega \in \mathcal{M}(\mathcal{X})$ is given by

$$U(\omega) = \sum_{\mathbf{X} \in \omega} U_1(\mathbf{X}) + \frac{1}{2} \sum_{\mathbf{X}, \mathbf{Y} \in \omega,\, \mathbf{X} \neq \mathbf{Y}} U_2(\mathbf{X}, \mathbf{Y}),$$

where

$$U_1(\mathbf{X}) = \frac{1}{2} \sum_{x_1, x_2 \in \mathbf{X},\, x_1 \neq x_2} \tilde{\Phi}(x_1 - x_2),$$

$$U_2(\mathbf{X}, \mathbf{Y}) = \sum_{x \in \mathbf{X},\, y \in \mathbf{Y}} \tilde{\Phi}(x - y).$$

The Boltzmann factor f is defined as

$$f(\omega) = \exp\{-U(\omega)\}, \quad \omega \in \mathcal{M}(\mathcal{X}).$$

The triple $(\mathcal{M}(\Lambda), W_{z,\Lambda}, \Phi)$ we will call the Ginibre gas in Λ with activity z, interaction Φ and Bose-Einstein statistics.

The main object of our interest is the grand partition function $\Xi(\Lambda, z)$ of the Ginibre gas in a bounded $\Lambda \subset \mathbb{R}^\nu$ which is defined by

$$\Xi(\Lambda, z) = W_{z,\Lambda}(f) = \int\limits_{\mathcal{M}(\Lambda)} \exp\{-U(\omega)\} W_{z,\Lambda}(d\omega), \tag{2.4}$$

We want to study the asymptotics of $\log \Xi(\Lambda, z)$ for large bounded Λ. As in the case of Maxwell-Boltzmann statistics (see [7]) the Brownian integral of a single loop constrained to a bounded domain Λ, $\int\limits_{\mathcal{X}(\Lambda)} \rho_{z,\Lambda}(d\mathbf{X})$ has its own, non-trivial contribution to the asymptotics of $\log \Xi(\Lambda, z)$. This is purely "quantum" effect which is not the case for the classical analogue of our model. Note that by (2.2),

$$\int\limits_{\mathcal{X}(\Lambda)} \rho_{z,\Lambda}(d\mathbf{X}) = \log W_{z,\Lambda}(\mathcal{M}(\Lambda)) = \log \Xi_{id}(\Lambda, z).$$

Thus we need to study first the asymptotics of $\log \Xi_{id}(\Lambda, z)$ for large Λ.

3. CONDITIONS ON THE POTENTIAL AND THE CLASS OF ADMISSIBLE DOMAINS

We suppose that the function Φ which defines the interaction $\tilde{\Phi}$ between loops (see (2.3)) satisfies the following conditions:

- **(a):** Φ is an even function: $\Phi(-u) = \Phi(u)$, $u \in \mathbb{R}^\nu$;
- **(b):** Φ is repulsive: $\Phi \geq 0$;
- **(c):** Φ has the following power decay at infinity:

$$\int\limits_{\mathbb{R}^\nu} |\Phi(u)| (1 + |u|)^l \, du, \quad l > 0.$$

The class of potentials Φ satisfying conditions (a)-(c) we denote by $\mathcal{P}_l^+$.

The class of admissible domains Λ consists of open bounded convex subsets of $\mathbb{R}^\nu$ with n, $n \geq 0$, convex closed holes. We assume that the boundary $\partial\Lambda$ of Λ consists of $n+1$ $(\nu - 1)$-dimensional closed C^3 manifolds. At each point $r \in \partial\Lambda$ we define local coordinates $(\eta, \xi_1, \ldots, \xi_{\nu-1})$ so that η is along the inward drawn unit normal $\mathbf{n}$ and $\xi_1, \ldots, \xi_{\nu-1}$ are along the directions of principal curvatures of $\partial\Lambda$ at the point r. In this local coordinates $\partial\Lambda$ is given by a C^3 function f_r:

$$\eta = f_r(\xi_1, \ldots, \xi_{\nu-1}) = f_r(\boldsymbol{\xi}), \quad ||\boldsymbol{\xi}|| < \delta \tag{3.1}$$

for some $\delta > 0$ small enough, $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_{\nu-1})$.

4. MAIN RESULTS

Let $F(\mathbf{X})$, $\mathbf{X} \in \mathcal{X}$ be a translation invariant function: $F(\mathbf{X} + \mathbf{u}) = F(\mathbf{X})$, for all $\mathbf{X} \in \mathcal{X}$ and $\mathbf{u} \in \mathbb{R}^\nu$. Hence we can think of F as a function on $\mathcal{X}^0$ and we assume that $F \in L_2(\mathcal{X}^0, P^0_{\overline{z}})$ for some $\overline{z} > 0$. Let

$$\Lambda_R = R \cdot \Lambda = \{R \cdot u | u \in \Lambda\}.$$

Theorem 1. *For any admissible domain* Λ *and for all* z *from the interval* $0 < z \leq \overline{z}$ *the following expansion holds true*

$$\int\limits_{\mathcal{X}(\Lambda_R)} F(\mathbf{X})\rho_{z,\Lambda_R}(d\mathbf{X}) = R^\nu|\Lambda|a_0(F,z) + R^{\nu-1}a_1(\Lambda,F,z) + R^{\nu-2}a_2(\Lambda,F,z) + o(R^{\nu-2})$$

as $R \to \infty$, *where* $|\Lambda|$ *is the volume of* Λ *and the coefficient* a_0, a_1 *and* a_2 *are given explicitly in terms of functional integrals by formulas (5.3), (5.29) and (5.30) respectively.*

In the case where the function F is in addition rotation invariant the coefficients a_1 and a_2 have simpler form.

