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Dedicated to the 65th Birthday of Professor R. V. Ambartzumian

LENGTH DISTRIBUTIONS OF EDGES IN PLANAR STATIONARY AND ISOTROPIC STIT TESSELLATIONS

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Stationary and isotropic random tessellations of the euclidean plane are studied which have the characteristic property to be stable with respect to iteration (or nesting), STIT for short. Since their cells are not in a face-to-face position, three different types of linear segments appear. For all the types the distribution of the length of the typical segment is given.

§1. INTRODUCTION

In this paper we study the length distributions of edges of a certain class of random planar tessellations – the STIT tessellations. These tessellations have the characteristic property to be **St**able with respect to **It**eration (also referred to as nesting) of tessellations.

The mathematical motivation for these tessellations goes back to a problem that was posed to two of the authors by R. V. Ambartzumian already in the 80-th. Also, we first learned from him the idea of the operation of iteration for tessellations.

The iteration generates a new tessellation $\mathcal{I}(Y^0, \mathcal{Y})$ from a 'frame' tessellation Y^0 and a sequence $\mathcal{Y} = \{Y^1, Y^2, ...\}$ of independent identically distributed (i.i.d.) tessellations by subdividing the i-th cell p_i of Y^0 by intersecting it with the cells of Y^i, $i = 1, 2, ...$ E.g., this operation can be applied to Poisson line tessellations in the plane and it results in another tessellation. This operation of iteration can be applied repeatedly, combined with an appropriate rescaling. The problem arises whether there exists a limit tessellation when the number of repetitions goes to infinity. A further question is how such limit tessellations can be described if they exist.

The existence of such tessellations was recently shown in [11], and their construction within bounded windows was described. They are STIT tessellations.

In the present paper we deal with these STIT tessellations without regarding the above mentioned process of repeated rescaled iteration.

Fig. 1. Simulation of a stationary and isotropic STIT tessellation (provided by J. Ohser).

An important feature of stationary STIT tessellations is that the interior of the typical cell – i.e. a random convex polygon – has the same distribution as the typical cell of a Poisson line tessellation. Hence, several results can easily be derived for stationary STIT tessellations.

STIT tessellations have T-shaped nodes only, and their cells are not necessarily in a face-to-face position. Therefore, when speaking about edges or linear segments of these tessellations, it is appropriate to apply Miles' classification which is introduced in [2], and to consider I-, J- and K-segments.

§2. STIT TESSELLATIONS

The tessellations are assumed to be stationary and isotropic which means the invariance of their distribution under translation and rotation. For an exact definition see [7], [8] or [13].

In the paper we consider random tessellations of the two-dimensional Euclidean space $\mathbb{R}^2$, i.e. random locally finite partitions of the plane into polytopes (compact convex polygons). These polygons are referred to as the cells of the tessellation. The vertices of the cells are called the nodes.

There are different ways to describe a random tessellation. We will write Y for the random closed set (RACS) of all boundary points of the cells, and $C(Y) = \{p_i, i = 1, 2, ...\}$ for the process that describes the cells p_i. Since Y is considered as a RACS its distribution is uniquely determined by its capacity functional.

For tessellations, the operation of iteration (also referred to as nesting) is defined as follows. Let $Y^1, Y^2, ...$ be a sequence of i.i.d. stationary tessellations in $\mathbb{R}^2$ and denote $\mathcal{Y} = \{Y^1, Y^2, ...\}$. Further assume that Y^0 is a stationary tessellation which is independent of $\mathcal{Y}$. Assume that the cells of Y^0 are numbered and that $C(Y^0) = \{p_1, p_2, ...\}$.

The iteration of the tessellation Y^0 and the sequence $\mathcal{Y}$ is defined as the tessellation

$$\mathcal{I}(Y^0, \mathcal{Y}) = Y^0 \cup \bigcup_{p_i \in C(Y^0)} (Y^i \cap p_i). \tag{1}$$

This formula describes the operation in terms of the boundaries of the cells. For the cells themselves it means that the cells p_i of the so called 'frame' tessellation Y^0 are independently subdivided by the cells p_{ik}, $k = 1, 2, ...$ (or their faces respectively) of the tessellations Y^i which intersect the interior of p_i, i.e. the new cells are of the type $p_i \cap p_{ik}$.

A list of references concerning iteration was given in [9].

For a real number $r > 0$ the homothetic tessellation rY is generated by transforming all points $x \in Y$ into rx. Accordingly, $r\mathcal{Y}$ means that this transformation is applied to all tessellations of the sequence $\mathcal{Y}$. Let L_A be the mean total length per unit area, the length intensity, of any of the tessellations $Y^0, Y^1, ...$ Then $\mathcal{I}(Y^0, \mathcal{Y})$ has the length intensity $2L_A$, and $\mathcal{I}(2Y^0, 2\mathcal{Y})$ has the length intensity L_A, respectively.