Theorem 2. *If under the conditions of Theorem 1 the function* F *is in addition rotation invariant, then*

$$\int\limits_{\mathcal{X}(\Lambda_R)} F(\mathbf{X})\rho_{z,\Lambda_R}(d\mathbf{X}) = R^\nu|\Lambda|a_0(F,z) + R^{\nu-1}|\partial\Lambda|\overline{a}_1(F,z) + R^{\nu-2}\int\limits_{\partial\Lambda} H_\Lambda(r)\sigma(dr)\overline{a}_2(F,z) + o(R^{\nu-2})$$

where $\overline{a}_1$ *and* $\overline{a}_2$ *are given by (5.31) and (5.32),* $H_\Lambda(r)$ *is the mean curvature of* $\partial\Lambda$ *at the point* $r \in \partial\Lambda$ *and* σ *is the* $\nu - 1$*-dimensional surface measure.*

Remark 1. *In dimension two,* $\nu = 2$, *according to Gauss-Bonnet theorem*

$$\int\limits_{\partial\Lambda} H_\Lambda(r)\sigma(dr) = 2\pi\Gamma(\Lambda),$$

where $\Gamma(\Lambda)$ *is the Euler-Poincaré characteristic of* Λ, $\Gamma(\Lambda) = 1 - n$, *if* Λ *has* n *holes. Therefore the corresponding term is purely topological.*

Remark 2. *In particular case where* $F \equiv 1$ *Theorem 2 gives an asymptotic expansion of the log-partition function* $\log \Xi_{id}(\Lambda_R, z)$ *of the ideal Ginibre gas in* Λ_R, *as* $R \to \infty$.

The next result gives the main term of the asympotic expansion of the log-partition function of the Ginibre gas in Λ_R with interaction Φ.

Theorem 3. *Let* $\Phi \in \mathcal{P}_l^+$, $l > 1$ *and* z *be from the interval*

$$0 < z < \exp\left\{ -\left[32\frac{\beta^{1-\nu/2}}{\pi^{\nu/2}} \sum_{j=1}^{\infty} j^{-(1+\nu/2)} \int_{\mathbb{R}^\nu} \Phi(u)du \right]^{1/2} \right\}. \tag{4.1}$$

then for any admissible domain $\Lambda \subset \mathbb{R}^\nu$

$$\log \Xi(\Lambda_R, z) = R^\nu \cdot p(\Phi,z)|\Lambda| + O(R^{\nu-1}) \quad as \quad R \to \infty,$$

where $p(\Phi, z)$ *is given by*

$$p(\Phi,z) = \int\limits_{\mathcal{X}^0} P_z^0(d\mathbf{X}) \int\limits_{\mathcal{M}(\mathcal{X})} \frac{g(\omega,\mathbf{X})}{|\omega|+1} W_{\rho_z}(d\omega),$$

and $\beta^{-1}p(\Phi,z)$ *is called pressure.*

Here and below, without confusing the reader, we write $(\mathbf{X},\omega)$ for the configuration $\{\mathbf{X}\}\cup\omega$. The Ursell function g is defined below by formula 6.1.

5. PROOF OF THEOREM 1

Let

$$I(R,z)=\int_{\mathcal{X}(\Lambda_R)} F(\mathbf{X})\rho_{z,\Lambda_R}(d\mathbf{X}).$$

By definition of the measure ρ_{z,Λ_R}

$$I(R,z)=\int\limits_{\Lambda_R} du \int\limits_{\mathcal{X}^0} \mathbb{I}_{\mathcal{X}(\Lambda_R)}(\mathbf{X}+u)F(\mathbf{X})P_z^0(d\mathbf{X})$$

where $\mathbb{I}_A$ is the indicator function of a set A.

We decompose this integral as follows

$$I(R,z)=I_0(R,z)-I_1(R,z), \tag{5.1}$$

where

$$I_0(R,z)=\int\limits_{\Lambda_R} du \int\limits_{\mathcal{X}^0} F(\mathbf{X})P_z^0(d\mathbf{X}),$$

$$I_1(R,z)=\int\limits_{\Lambda_R} du \int\limits_{\mathcal{X}^0} \Big(1-\mathbb{I}_{\mathcal{X}(\Lambda_R)}(\mathbf{X}+u)\Big) F(\mathbf{X})P_z^0(d\mathbf{X}).$$

This gives the volume term:

$$I_0(R,z)=R^\nu\cdot|\Lambda|\cdot a_0(F,z) \tag{5.2}$$

with

$$a_0(F,z)=\int\limits_{\mathcal{X}^0} F(\mathbf{X})P_z^0(d\mathbf{X}). \tag{5.3}$$

To study I_1 from (5.1) we put

$$\Lambda_{R,\delta}=\Big\{u\in\Lambda_R | d(u,\partial\Lambda)<\delta\sqrt{R}\Big\}$$

where d is the Euclidean distance in $\mathbb{R}^\nu$. Then

$$I_1(R,z)=I_2(R,z)+I_2'(R,z), \tag{5.4}$$

where

$$I_2(R,z)=\int\limits_{\Lambda_{R,\delta}} du \int\limits_{\mathcal{X}^0} \Big(1-\mathbb{I}_{\mathcal{X}(\Lambda_R)}(\mathbf{X}+u)\Big) F(\mathbf{X})P_z^0(d\mathbf{X}),$$

$$I_2'(R,z)=\int\limits_{\Lambda_R\setminus\Lambda_{R,\delta}} du \int\limits_{\mathcal{X}^0} \Big(1-\mathbb{I}_{\mathcal{X}(\Lambda_R)}(\mathbf{X}+u)\Big) F(\mathbf{X})P_z^0(d\mathbf{X}).$$

Evidently

$$|I_2'(R,z)| \le \int\limits_{\Lambda_R\setminus\Lambda_{R,\delta}} du \int\limits_{\mathcal{X}^0} \mathbb{I}_{\{\sup||\mathbf{X}||\ge\delta\sqrt{R}\}}(\mathbf{X})\,|F(\mathbf{X})|\,P_z^0(d\mathbf{X})$$

By Schwarz inequality and Lemma 1 from [6]

$$\int\limits_{\mathcal{X}^0} \mathbb{I}_{\{\sup||\mathbf{X}||\ge\delta\sqrt{R}\}}(\mathbf{X})\,|F(\mathbf{X})|\,P_z^0(d\mathbf{X}) \tag{5.5}$$

$$\le \|F\|_{L_2}\left[P_z^0\left(\sup\|\mathbf{X}\|\ge\delta\sqrt{R}\right)\right]^{1/2} \le C\,(\nu)\,\beta^{-\nu/4}\|F\|_{L_2}\exp[-C(\beta,z)\delta R]$$

for all z, $0<z<1$, where $C(\beta,z)=\left(\frac{|\ln z|}{32\beta}\right)^{1/2}$ and

$$\|F\|_{L_2}=\left(\int\limits_{\mathcal{X}^0} F^2(\mathbf{X})P_{+,z}^0(d\mathbf{X})\right)^{1/2}.$$

Hence (for simplicity we denote all the constants by the same letter C indicating only the dependence on the parameters)

$$|I_2'(R,z)| \le |\Lambda|C\,(\nu,\beta,z)\,\|F\|_{L_2}\exp[-C(\beta,z)\delta R]. \tag{5.6}$$

Now consider $I_2(R,z)$. We have

$$I_2(R,z)=I_3(R,z)+I_3'(R,z), \tag{5.7}$$

where

$$I_3(R,z)=\int\limits_{\Lambda_{R,\delta}} du \int\limits_{\mathcal{X}^0}\left(1-\mathbb{I}_{\mathcal{X}(\Lambda_R)}(\mathbf{X}+u)\right)\mathbb{I}_{\{\sup||\mathbf{X}||<\delta\sqrt{R}\}}(\mathbf{X})F(\mathbf{X})P_z^0(d\mathbf{X}),$$

$$I_3'(R,z)=\int\limits_{\Lambda_{R,\delta}} du \int\limits_{\mathcal{X}^0}\left(1-\mathbb{I}_{\mathcal{X}(\Lambda_R)}(\mathbf{X}+u)\right)\mathbb{I}_{\{\sup||\mathbf{X}||\ge\delta\sqrt{R}\}}(\mathbf{X})F(\mathbf{X})P_z^0(d\mathbf{X}).$$

According to (5.5)

$$|I_3'(R,z)| \le |\Lambda|C\,(\nu,\beta,z)\,\|F\|_{L_2}\exp[-C(\beta,z)\delta R]. \tag{5.8}$$

To estimate $I_3(R,z)$ we use the local coordinates. Similarly to (3.1) $\partial\Lambda_R$ is given locally by

$$\eta=f_{r,R}(\xi),\quad \|\xi\|<\delta\sqrt{R}.$$

We have the following relations between the functions $f_{r,R}$ and $f_r\equiv f_{r,1}$:

$$f_{r,R}(\xi)=Rf_{r,1}(R^{-1}\xi). \tag{5.9}$$

Let $k_i(r|R)$, $i=1,\dots,\nu-1$, be the principal curvatures of $\partial\Lambda_R$ at the point $r\in\partial\Lambda_R$. From (5.9) it follows that

$$k_i(r|R)=R^{-1}k_i(r|1),\quad i=1,\dots,\nu-1. \tag{5.10}$$

Then (see, for example, [8])

$$I_3(R,z) = \int\limits_{\partial\Lambda_R} \sigma_R(dr) \int\limits_0^{\delta\sqrt{R}} \prod_{i=1}^{\nu-1} (1-\tau k_i(r|R))\, d\tau$$

$$\cdot \int\limits_{\mathcal{X}^0} \left(1 - \mathbb{I}_{\mathcal{X}(\Lambda_R)}(\mathbf{X}+r+\tau\mathbf{n})\right) \mathbb{I}_{\sup\|\mathbf{X}\|<\delta\sqrt{R}}(\mathbf{X}) F(\mathbf{X}) P_z^0(d\mathbf{X}).$$

For each $\mathbf{X} \in \mathcal{X}^0$ such that $\sup\|\mathbf{X}\| < \delta\sqrt{R}$ we put

$$\gamma(\mathbf{X}) \equiv \gamma_{r,R}(\mathbf{X}) = \inf_t \left[\mathbf{X}_\mathbf{n}(t) - f_{r,R}(\mathbf{X}_T(t))\right].$$

Here

$$\mathbf{X}_\mathbf{n}(t) = \langle \mathbf{X}(t), \mathbf{n}\rangle, \quad \mathbf{X}_T(t) = \mathbf{X}(t) - \langle \mathbf{X}(t), \mathbf{n}\rangle \mathbf{n} \tag{5.11}$$

where $\langle\cdot,\cdot\rangle$ stands for the scalar product in $\mathbb{R}^\nu$. It is easy to check that, for any $\mathbf{X} \in \mathcal{X}^0$ with $\sup\|\mathbf{X}\| < \delta\sqrt{R}$, $\mathbb{I}_{\mathcal{X}(\Lambda_R)}(\mathbf{X}+r+\tau\mathbf{n}) = 0$ iff $\tau + \gamma(\mathbf{X}) < 0$. Therefore

$$\begin{aligned} I_3(R,z) = & \int\limits_{\partial\Lambda_R} \sigma_R(dr) \int\limits_0^{\delta\sqrt{R}} \prod_{i=1}^{\nu-1} (1-\tau k_i(r|R))\, d\tau \cdot \\ & \cdot \int\limits_{\mathcal{X}^0} \mathbb{I}_{\tau+\gamma(\mathbf{X})<0}(\mathbf{X}) \mathbb{I}_{\sup\|\mathbf{X}\|<\delta\sqrt{R}}(\mathbf{X}) F(\mathbf{X}) P_z^0(d\mathbf{X}). \end{aligned} \tag{5.12}$$

Using the equality

$$\mathbb{I}_{\sup\|\mathbf{X}_\mathbf{n}\|<\delta\sqrt{R}} = \mathbb{I}_{\sup\|\mathbf{X}\|<\delta\sqrt{R}} + \mathbb{I}_{\sup\|\mathbf{X}\|\geq\delta\sqrt{R}} \mathbb{I}_{\sup\|\mathbf{X}_\mathbf{n}\|<\delta\sqrt{R}},$$

we can rewrite (5.12) as

$$\begin{aligned} I_3(R,z) = & \int\limits_{\partial\Lambda_R} \sigma_R(dr) \int\limits_0^{\delta\sqrt{R}} \prod_{i=1}^{\nu-1} (1-\tau k_i(r|R))\, d\tau \cdot \\ & \cdot \int\limits_{\mathcal{X}^0} \mathbb{I}_{\tau+\gamma(\mathbf{X})<0}(\mathbf{X}) \mathbb{I}_{\sup\|\mathbf{X}_\mathbf{n}\|<\delta\sqrt{R}}(\mathbf{X}) F(\mathbf{X}) P_z^0(d\mathbf{X}) - \\ & - \int\limits_{\partial\Lambda_R} \sigma_R(dr) \int\limits_0^{\delta\sqrt{R}} \prod_{i=1}^{\nu-1} (1-\tau k_i(r|R))\, d\tau \cdot \\ & \cdot \int\limits_{\mathcal{X}^0} \mathbb{I}_{\tau+\gamma(\mathbf{X})<0}(\mathbf{X}) \mathbb{I}_{\sup\|\mathbf{X}\|\geq\delta\sqrt{R}}(\mathbf{X}) \mathbb{I}_{\sup\|\mathbf{X}_\mathbf{n}\|<\delta\sqrt{R}}(\mathbf{X}) F(\mathbf{X}) P_z^0(d\mathbf{X}) \equiv \\ & \equiv I^A(R,z) + \tilde{I}^A(R,z). \end{aligned} \tag{5.13}$$