Let Y^0 be a stationary tessellation and $\mathcal{Y}^1, \mathcal{Y}^2, ...$ a sequence of sequences of tessellations such that all the occurring tessellations (including Y^0) are i.i.d. Then the sequence $\mathcal{I}_2(Y^0), \mathcal{I}_3(Y^0), ...$ of rescaled iterations is defined by (see [9])

$$\begin{aligned} \mathcal{I}_2(Y^0) &= \mathcal{I}(2Y^0, 2\mathcal{Y}^1) \\ \mathcal{I}_m(Y^0) &= \mathcal{I}(mY^0, m\mathcal{Y}^1, ..., m\mathcal{Y}^{m-1}) \\ &= \mathcal{I}(\mathcal{I}(mY^0, m\mathcal{Y}^1, ..., m\mathcal{Y}^{m-2}), m\mathcal{Y}^{m-1}) \ , \qquad m = 3, 4, ... \end{aligned}$$

Here m is the rescaling factor which is chosen to keep the parameter L_A of the tessellation $\mathcal{I}_m(Y^0)$ constant for all m. We use the abbreviation $\mathcal{I}_m(Y^0)$ since it is assumed that all the other tessellations in the sequences $\mathcal{Y}^1, \mathcal{Y}^2, ...$ are independent and have the same distribution as Y^0.

The symbol $\stackrel{D}{=}$ is used for the identity relation of distributions.

Definition 1. A stationary tessellation Y is said to be stable with respect to iteration (STIT) if

$$Y \stackrel{D}{=} \mathcal{I}_m(Y) \quad \text{for} \quad \text{all} \quad m = 2, 3, \ldots ,$$

i.e. if its distribution is not changed by repeated rescaled iteration with sequences of tessellations with the same distribution.

2.1. Description of the STIT tessellations – construction in bounded windows.

Here we give a short sketch of the construction which was described in [11] in full detail, in arbitrary dimension and also for the non-isotropic case. Here we restrict it to the planar and isotropic case. Let $W \subset \mathbb{R}^2$ be a bounded rectangular window and $a > 0$ a positive real number.

The intuitive idea of the construction is the following : The window W has an exponentially distributed 'life time'. At the end of this time interval a random line is thrown onto W, which divides W into two new 'cells'. These two cells have independent and exponentially distributed life times until they are divided further by random lines. After any division, exponentially distributed life times of the new cells begin, and they are independent of all the other life times. Special attention has to be paid to the adjustment of the parameters of these exponential distributions. In the isotropic case, these parameters are proportional to the perimeter of the respective cells such that smaller cells have a stochastically longer life than larger ones. This procedure of repeated cell division is stopped at the fixed time $a > 0$ and the state at this time is interpreted as a realization of the tessellation $Y(a, W)$.

More formally, let (τ_i, γ_i), $i = 1, 2, \ldots$ be i.i.d. pairs of random variables τ_i that are exponentially distributed with parameter $a/2$, and isotropic uniform random (IUR) lines γ_i on W ; τ_i and γ_i independent. The perimeter of a convex polygon p is denoted by $U(p)$. We define a set-valued process $Y(t, W)$, $t \geq 0$, with $Y(t, W) = \emptyset$ for $0 \leq t < \tilde{\tau}_1$ with $\tilde{\tau}_1 = \tau_1/U(W)$. Then $Y(\tilde{\tau}_1, W) = W \cap \tilde{\gamma}_1$ with $\tilde{\gamma}_1 = \gamma_1$, and $Y(t, W)$ remains constant until the next update. The chord $W \cap \tilde{\gamma}_1$ divides W into two polygons, q_2 and q_3 say. Their lives start at time $\tilde{\tau}_1$ and last $\tilde{\tau}_2 = \tau_2/U(q_2)$ and $\tilde{\tau}_3 = \tau_3/U(q_3)$ respectively. Now consider a general polygon q_i that is generated by the construction at a time τ. It lives from τ to $\tau + \tilde{\tau}_i$ with $\tilde{\tau}_i = \tau_i/U(q_i)$. At the end of its life it is divided by the line $\tilde{\gamma}_i$ which has the distribution of γ_i restricted to $[q_i]$, the set of all lines that hit q_i. The update at that time is $Y(\tau + \tilde{\tau}_i, W) = \bigcup_{t < \tau + \tilde{\tau}_i} Y(t, W) \cup (q_i \cap \tilde{\gamma}_i)$.

Thus the process is updated by adding the new linear segment $q_i \cap \tilde{\gamma}_i$ whenever the lifetime of a polygon q_i is over. There arise two new polygons that are generated

from q_i by the chord $q_i \cap \tilde{\gamma}_i$, and they are then treated as described above. The state $Y(a, W)$ of the process at time a is a random tessellation in W.
Figure 2 illustrates the result of the construction with a small number of edges. A larger simulation is shown in Figure 1.

Fig. 2. Illustration of the construction of a STIT tessellation (provided by J. Ohser).

2.2. Review of essential properties.

We summarize some results for the described construction. The proofs were given in [11]. For a fixed value a the construction yields a non-degenerate tessellation in any bounded (rectangular) window W. Moreover, for any $a > 0$ there exists a random stationary tessellation $Y(a)$ in $\mathbb{R}^2$ such that $Y(a) \cap W \stackrel{D}{=} Y(a, W)$, i.e. the restriction of $Y(a)$ to the window W has the same distribution as the construction inside W as described above. These tessellations $Y(a)$, $a > 0$, are STIT. The length intensity L_A is proportional to the parameter a, and in the isotropic case we have $L_A = \pi a/2$. Even for the whole distribution of the tessellations the parameter a commutes with the homothetic transform such that

$$a\,Y(a) \stackrel{D}{=} Y(1). \tag{2}$$

Therefore we will mostly consider $Y(1)$ only, and denote it by Y.