Let us estimate the second term $\tilde{I}^A(R,z)$. It is clear that for each admissible domain Λ

$$\overline{k} = \max_{1\leq i\leq\nu-1} \sup_{r\in\partial\Lambda} |k_i(r|1)| < \infty.$$

Assuming $\delta < \overline{k}^{-1}$, we have that

$$\left|\prod_{i=1}^{\nu-1}(1-\tau k_i(r|R)\right| < 2^{\nu-1}, \quad 0 < \tau < \delta R.$$

Hence using (5.5) we see that

$$\left|\tilde{I}^A(R,z)\right| \leq |\Lambda| C\left(\nu,\beta,z\right) \|F\|_{L_2} \exp[-C(\beta,z)\delta R]. \tag{5.14}$$

The first term $I^A(R,z)$ in (5.13) we decompose as

$$\begin{aligned} I^A(R,z) = \int_{\partial\Lambda_R} \sigma_R(dr) \int_0^\infty \prod_{i=1}^{\nu-1} (1-\tau k_i(r|R))\, d\tau \cdot \\ \cdot \int_{\mathcal{X}^0} \mathbb{I}_{\tau+\gamma(\mathbf{X})<0}(\mathbf{X}) \mathbb{I}_{\sup \|\mathbf{X_n}\|<\delta\sqrt{R}}(\mathbf{X}) F(\mathbf{X}) P_z^0(d\mathbf{X}) - \\ - \int_{\partial\Lambda_R} \sigma_R(dr) \int_{\delta\sqrt{R}}^\infty \prod_{i=1}^{\nu-1} (1-\tau k_i(r|R))\, d\tau \cdot \\ \cdot \int_{\mathcal{X}^0} \mathbb{I}_{\tau+\gamma(\mathbf{X})<0}(\mathbf{X}) \mathbb{I}_{\sup \|\mathbf{X_n}\|<\delta\sqrt{R}}(\mathbf{X}) F(\mathbf{X}) P_z^0(d\mathbf{X}) \equiv \\ \equiv I_1^A(R,z) + \hat{I}^A(R,z) \end{aligned} \tag{5.15}$$

Let us show that

$$\left|\hat{I}^A(R,z)\right| \leq C \exp\left(-C(\beta,z)\delta\sqrt{R}\right) \tag{5.16}$$

where $C = C(\nu,\beta,z,\Lambda,F,\delta)$ does not depend on R. >From (5.10) it follows that

$$\prod_{i=1}^{\nu-1} (1-\tau k_i(r|R)) = \sum_{s=0}^{\nu-1} \tau^s a_s(r|R) = \sum_{s=0}^{\nu-1} R^{-s}\tau^s a_s(R^{-1}r|1), \tag{5.17}$$

where $a_0(r|R) = 1$,

$$a_s(r|R) = (-1)^s \sum_{1\leq i_1<\cdots<i_s\leq\nu-1} k_{i_1}(r|R)\cdots k_{i_s}(r|R), \quad s = 1,\cdots,\nu-1.$$

Hence

$$\begin{aligned} \left|\hat{I}^A(R,z)\right| \leq \sum_{s=0}^{\nu-1} \int_{\partial\Lambda_R} |a_s(r|R)| \sigma_R(dr) \int_{\delta\sqrt{R}}^\infty \tau^s\, d\tau \\ \cdot \int_{\mathcal{X}^0} \mathbb{I}_{\tau+\gamma(\mathbf{X})<0}(\mathbf{X}) \mathbb{I}_{\sup \|\mathbf{X_n}\|<\delta\sqrt{R}}(\mathbf{X}) |F(\mathbf{X})| P_z^0(d\mathbf{X}). \end{aligned}$$

Now with the help of (5.9) and the condition that $f_{r,R}$ is of class C^3 one can easily obtain that

$$f_{r,R}(\xi) = R^{-1}\frac{1}{2}\sum_{s=0}^{\nu-1} k_i(R^{-1}r|1)\xi_i^2 + R^{-2}\epsilon_{r,R}(\xi), \ \|\xi\| < \delta\sqrt{R} \tag{5.18}$$

where

$$|\epsilon_{r,R}(\xi)| \leq C(\nu)C(\Lambda)\|\xi\|^3 \tag{5.19}$$

uniformly in $r \in \partial\Lambda_R$ and $R \geq 1$. This implies that for all ξ, $\|\xi\| < \delta R$ and R large enough

$$|f_{r,R}(\xi)| \leq \overline{k}\delta^2.$$

Using the fact that $\sup_t \|\mathbf{X}(t)\| > \tau - \overline{k}\delta^2$ for any loop $\mathbf{X}$ starting at the point $r + \tau\mathbf{n}$ with $\tau > \delta\sqrt{R}$ and such that $\tau + \gamma(\mathbf{X}) < 0$, we can write:

$$\int\limits_{\mathcal{X}^0} \mathbb{I}_{\tau+\gamma(\mathbf{X})<0}(\mathbf{X})\mathbb{I}_{\sup \|\mathbf{X_n}\|<\sigma\sqrt{R}}(\mathbf{X})|F(\mathbf{X})|P_z^0(d\mathbf{X})$$

$$\leq \int\limits_{\mathcal{X}^0} \mathbb{I}_{\sup \|\mathbf{X_n}\|>\tau-\overline{k}\delta^2}(\mathbf{X})|F(\mathbf{X})|P_z^0(d\mathbf{X})$$

$$\leq \|F\|_{L_2}\left[P_{+,z}^0\left(\sup \|\mathbf{X_n}\| > \tau - \overline{k}\delta^2\right)\right]^{1/2}$$

$$\leq C(\nu)\beta^{-\nu/4}\|F\|_{L_2}\exp[C(\beta,z)\overline{k}\delta^2]\exp[-C(\beta,z)\tau].$$

Hence

$$|\hat{I}^A(R,z)| \leq C(\nu,\beta,z,\overline{k},\delta)\|F\|_{L_2}\sum_{s=0}^{\nu-1}\int\limits_{\partial\Lambda_R} a_s(r|R)\sigma_R(dr)\int\limits_{\sigma\sqrt{R}}^{\infty}\tau^s\exp[-C(\beta,z)\tau]d\tau$$

$$\leq C(\nu,\beta,z,\Lambda,\delta)\|F\|_{L_2}\exp[-C(\beta,z)\delta\sqrt{R}], \tag{5.20}$$

which proves the formula (5.16).