It is intuitively clear for the construction and it can be shown strictly, that the continuation of the process from time a to time $a + b$ corresponds to an iteration of tessellations with parameter b, i.e.

$$Y(a + b) \stackrel{D}{=} \mathcal{I}(Y(a), \mathcal{Y}(b)), \quad a > 0,\ b > 0, \tag{3}$$

where $\mathcal{Y}(b) = \{Y^1(b), Y^2(b), ...\}$ is a sequence of i.i.d. tessellations, distributed as $Y(b)$.
Given the length intensity L_A and the directional distribution of the edges – in our case the uniform distribution on $[0, \pi)$ – the distribution of a stationary tessellation is uniquely determined by the STIT property.

We emphasize two further properties which we will make use of in the following.

Lemma 1. The interior of the typical cell of the stationary and isotropic STIT tessellation has the same distribution as the interior of the typical cell of the stationary and isotropic Poisson line tessellation with the same length intensity L_A.

The difference in the distributions of the two mentioned cells arises when the nodes on the boundaries are taken into account, cf. subsection 2.3.
It is easy to see that the STIT property of a tessellation $Y(a)$ transfers to intersections with lines. This yields

Lemma 2. The intersection point process that is induced by the STIT tessellation $Y(a)$ on any line is a stationary Poisson point process, and its intensity is $2\, L_A/\pi = a$.

2.3. Relations for mean values.

A variety of mean values for general stationary random tessellations was studied systematically in [6], cf. also [14] and the references given there. The properties of STIT tessellations mentioned in the previous paragraph yield several particular relations for those parameters.
The following formulas have already been published, also for non-isotropic stationary STIT tessellations in [12] and [10]. We summarize them briefly, since they provide a useful tool to check results for distributions.
As usual in the theory of tessellations, an edge is a linear segment in $Y(a)$ between two nodes and without further nodes in its relative interior. Later in this paper these edges will also be referred to as K-segments. We use the following notation for mean values.

L_A — mean total edge length per unit area, edge length intensity ;
L^K — mean length of the typical edge (i.e. K-segment) ;
U_2, A_2 — mean perimeter and mean area, respectively, of the typical cell ;
N_0 — mean number of nodes per unit area ;
N_1 — mean number of edge centers per unit area ;
N_2 — mean number of cell centroids per unit area ;

$N_{01} = N_{02}$ — mean number of edges emanating from the typical node = mean number of cells which contain the typical node ;

$N_{20} = N_{21}$ — mean number of nodes = mean number of edges on the boundary of the typical cell.

We present the formulas together with well known results for Poisson line tessellations in order to compare both tessellations. We will use the upper index $\times$ to indicate that a symbol refers to the Poisson line tessellation $Y^\times$. The Poisson line tessellations have X-shaped nodes (crossings) only.

Lemma 3. If $Y(a)$ is the stationary and isotropic STIT tessellation described above and $Y^\times$ the stationary and isotropic Poisson line tessellation with the same intensity L_A then

(1). $L^K = \frac{\pi}{3L_A} = \frac{2}{3} L^{K\times}$,
(2). $U_2 = \frac{2\pi}{L_A} = U_2^\times$,
(3). $A_2 = \frac{\pi}{L_A^2} = A_2^\times$,
(4). $N_0 = \frac{2}{\pi} L_A^2 = 2N_0^\times$,
(5). $N_1 = \frac{3}{\pi} L_A^2 = \frac{3}{2} N_1^\times$,
(6). $N_2 = \frac{1}{\pi} L_A^2 = N_2^\times$,
(7). $N_{01} = N_{02} = 3, \quad N_{01}^\times = N_{02}^\times = 4$,
(8). $N_{20} = N_{21} = 6, \quad N_{20}^\times = N_{21}^\times = 4$.

2.4. I-, J- and K-segments.

For a tessellation Y where the cells are not necessarily face-to-face, it is useful to consider different types of edges, namely I-, J- and K-segments, according to [2]. A K-segment is an edge of the tessellation without any node in its relative interior. A J-segment is a one-dimensional face of a cell of Y. Thus different J-segments can overlap, e.g. $\overline{BD}$ and $\overline{AC}$ in Figure 3. Any point in Y which is not a node belongs to exactly two J-segments. An I-segment is convex and a union of collinear K-segments, that cannot be lengthened by an additional K-segment of Y. The I-segments are the chords $q_i \cap \tilde{\gamma}_i$ that appear during the construction of the STIT tessellations as described in subsection 2.1. For an illustration see Figure 1.

For the stationary and isotropic STIT tessellation $Y(a)$ denote by N^K, N^J, N^I the mean number of centers of K-, J-, I-segments respectively, per unit area. By L^K, L^J, L^I denote the mean length of the typical segments of the respective types. For the J-segments their multiplicity has to be taken into account.

Lemma 4.

(1). $N^K = N_1 = \frac{3}{\pi} L_A^2$,

Fig. 3. Illustration of different types of segments : $\overline{AE}$ is the only I-segment. $\overline{AB}$, $\overline{BD}$, $\overline{DE}$, $\overline{AC}$, $\overline{CE}$ are the J-segments. $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$ are the K-segments in this example where A, B, C, D, E are nodes of a tessellation.