Hence combining the formulas (5.1), (5.4), (5.6)-(5.8), (5.13) and (5.16) we find that

$$I_1(R,z) = \int\limits_{\partial\Lambda_R}\sigma_R(dr)\int\limits_0^{\infty}\prod_{i=1}^{\nu-1}(1-\tau k_i(r|R))\,d\tau\,\cdot$$

$$\cdot\int\limits_{\mathcal{X}^0}\mathbb{I}_{\tau+\gamma(\mathbf{X})<0}(\mathbf{X})\mathbb{I}_{\sup\|\mathbf{X_n}\|<\delta\sqrt{R}}(\mathbf{X})F(\mathbf{X})P_z^0(d\mathbf{X}) + O(e^{-C\sqrt{R}}).$$

Applying Fubini's theorem and formula (5.17) to the last integral we have

$$I_1(R,z) = \sum_{s=0}^{\nu-1}\int\limits_{\partial\Lambda_R} a_s(r|R)\sigma_R(dr)\int\limits_{\mathcal{X}^0}\mathbb{I}_{\sup\|\mathbf{X_n}\|<\delta\sqrt{R}}(\mathbf{X})F(\mathbf{X})P_z^0(d\mathbf{X})\,\cdot$$

$$\cdot\int\limits_0^{-\gamma(\mathbf{X})}\tau^s d\tau + O(e^{-C\sqrt{R}})$$

or

$$I_1(R,z) = \sum_{s=0}^{\nu-1} L_s(z,R) + O(e^{-C\sqrt{R}}). \tag{5.21}$$

with

$$L_s(z,R) = \frac{1}{s+1}\int\limits_{\partial\Lambda_R} a_s(r|R)\sigma_R(dr)\,\cdot \tag{5.22}$$

$$\cdot\int\limits_{\mathcal{X}^0}\mathbb{I}_{\sup\|\mathbf{X_n}\|<\delta\sqrt{R}}(\mathbf{X})F(\mathbf{X})\left(-\gamma(\mathbf{X})\right)^{s+1}P_z^0(d\mathbf{X}).$$

Let $e_1,\ldots,e_{\nu-1}$ be unit vectors drawn along the directions of the principal curvatures of $\partial\Lambda_R$ at the point $r \in \partial\Lambda_R$. For each $\mathbf{X} \in \mathcal{X}^0$, with $\sup\|\mathbf{X}\| < \delta R$, we choose

$t_{\mathbf{n}} = t_{\mathbf{n}}(\mathbf{X})$ and $t_R = t_R(\mathbf{X})$ from the interval $[0, |\mathbf{X}|\beta]$ so that $\mathbf{X}_{\mathbf{n}}(t_{\mathbf{n}}) = \inf_t \mathbf{X}_{\mathbf{n}}(t)$ and

$$\mathbf{X}_{\mathbf{n}}(t_R) - f_{r,R}(\mathbf{X}_T(t_R)) = \inf_t(\mathbf{X}_{\mathbf{n}}(t) - f_{r,R}(\mathbf{X}_T(t))).$$

By Proposition 1 from Appendix 1 $t_{\mathbf{n}}$ is P_z^0-almost surely unique and by Proposition 2 from Appendix 2 $t_{\mathbf{n}} \to t_R$, as $R \to \infty$, P_z^0-almost surely for all z, $0 < z \leq 1$.

Let us show that the following representation of $\gamma(\mathbf{X})$ is valid:

$$-\gamma(\mathbf{X}) = -\mathbf{X}_{\mathbf{n}}(t_{\mathbf{n}}) + \frac{R^{-1}}{2} \sum_{i=1}^{\nu-1} k_i(R^{-1}r|1) \langle \mathbf{X}_T(t_{\mathbf{n}}), e_i \rangle^2 + R^{-1} \tilde{\epsilon}_{r,R}(\mathbf{X}), \tag{5.23}$$

where

$$|\tilde{\epsilon}_{r,R}(\mathbf{X})| \leq C(\nu, \beta) \left\{ \sum_{i=1}^{\nu-1} \left(\langle \mathbf{X}_T(t_R), e_i \rangle^2 - \langle \mathbf{X}_T(t_{\mathbf{n}}), e_i \rangle^2 \right) + R^{-1} \|\mathbf{X}\|^3 \right\}. \tag{5.24}$$

Note that from Proposition 2 and the Lebesgue dominant convergence theorem it follows that

$$\int_{\partial\Lambda_R} \sigma_R(dr) \int_{\mathcal{X}^0} \tilde{\epsilon}_{r,R}(\mathbf{X}) P_z^0(d\mathbf{X}) = o\left(R^{\nu-1}\right), \quad \text{as } R \to \infty. \tag{5.25}$$

Let us prove (5.23). We have that

$$-\gamma(\mathbf{X}) = f_{r,R}(\mathbf{X}_T(t_{\mathbf{n}})) - \mathbf{X}_{\mathbf{n}}(t_{\mathbf{n}}) + \Delta\left(\mathbf{X}|r, R\right),$$

where

$$0 \leq \Delta\left(\mathbf{X}|r, R\right) = f_{r,R}(\mathbf{X}_T(t_R)) - \mathbf{X}_{\mathbf{n}}(t_R) - f_{r,R}(\mathbf{X}_T(t_{\mathbf{n}})) + \mathbf{X}_{\mathbf{n}}(t_{\mathbf{n}}).$$

Using 5.18 we find that

$$\Delta\left(\mathbf{X}|r, R\right) \leq f_{r,R}(\mathbf{X}_T(t_R)) - f_{r,R}(\mathbf{X}_T(t_{\mathbf{n}})) = \frac{R^{-1}}{2} \sum_{i=1}^{\nu-1} k_i\left(R^{-1}r|1\right) \cdot$$

$$\cdot \left[\langle \mathbf{X}_T(t_R), e_i \rangle^2 - \langle \mathbf{X}_T(t_{\mathbf{n}}), e_i \rangle^2 \right] + R^{-2}\left[\epsilon_{r,R}(\mathbf{X}_T(t_R)) - \epsilon_{r,R}(\mathbf{X}_T(t_{\mathbf{n}}))\right].$$

This according to 5.19 and Proposition 2 implies (5.23) and (5.24).