(2). $N^J = 2N_1^\times = \frac{4}{\pi} L_A^2$,
(3). $N^I = \frac{1}{\pi} L_A^2$,
(4). $L^K = \frac{\pi}{3L_A}$,
(5). $L^J = L^{K\times} = \frac{\pi}{2L_A}$,
(6). $L^I = \frac{\pi}{L_A}$.

One can conclude from this lemma that in the stationary and isotropic STIT tessellation $Y(a)$ the mean number of nodes in the relative interior of the typical I-segment is 2 and for the typical J-segment it is 1/2. The mean number of J-segments on the boundary of the typical cell of Y is 4 and that one of K-segments (edges) is 6.

3. LENGTH DISTRIBUTIONS OF I- AND OF K–SEGMENTS

In this section we consider the length distributions of the different types of segments for the stationary and isotropic STIT tessellation $Y = Y(1)$ only. The results can easily be transfered to $Y(a)$ for any $a > 0$, due to (2).

As a corollary of Lemma 1 we obtain immediately that the distribution of the length of the typical J-segment is the exponential distribution with parameter $a = 1$.

3.1. The length distribution of the 'remaining I-segment'.

In order to find the length distribution of the typical I-segment, we start with a functional equation for the survival function of the length of the 'remaining I-segment'. Let the tessellation y be a realization of Y, the point $z \in y$ and s the I-segment of $y - z$ with the origin $o \in s$. The remaining I-segment $\bar{s}$ is the intersection of s with the closed upper half-plane. For stationary and isotropic STIT tessellations these objects are a.s. uniquely defined. The length of a segment s is denoted by $|s|$.

Let $Y^0, Y^1, \ldots$ be i.i.d. STIT tessellations, distributed as Y and $\mathcal{Y} = (Y^1, Y^2, \ldots)$.

Denote $Y_1 = Y^0$, $Y_2 = \mathcal{I}(Y_1, \mathcal{Y})$, $Y_3 = 2\,Y_2$ and $Y_4 = Y_2 \setminus Y_1$. All these are RACS but Y_4 is not a tessellation ; it consists of those parts of tessellations that are nested into the cells of Y_1. The length intensity of Y_2 is $2L_A = \pi$, the other RACS have length intensity L_A. The tessellation Y_1 is referred to as the 'frame' for the iteration. Any Y_i, $i = 1, ..., 4$, induces a random measure on $\mathbb{R}^2$, that corresponds to the one-dimensional Hausdorff measure, i.e. a random length measure. We make use of Palm distributions w.r.t. these measures. Denote by $\mathbf{P} = P_{Y_1,\mathcal{Y}}$ the joint distribution of Y_1 and the sequence $\mathcal{Y}$. In the integrals below, the realizations of the RACS Y_i are denoted by y_i, and λ_i denotes the length measure on $\mathbb{R}^2$ that is induced by y_i. For a Borel set $B \subset \mathbb{R}^2$,

$$\lambda_i(B) \text{... the total length of } y_i \cap B,$$

cf. [13], p.131.

Let C be a measurable set of closed subsets of $\mathbb{R}^2$, i.e. an element of the σ-algebra, sometimes referred to as the 'hitting σ-algebra' which is used for the definition of RACS, cf. [3], [14] or [13].
The value of the function $\mathbf{1}(\cdot)$ is defined as 1 if the statement in brackets is valid and 0 otherwise, i.e. it indicates whether a condition is satisfied or not. We denote the realizations of the sequence $\mathcal{Y}$ by $\mathbf{y}$.

For $i = 1, 2, 3$ we define the Palm distributions Q_i by

$$Q_i(C) = \frac{1}{L_A} \int \mathbf{P}(\mathrm{d}(y_1, \mathbf{y})) \int \lambda_i(\mathrm{d}z)\, \mathbf{1}(y_i - z \in C)\mathbf{1}(z \in [0,1]^2).$$

On the other hand, Q_4 is defined as the Palm distribution of Y_2 w.r.t. the length measure of Y_4 ; intuitively, the 'distribution of Y_2 under the condition that a point of the subset Y_4 is in the origin'. Accordingly, Q_5 is the Palm distribution of Y_2 w.r.t. the length measure of the frame Y_1, i.e. the 'distribution of Y_2 under the condition that a point of the frame is in the origin'. This can be written as

$$Q_i(C) = \frac{2}{L_A} \int \mathbf{P}(\mathrm{d}(y_1, \mathbf{y})) \int \lambda_i(\mathrm{d}z)\, \mathbf{1}(y_2 - z \in C)\mathbf{1}(z \in [0,1]^2)$$

for $i = 4, 5$ with $\lambda_5 = \lambda_1$. Notice that there is y_2 instead of y_i in the integrand.
The corresponding survival functions of the length of the remaining I-segments are denoted by $\mathcal{H}_i$, $i = 1, ..., 5$, such that

$$\mathcal{H}_i(x) = Q_i(|\bar{s}| \geq x) \quad \text{for } x > 0.$$

In particular, $\mathcal{H} = \mathcal{H}_1$ is the survival function of the length of the remaining I-segment of the stationary and isotropic STIT tessellation with $L_A = \frac{\pi}{2}$.

Lemma 5.

$$\mathcal{H}(2x) = \frac{1}{2}\mathcal{H}(x) \cdot \mathrm{e}^{-x} + \frac{1}{2}\mathcal{H}(x) \quad \text{for all } x > 0. \tag{4}$$

Proof : Obviously, for any realization $(y_1, \mathbf{y})$ of $(Y_1, \mathcal{Y})$ we have the equation $\lambda_2 = \lambda_4 + \lambda_5$. This yields

$$Q_2 = \frac{1}{2}Q_4 + \frac{1}{2}Q_5. \tag{5}$$

Hence

$$\mathcal{H}_2 = \frac{1}{2}\mathcal{H}_4 + \frac{1}{2}\mathcal{H}_5, \tag{6}$$

Since any I-segment of the frame tessellation remains an I-segment after iteration, we have

$$\mathcal{H}_5 = \mathcal{H}_1 = \mathcal{H}. \tag{7}$$

Since all the considered tessellations are STIT, we obtain $\mathcal{H}_3 = \mathcal{H}_1 = \mathcal{H}$. Thus the equation $Y_3 = 2\,Y_2$ yields

$$\mathcal{H}_2(x) = \mathcal{H}_3(2x) = \mathcal{H}(2x). \tag{8}$$

Finally, we derive an expression for $\mathcal{H}_4$, i.e. we consider the remaining I-segment $\bar{s}$ in the origin that is located in the cell p_0 of the frame Y_1 with $o \in p_0$. The definition of iteration provides $\bar{s} = \bar{u} \cap p_0$, where $\bar{u}$ is the remaining I-segment of the tessellation from the sequence $\mathcal{Y}$ that is nested into the cell p_0. Thus $|\bar{s}| = |\bar{u} \cap p_0| \stackrel{D}{=} \min\{|\bar{u}|, \xi\}$, and ξ is a random variable, exponentially distributed with parameter 1 and independent of $|\bar{u}|$. The independence is a consequence of the independence assumptions for the tessellations $Y^0, Y^1, \ldots$. The exponential distribution of ξ follows from Lemma 2, applied to the frame tessellation Y_1 and its intersection with the line through $\bar{u}$. Hence, for $x > 0$

$$\mathcal{H}_4(x) = \mathcal{P}(|\bar{u} \cap p_0| \geq x) = \mathcal{P}(\min\{|\bar{u}|, \xi\} \geq x) = \mathcal{P}(|\bar{u}| \geq x) \cdot \mathcal{P}(\xi \geq x) = \mathcal{H}(x) \cdot \mathrm{e}^{-x}$$

and thus

$$\mathcal{H}_4(x) = \mathcal{H}(x) \cdot \mathrm{e}^{-x}. \tag{9}$$

Plugging equations (7)— (9) into (6) yields (4). Lemma 5 is proved.
Since $\mathcal{H}$ is a survival function of a positive random variable the additional condition

$$\mathcal{H}(0+) = 1 \tag{10}$$

has to be satisfied where $\mathcal{H}(0+) = \lim\limits_{x \downarrow 0} \mathcal{H}(x)$.

Lemma 6. The unique solution of the functional equation (4) with the restriction (10) is

$$\mathcal{H}(x) = \frac{1-\mathrm{e}^{-x}}{x} \qquad \textit{for all } x > 0.$$

Proof : A straightforward calculation shows that the function $\mathcal{H}$ given in (11) satisfies (4) as well as (10), i.e. $\mathcal{H}(2x) = \frac{1}{2}\mathcal{H}(x)(1+\mathrm{e}^{-x})$. In order to prove the uniqueness, assume that $\mathcal{G}$ is a solution of (4), i.e. $\mathcal{G}(2x) = \frac{1}{2}\mathcal{G}(x)(1+\mathrm{e}^{-x})$ and $\mathcal{G}(0+) = 1$. This yields

$$\frac{\mathcal{G}(2x)}{\mathcal{H}(2x)} = \frac{\mathcal{G}(x)}{\mathcal{H}(x)}, \qquad \text{for } x > 0.$$

A repeated application of this equation leads to

$$\frac{\mathcal{G}(2^n x)}{\mathcal{H}(2^n x)} = \frac{\mathcal{G}(x)}{\mathcal{H}(x)}, \qquad \text{for } x > 0,\ n = 1, 2, \ldots$$

or, equivalently,

$$\frac{\mathcal{G}(t)}{\mathcal{H}(t)} = \frac{\mathcal{G}(2^{-n}t)}{\mathcal{H}(2^{-n}t)}, \qquad \text{for } t > 0.$$

With the restriction (10) we obtain

$$\lim_{n\to\infty} \frac{\mathcal{G}(2^{-n}t)}{\mathcal{H}(2^{-n}t)} = 1, \qquad \text{for all} \qquad t > 0,$$

and thus $\mathcal{G} = \mathcal{H}$. Lemma 6 is proved.

Summarizing the results of this section yields

Theorem 1. The survival function of the length of the remaining I-segment of the stationary and isotropic STIT tessellation with $L_A = \frac{\pi}{2}$ is

$$\mathcal{H}(x) = \frac{1-\mathrm{e}^{-x}}{x} \qquad \text{for all} \qquad x > 0. \tag{11}$$

3.2. Alternative representations of $\mathcal{H}$

Lemma 7. Let P_λ be the distribution of a stationary Poisson point process with intensity λ on the real line $\mathbb{R}$ and Φ be a mixed Poisson process with the distribution $\int_0^1 P_\lambda \,\mathrm{d}\lambda$. Then

$$\mathcal{H}(x) = \mathcal{P}(\Phi \cap [0, x) = \emptyset) \qquad \textit{for} \qquad x > 0.$$

Proof : A straightforward calculation yields

$$\mathcal{P}(\Phi \cap [0, x) = \emptyset) = \int_0^1 \mathrm{e}^{-\lambda x}\,\mathrm{d}\lambda = \frac{1-\mathrm{e}^{-x}}{x} = \mathcal{H}(x).$$

Lemma 7 is proved.