With the help of (5.23) we can treat the terms $L_s(z, R)$ from (5.22). Consider $L_0(z, R)$. We have that

$$\begin{aligned} L_0(z, R) &= R^{\nu-1} \int_{\partial\Lambda} \sigma(dr) \int_{\mathcal{X}^0} F(\mathbf{X}) \Big(-\mathbf{X}_{\mathbf{n}}(t_{\mathbf{n}}) + \tfrac{R^{-1}}{2} \sum_{i=1}^{\nu-1} k_i(r|1) \cdot \\ &\quad \cdot \langle \mathbf{X}_T(t_{\mathbf{n}}), e_i \rangle^2 + R^{-1} \tilde{\epsilon}_{r,R}(\mathbf{X}) \Big) P_z^0(d\mathbf{X}) = \\ &= -R^{\nu-1} \int_{\partial\Lambda} \sigma(dr) \int_{\mathcal{X}^0} F(\mathbf{X}) \inf \mathbf{X}_{\mathbf{n}} P_z^0(d\mathbf{X}) + \\ &\quad + \tfrac{R^{\nu-2}}{2} \int_{\partial\Lambda} \sigma(dr) \int_{\mathcal{X}^0} \sum_{i=1}^{\nu-1} k_i(r|1) F(\mathbf{X}) \langle \mathbf{X}_T(t_{\mathbf{n}}), e_i \rangle^2 P_z^0(d\mathbf{X}) + o(R^{\nu-2}) \end{aligned} \tag{5.26}$$

In a similar way, according to (5.17),

$$L_1(z,R) = -\tfrac{1}{2}R^{\nu-2}\int_{\partial\Lambda}\sum_{i=1}^{\nu-1} k_i(r|1)\sigma(dr)\cdot \cdot\int_{\mathcal{X}^0} F(\mathbf{X})\mathbf{X}^2_{\mathbf{n}}(t_{\mathbf{n}})P^0_z(d\mathbf{X}) + O(R^{\nu-3}). \tag{5.27}$$

It is easy to check that

$$\sum_{s=2}^{\nu-1} L_s(z,R) = O(R^{\nu-3}). \tag{5.28}$$

Indeed for R large enough $|\gamma(\mathbf{X})|^s \le C\sup\|\mathbf{X}\|^s$.

By Lemma 1 from [6] it is easy to check that $\sup\|\mathbf{X}\|^s \in L_1(\mathcal{X}^0, P^0_z)$. Therefore

$$\sum_{s=2}^{\nu-1}|L_s(z,R)| \le \sum_{s=2}^{\nu-1}\tfrac{R^{\nu-1}}{s+1}\int_{\partial\Lambda} R^{-s}a_s(r|1)\sigma(dr)\cdot \cdot\int_{\mathcal{X}^0} F(\mathbf{X})\sup\|\mathbf{X}\|^{s+1}P^0_z(d\mathbf{X}) = O(R^{\nu-3}).$$

Now from (5.21), (5.26)-(5.28) it follows that

$$I_1(R,z) = R^{\nu-1}a_1(\Lambda,F,z) + R^{\nu-2}a_2(\Lambda,F,z) + o(R^{\nu-2})$$

where

$$a_1 = -\int_{\partial\Lambda}\sigma(dr)\int_{\mathcal{X}^0} F(\mathbf{X})\inf\mathbf{X}_{\mathbf{n}}P^0_z(d\mathbf{X}), \tag{5.29}$$

$$a_2 = \frac{1}{2}\int_{\partial\Lambda}\sigma(dr)\int_{\mathcal{X}^0} F(\mathbf{X})\sum_{i=1}^{\nu-1}k_i(r|1)\left[\langle\mathbf{X}_T(t_{\mathbf{n}}),e_i\rangle^2 - \mathbf{X}^2_{\mathbf{n}}(t_{\mathbf{n}})\right]P^0_z(d\mathbf{X}) \tag{5.30}$$

This together with (5.1) completes the proof of Theorem 1.

Now suppose that the function $F(\mathbf{X})$ is in addition rotation invariant. Then the integral

$$\int_{\mathbf{X}^0} F(\mathbf{X})\inf\mathbf{X}_{\mathbf{n}}P^0_z(d\mathbf{X})$$

does not depend on the orientation of the unit normal $\mathbf{n}$ in $\mathbb{R}^\nu$, because the measure P^0_z also is rotation invariant. Hence a_1 takes a simple form:

$$a_1 = |\partial\Lambda|\overline{a}_1(F,z)$$

with

$$\overline{a}_1(F,z) = -\int_{\mathbf{X}^0} F(\mathbf{X})\inf\langle\mathbf{X},\mathbf{d}_1\rangle P^0_z(d\mathbf{X}) \tag{5.31}$$

where $\mathbf{d}_1$ is any fixed unit vector in $\mathbb{R}^\nu$. In the same way

$$a_2 = \frac{1}{2}\int_{\partial\Lambda}\sum_{i=1}^{\nu-1}k_i(r|1)\sigma(dr)\int_{\mathcal{X}^0} F(\mathbf{X})\left[\left\langle\mathbf{X}_T(\overline{t}),\mathbf{d}_2\right\rangle^2 - \left\langle\mathbf{X}_{\mathbf{n}}(\overline{t}),\mathbf{d}_1\right\rangle^2\right]P^0_z(d\mathbf{X}),$$

or

$$a_2 = \int_{\partial\Lambda} H_\Lambda(r)\sigma(dr)\overline{a}_2(F,z),$$

where

$$H_\Lambda(r) = \frac{1}{\nu-1}\sum_{i=1}^{\nu-1} k_i(r|1)$$

is the mean curvature of $\partial\Lambda$ at the point r and

$$\overline{a}_2(F,z) = \frac{\nu-1}{2}\int_{\mathcal{X}^0} F(\mathbf{X})\left[\left\langle \mathbf{X}_T(\overline{t}),\mathbf{d}_2\right\rangle^2 - \left\langle \mathbf{X}(\overline{t}),\mathbf{d}_1\right\rangle^2\right] P_z^0(d\mathbf{X}). \tag{5.32}$$