Lemma 8. Let η and ξ be independent random variables, η exponentially distributed with parameter 1 and ξ uniformly distributed on the interval $(0,1)$. Then the survival function of the random variable η/ξ is equal to $\mathcal{H}$.

Proof : For $x > 0$ we have

$$\mathcal{P}(\frac{\eta}{\xi} \geq x) = \mathcal{P}(\eta \geq x\,\xi) = \int_0^1 \mathcal{P}(\eta \geq x\,t)\mathrm{d}t = \int_0^1 \mathrm{e}^{-tx}\mathrm{d}t = \mathcal{H}(x).$$

Lemma 8 is proved.

3.3. The length distribution of the typical I-segment.

Now we consider the set of the centers of I-segments of a STIT tessellation Y as a point process in $\mathbb{R}^2$. This is a stationary point process and we are interested in the distribution of the length of the I-segment that has the typical point as its center. For a realization y of Y denote by φ_y the measure on $\mathbb{R}^2$ which is the sum of the Dirac measures for all centers of I-segments of y. Then the respective Palm distribution is given by

$$P^0(C) = \frac{1}{N^I}\int P(\mathrm{d}y)\int \varphi_y(\mathrm{d}z)\mathbf{1}(y - z \in C)\mathbf{1}(z \in [0,1]^2)$$

for measurable sets C of closed subsets of $\mathbb{R}^2$. A systematic approach to Palm methods for stationary random tessellations was introduced in [5].
The survival function of the length of the typical I-segment can be defined as

$$\mathcal{F}(x) = P^0(\text{'length of the I-segment in } o\text{'} \geq x) \quad \text{for} \quad x > 0.$$

Theorem 2. The survival function of the length of the typical I-segment of a stationary and isotropic STIT tessellation with $L_A = \frac{\pi}{2}$ is

$$\mathcal{F}(x) = \frac{2}{x^2}\left(1 - \mathrm{e}^{-x} - x\mathrm{e}^{-x}\right) \qquad \textit{for} \quad x > 0. \tag{12}$$

Proof : Let Q be the Palm distribution of Y with respect to the length measure induced by Y ; cf. Q_1 in subsection 3.1. Then Q can be related to P^0 by

$$L^I\,Q(C) = \int P^0(\mathrm{d}y)\int \lambda_y(\mathrm{d}a)\,\mathbf{1}(y - a \in C), \tag{13}$$

where λ_y denotes the length measure induced by the I-segment of y that contains the origin, and C a measurable set of closed subsets of $\mathbb{R}^2$. This equation can be deduced from the general theory of interrelations between Palm measures of a stationary random measure, see [4].

Denote by $\ell(y)$ the length of the I-segment of y that contains o (in Section 3.1 this length was denoted by $|s|$) and by $u(y)$ the length of the intersection of that segment with the closed upper half-plane (above denoted by $|\bar{s}|$), i.e. the remaining I-segment (defined Q-a.s. and P^0-a.s.). Then it is possible to derive from (13) that for $x > 0$

$$L^I \int Q(\mathrm{d}y)\mathbf{1}(u(y) \geq x) = \int P^0(\mathrm{d}y) \int_0^{\ell(y)} \mathrm{d}s\, \mathbf{1}(s \geq x).$$

Hence,

$$L^I\,\mathcal{H}(x) = L^I \int Q(\mathrm{d}y)\mathbf{1}(u(y) \geq x) = \int P^0(\mathrm{d}y) \int_0^{\infty} \mathrm{d}s\, \mathbf{1}(x \leq s \leq \ell(y)) =$$

$$= \int_x^{\infty} \mathrm{d}s \int P^0(\mathrm{d}y)\, \mathbf{1}(s \leq \ell(y)).$$

Since

$$\int P^0(\mathrm{d}y)\, \mathbf{1}(0 \leq s \leq \ell(y)) = \mathcal{F}(s)$$

we finally obtain

$$L^I\,\mathcal{H}(x) = \int_x^{\infty} \mathrm{d}s\, \mathcal{F}(s) \quad \text{for all} \quad x > 0.$$

According to (11), the function $\mathcal{H}$ is continuously differentiable, and this yields

$$\mathcal{F}(x) = -L^I\,\mathcal{H}'(x) \quad \text{for all} \quad x > 0. \tag{14}$$

As $\mathcal{F}$ is the survival function of a **positive** random variable it satisfies $\mathcal{F}(0+) = 1$ and hence (14) implies

$$-L^I\,\mathcal{H}'(0+) = 1 \tag{15}$$

and thus

$$\mathcal{F} = \frac{1}{\mathcal{H}'(0+)}\mathcal{H}'. \tag{16}$$

»From Theorem 1 we derive

$$\mathcal{H}'(x) = \frac{x\mathrm{e}^{-x} + \mathrm{e}^{-x} - 1}{x^2} \tag{17}$$

and

$$\mathcal{H}'(0+) = -\frac{1}{2}. \tag{18}$$

The combination of (16) — (18) yields (12).