Here $\mathbf{d}_1,\mathbf{d}_2$ is an arbitrary fixed pair of orthogonal unit vectors in $\mathbb{R}^\nu$ and $\overline{t}$ is defined by $\left\langle \mathbf{X}_\mathbf{n}(\overline{t}),\mathbf{d}_1\right\rangle = \inf\left\langle \mathbf{X}_\mathbf{n}(t),\mathbf{d}_1\right\rangle$. Theorem 2 is proved.

6. PROOF OF THEOREM 3

Let g be the Ursell function given by the formula:

$$g(\omega) = \prod_{\mathbf{X}\in\omega} e^{-U_1(\mathbf{X})} \sum_{\gamma\in\Gamma_{con}(\omega)} \prod_{[\mathbf{X},\mathbf{X}]\in E(\gamma)} \left(e^{-U_2(\mathbf{X},\mathbf{X})} - 1\right), \tag{6.1}$$

where $\gamma\in\Gamma_{con}(\omega)$ is the set of all connected graphs consructed on ω, $E(\gamma)$ is the set of edges of the graph γ.

To develop the large volume asymptotics of the log-partition function $\log\Xi(\Lambda_R,z)$ of the Ginibre gas with interaction we use the cluster representation $\log\Xi(\Lambda_R,z)$ in terms of the Ursell function:

$$\log\Xi(\Lambda_R,z) = \int_{\mathcal{M}(\Lambda_R)} g(\omega)W_{z,\Lambda_R}(d\omega)$$

(See for details [6]). It follows from Corollary 3 and formulas (12) and (32) in [6] that the Ursell function $g\in L^1(\mathcal{M}(\Lambda_R),W_{z,\Lambda})$, $R\geq 1$, for all z from the intervall (4.1).

An appplication of formula (4) from [6] gives

$$\log\Xi(\Lambda_R,z) = \int_{\mathcal{X}(\Lambda_R)} \rho_z(d\mathbf{X}) \int_{\mathcal{M}(\Lambda_R)} g_1(\mathbf{X},\omega)W_{z,\Lambda_R}(d\omega)$$

where $g_1(\omega) = \frac{g(\omega)}{|\omega|}$, $\omega\in\mathcal{M}\backslash\{\emptyset\}$. This implies

$$\log\Xi(\Lambda_R,z) = A_0(R,z) - A_1(R,z), \tag{6.2}$$

where

$$A_0(R,z) = \int_{\mathcal{X}(\Lambda_R)} G_z(\mathbf{X})\rho_z(d\mathbf{X}),$$

$$A_1(R,z) = \int_{\mathcal{X}(\Lambda_R)} \rho_z(d\mathbf{X}) \int_{\mathcal{M}^c(\Lambda_R)} g_1(\mathbf{X},\omega)W_{\rho_z}(d\omega),$$

$$G_z(\mathbf{X}) = \int_{\mathcal{M}} g_1(\mathbf{X},\omega)W_{\rho_z}(d\omega).$$

Note that G_z is translation invariant function: $G_z(\mathbf{X}+u)=G_z(\mathbf{X})$, for any $u\in\mathbb{R}^\nu$ and $\mathbf{X}\in\mathcal{M}$. This follows from the translation invariance of the Ursell function and the measure W_{ρ_z}. By [6], Lemma 4, $G_z\in L^2(\mathcal{X}^0,P_z^0)$ for all z from the intervall (4.1). According to Theorem 1

$$A_0(R,z)=R^\nu|\Lambda|a_0(G_z)+R^{\nu-1}a_1(\Lambda,G_z)+R^{\nu-2}a_2(\Lambda,G_z)+o(R^{\nu-2}).$$

Now consider $A_1(R,z)$. We will show below that $A_1(R,z)=O(R^{\nu-1})$. Similarly to (5.4) we decompose A_1 as:

$$A_1(R,z)=A_2(R,z)+A_2'(R,z), \tag{6.3}$$

where

$$A_2(R,z)=\int\limits_{\Lambda_{R,\delta}}du\int\limits_{\mathcal{X}^u}\mathbb{I}_{\mathcal{X}(\Lambda_R)}(\mathbf{X})P_z^u d(\mathbf{X})\int\limits_{\mathcal{M}^c(\Lambda_R)}g_1(\mathbf{X},\omega)W_{\rho_z}(d\omega),$$

$$A_2'(R,z)=\int\limits_{\Lambda_R\setminus\Lambda_{R,\delta}}du\int\limits_{\mathcal{X}^u}\mathbb{I}_{\mathcal{X}(\Lambda_R)}(\mathbf{X})P_z^u d(\mathbf{X})\int\limits_{\mathcal{M}^c(\Lambda_R)}g_1(\mathbf{X},\omega)W_{\rho_z}(d\omega).$$

Applying Corollary 1 from [6] we find that

$$\begin{aligned}|A_2'(R,z)|&\le\int\limits_{\Lambda_R\setminus\Lambda_{R,\delta}}du\int\limits_{\mathcal{X}^u}P_z^u d(\mathbf{X})\int\limits_{\mathcal{M}^c(B_u(\delta R)}|g_1(\mathbf{X},\omega)|\,W_{\rho_z}(d\omega)\le\\ &\le C\,(1+\delta R)^{-l}\,R^\nu|\Lambda|=O(R^{\nu-1}),\end{aligned}\tag{6.4}$$

where $C=C\,(\Phi,\beta,\nu,z,l)>0$ and $B_u(R)$ is a ball in $\mathbb{R}^\nu$ of radius R centered at $u\in\mathbb{R}^\nu$.