Theorem 2 is proved.

Formulas (15) and (18) imply $L^I = 2$ what can also be deduced from Lemma 4.

3.4. Alternative representations of the length distribution.

The distribution function (12) does not belong to one of the standard types in probability theory. Therefore, it is of interest to find interpretations and relations to well-known distributions.

Lemma 9. Let P_λ be the distribution of a stationary Poisson point process with intensity λ on the real line $\mathbb{R}$ and Φ be a mixed Poisson process with the distribution $\int_0^1 P_\lambda \, \mathrm{d}\lambda$. Then the survival function of the length of the typical interval of Φ is equal to $\mathcal{F}$.

Proof : The relation (16) between the length distributions of the remaining I-segment and of the typical I-segment is essentially the same as the relation between the distribution of length of the interval from o to the next point on the right and the distribution of the length of the typical interval of a stationary point process on the line. Thus the proof follows from Lemma 7. The proof is complete.

Lemma 10. Let η and ξ be independent random variables, η exponentially distributed with parameter 1 and ξ uniformly distributed on the interval $(0,1)$. Then the survival function of the random variable $\eta/\sqrt{\xi}$ is equal to $\mathcal{F}$.

Proof : For $x > 0$ we have

$$\mathcal{P}\left(\frac{\eta}{\sqrt{\xi}} \geq x\right) = \int_0^1 \mathcal{P}(\eta \geq x\,\sqrt{t})\mathrm{d}t = \int_0^1 \mathrm{e}^{-x\sqrt{t}}\mathrm{d}t = 2\int_0^1 s\,\mathrm{e}^{-xs}\mathrm{d}s = \mathcal{F}(x).$$

Lemma 10 is proved.

3.5. The length distribution of the 'remaining K-segment'.

The length distribution of the typical K-segment can be found in an analogous manner as that one of the I-segments. Again, we start with a functional equation for the survival function of the length of the 'remaining K-segment'. Let the tessellation y be a realization of Y, the point $z \in y$ and s_K the K-segment of $y - z$ with the origin $o \in s_K$. The remaining K-segment $\bar{s}_K$ is the intersection of s_K with the closed upper half-plane. For stationary and isotropic STIT tessellations these objects are a.s. uniquely defined. We consider exactly the same Palm distributions $Q_1, ..., Q_5$ as they were introduced in subsection 3.1. The survival functions w.r.t. the remaining K-segment are now

$$\mathcal{H}_i^K(x) = Q_i(|\bar{s}_K| \geq x), \quad \text{for} \quad x > 0,$$

$i = 1, ..., 5$. In particular, $\mathcal{H}^K = \mathcal{H}_1^K$ is the survival function of the length of the remaining K-segment of the stationary and isotropic STIT tessellation with $L_A = \frac{\pi}{2}$.

Lemma 11.

$$\mathcal{H}^K(2x) = \frac{1}{2}\mathcal{H}^K(x)\cdot \mathrm{e}^{-x} + \frac{1}{2}\mathcal{H}^K(x)\cdot \mathrm{e}^{-2x} \qquad \textit{for all} \quad x > 0. \tag{19}$$

Proof : Obviously, an analogous formula to (6) follows from (5), i.e.

$$\mathcal{H}_2^K = \frac{1}{2}\mathcal{H}_4^K + \frac{1}{2}\mathcal{H}_5^K, \tag{20}$$

Also (8) transfers to the remaining K-segment,

$$\mathcal{H}_2^K(x) = \mathcal{H}_3^K(2x) = \mathcal{H}^K(2x). \tag{21}$$

Almost literally the arguments for $\mathcal{H}_4$ can be repeated, and this yields an analog to (9)

$$\mathcal{H}_4^K(x) = \mathcal{H}^K(x)\cdot \mathrm{e}^{-x}. \tag{22}$$

The difference between I- and K-segments appears in $\mathcal{H}_5^K$. A K-segment on the frame tessellation can be modified by iteration : it becomes shorter if the edges of the nested tessellations – of the RACS Y_4 in our notation – generate additional nodes on the K-segment. Consider a shifted realization $y_2 - z$ of the tessellation Y_2 with $z \in y_1$, i.e. on the frame. Let $\bar{u}_K$ denote the remaining K-segment w.r.t. y_1. Assume that in the two cells (of the frame tessellation) which are adjacent to $\bar{u}_K$ the tessellations Y^i and Y^j are nested with the realizations y^i and y^j, respectively. Each of them generates the realization of a stationary Poisson point process of intensity 1 on the line through $\bar{u}_K$. Independence implies that their superposition is a stationary Poisson point process of intensity 2. By definition $\bar{s}_K$ is either the segment from o to the next point of the mentioned Poisson process, if there is such a point on $\bar{u}_K$ or, otherwise $\bar{u}_K$ itself. Thus $|\bar{s}_K| \stackrel{D}{=} \min\{|\bar{u}_K|, \xi_K\}$, and ξ_K is a random variable, exponentially distributed with parameter 2 and independent of $|\bar{u}_K|$. The independence is a consequence of the independence assumptions for the tessellations $Y^0, Y^1, \ldots$. Hence, for $x > 0$ analogous calculations as for $\mathcal{H}_4$ yield

$$\mathcal{H}_5^K(x) = \mathcal{H}^K(x)\cdot \mathrm{e}^{-2x}. \tag{23}$$

Plugging equations (23), (21), (22) into (20) yields (19). The proof is complete.