Consider $A_2(R,z)$. Using the local coordinate system we can write

$$\begin{aligned}A_2(R,z)&=\int\limits_{\partial\Lambda_R}\sigma_R(dr)\int\limits_0^{\delta R}\prod_{i=1}^{\nu-1}(1-tk_i(r|R))\,dt\,\cdot\\ &\cdot\int\limits_{\mathcal{X}^0}\mathbb{I}_{\mathcal{X}(\Lambda_R)}(\mathbf{X}^0+r+t\mathbf{n})P_z^0(d\mathbf{X}^0)\int\limits_{\mathcal{M}^c(\Lambda_R)}g_1(\mathbf{X}^0+r+t\mathbf{n},\omega)W_{\rho_z}(d\omega).\end{aligned}\tag{6.5}$$

Again applying Corollary 1 from [6] we have that

$$\begin{aligned}|A_2(R,z)|&\le 2^{\nu-1}\int\limits_{\partial\Lambda_R}\sigma_R(dr)\int\limits_0^{\delta R}dt\int\limits_{\mathcal{X}^0}P_z^0(d\mathbf{X}^0)\,\cdot\\ &\cdot\int\limits_{\mathcal{M}^c(B_{r+t\mathbf{n}}(t))}\left|g_1(\mathbf{X}^0+r+t\mathbf{n},\omega)\right|W_{\rho_z}(d\omega)\le\\ &\le C\,(\Phi,\beta,\nu,z,l)\int\limits_{\partial\Lambda_R}\sigma_R(dr)\int\limits_0^\infty(1+t)^{-l}\,dt=O(R^{\nu-1}).\end{aligned}\tag{6.6}$$

This completes the proof of Theorem 3.

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7. APPENDIX

Proposition 1. *The time $\underline{t}$ at which one dimensional composite Brownian loop attains its infimum is P_z^0-almost surely unique for all $z: 0 < z \leq 1$.*

Proof: Let $\mathcal{X}^0$ be the space of all one dimensional composite Brownian loops. Let

$$T(\mathbf{X}) = \left\{ t \in [0, |\mathbf{X}|\beta] | \mathbf{X}(t) = \inf_s \mathbf{X}(s) \right\}.$$

We need to show that

$$P_z^0\{\mathbf{X} \in \mathcal{X} | card T(\mathbf{X}) > 1\} = 0.$$

Let $\overline{h}(\mathbf{X}) = \sup T(\mathbf{X})$ and $\underline{h}(\mathbf{X}) = \inf T(\mathbf{X})$, $\mathbf{X} \in \mathcal{X}^0$. For each $\mathbf{X} \in \mathcal{X}^0$ let $\hat{\mathbf{X}} \in \mathcal{X}^0$ be defined by $\hat{\mathbf{X}}(t) = \mathbf{X}(j\beta - t)$ if $\mathbf{X} \in \mathcal{X}^0_{j\beta}$. Evidently $\hat{} : \ \mathcal{X}^0 \to \mathcal{X}^0$ is one to one mapping which preserves the measure $P^0_{j\beta}$, $j = 1, 2, \ldots$, on each $\mathcal{X}_{j\beta}$. Therefore $\hat{\mathbf{X}}$ preserves the measure P_z^0. Taking into account that $\overline{h}(\hat{\mathbf{X}}) = \underline{h}(\mathbf{X})$ we have that

$$\int_{\mathcal{X}^0} \overline{h}(\mathbf{X}) P_z^0(d\mathbf{X}) = \int_{\mathcal{X}^0} \overline{h}(\hat{\mathbf{X}}) P_z^0(d\hat{\mathbf{X}}) = \int_{\mathcal{X}^0} \underline{h}(\mathbf{X}) P_z^0(d\mathbf{X})$$

Thus $\overline{h} - \underline{h} \geq 0$ with

$$\int_{\mathcal{X}^0} (\overline{h}(\mathbf{X}) - \underline{h}(\mathbf{X})) P_z^0(d\mathbf{X}) = 0$$

which implies that $P_z^0\{\mathbf{X} \in \mathcal{X}^0 | card T(\mathbf{X}) > 1\} = 0$.

Proposition 2. *For each $\mathbf{X} \in \mathcal{X}^0$, and all z, $0 < z \leq 1$, $t_R(\mathbf{X}) \to t_{\mathbf{n}}(\mathbf{X})$, as $R \to \infty$, P_z^0-almost surely.*

Proof: It is sufficient to show that

$$|\mathbf{X_n}(t_R) - \mathbf{X_n}(t_{\mathbf{n}})| \to 0, \quad \text{as} \quad R \to \infty, \tag{7.1}$$

for each $\mathbf{X} \in \mathcal{X}^0$. Indeed, if $\overline{\tau}(\mathbf{X})$ is a limiting point for the set $\{t_R(\mathbf{X}),\ R \geq 1\}$ then (7.1) implies that $\langle \mathbf{X} \cdot \mathbf{n} \rangle(\overline{\tau}) = \langle \mathbf{X} \cdot \mathbf{n} \rangle(t_{\mathbf{n}}) = \inf \langle \mathbf{X} \cdot \mathbf{n} \rangle$ and by Proposition 1 $\overline{\tau}(\mathbf{X}) = t_{\mathbf{n}}(\mathbf{X})$ P_z^0-almost surely.

Let us prove (7.1). By definitions of t_R and $t_{\mathbf{n}}$

$$\inf_t (\mathbf{X}(t_{\mathbf{n}}) - f_{r,R}(\mathbf{X}_T(t))) - \inf_t \mathbf{X_n}(t) \leq \mathbf{X_n}(t_{\mathbf{n}}) - f_{r,R}(\mathbf{X}_T(t_{\mathbf{n}})) - \mathbf{X_n}(t_{\mathbf{n}})$$

which implies

$$\mathbf{X_n}(t_R) - f_{r,R}(\mathbf{X}_T(t_R)) - \mathbf{X_n}(t_{\mathbf{n}}) \leq -f_{r,R}(\mathbf{X}_T(t_{\mathbf{n}})),$$

which together with the bound

$$|f_{r,R}(\xi)| \leq CR^{-1}\|\xi\|^2,$$

(see (5.18)) gives

$$0 \leq \mathbf{X_n}(t_R) - \mathbf{X_n}(t_{\mathbf{n}}) \leq 2CR^{-1}\|\mathbf{X}\|^2.$$

Formula (7.1) is proved.

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