Theorem 3. The unique solution of the functional equation (19) with the restriction $\mathcal{H}^K(0+) = 1$, and hence the survival function of the length of the typical K-segment, is

$$\mathcal{H}^K(x) = \frac{1-\mathrm{e}^{-x}}{x}\cdot \mathrm{e}^{-x} \qquad \textit{for all} \quad x > 0. \tag{24}$$

Proof : For $x > 0$ denote $\mathcal{U}(x) = \mathcal{H}^K(x) \cdot \mathrm{e}^x$. Then

$$\mathcal{U}(2x) = \mathcal{H}^K(2x) \cdot \mathrm{e}^{2x} = \frac{1}{2}\mathcal{H}^K(x) \cdot \mathrm{e}^x + \frac{1}{2}\mathcal{H}^K(x)$$

and hence

$$\mathcal{U}(2x) = \frac{1}{2}\mathcal{U}(x) + \frac{1}{2}\mathcal{U}(x) \cdot \mathrm{e}^{-x}.$$

This is the same functional equation as (4), and also $\mathcal{U}(0+) = 1$. Thus Lemma 6 yields the result. The proof is complete.

There is a simple relation between the survival functions of the remaining K- and I-segments, namely

$$\mathcal{H}^K(x) = \mathcal{H}(x) \cdot \mathrm{e}^{-x} \quad \text{for all} \quad x > 0.$$

Since the length of the remaining J-segments is exponentially distributed with parameter 1, we obtain

$$\mathcal{H}^K(x) = \mathcal{H}^I(x) \cdot \mathcal{H}^J(x) \quad \text{for all} \quad x > 0,$$

where $\mathcal{H}^I = \mathcal{H}$, and $\mathcal{H}^J$ denotes the survival function of the remaining J-segment. This can also be expressed in the language of random variables. Let η^I, η^J, η^K be random variables that have the same distribution as the length of the remaining I-, J- or K-segment, respectively. If η^I and η^J are independent, then $\eta^K \overset{D}{=} \min\{\eta^I, \eta^J\}$.

3.6. The length distribution of the typical K-segment.

Analogously as for the I-segments we consider now the set of the centers of K-segments of a STIT tessellation Y as a point process in $\mathbb{R}^2$ and the distribution of the length of the K-segment that has the typical point as its center. For a realization y of Y denote by φ_y^K the measure on $\mathbb{R}^2$ which is the sum of the Dirac measures for all centers of K-segments of y. Then the respective Palm distribution is given by

$$P^{0K}(C) = \frac{1}{N^K} \int P(\mathrm{d}y) \int \varphi_y^K(\mathrm{d}z) \mathbf{1}(y - z \in C) \mathbf{1}(z \in [0,1]^2)$$

for measurable sets C of closed subsets of $\mathbb{R}^2$. The survival function of the length of the typical K-segment can be defined as

$$\mathcal{F}^K(x) = P^{0K}(\text{'length of the K-segment in } o\text{'} \geq x) \quad \text{for} \quad x > 0.$$

The same method as it was used in the proof of Theorem 2 leads to

Theorem 4. The survival function of the length of the typical K-segment of a stationary and isotropic STIT tessellation with $L_A = \frac{\pi}{2}$ *is*

$$\mathcal{F}^K(x) = \frac{2}{3x^2}\left(1 + x - (1+2x)\mathrm{e}^{-x}\right) \cdot \mathrm{e}^{-x} \quad \textit{for} \quad x > 0. \tag{25}$$

3.7. Alternative representations of the length distribution

The expression in (25) for the survival function $\mathcal{F}^K$ does not have an intuitive meaning. But it is possible to show that it can be represented as the Laplace transform of a standard function. This allows a better insight into the length distribution of the typical K-segment. The following three statements can be shown by straightforward calculations.

Lemma 12. The survival function $\mathcal{H}^K : (0,\infty) \to [0,1]$ *of the length of the remaining K-segment given by (24) can be represented as*

$$\mathcal{H}^K(x) = \int_1^2 \mathrm{e}^{-xt}\,\mathrm{d}t \quad \textit{for} \quad x > 0.$$

This means that $\mathcal{H}^K$ is the Laplace transform of the indicator $t \to \mathbf{1}(1 < t < 2)$ of the interval $(1,2)$, and this function is a probability density.

Corollary 1. Let P_λ be the distribution of a stationary Poisson point process with intensity λ on the real line $\mathbb{R}$ and let Φ be a mixed Poisson process with the distribution $\int_1^2 \mathrm{d}\lambda\, P_\lambda$. Then

$$\mathcal{H}^K(x) = \mathcal{P}(\Phi \cap [0,x) = \emptyset) \quad \text{for} \quad x > 0.$$

Corollary 2. Let η and ξ be independent random variables, η exponentially distributed with parameter 1 and ξ uniformly distributed on the interval $(0,1)$. Then the survival function of the random variable $\eta/(1+\xi)$ is equal to $\mathcal{H}^K$.

In the following lemma it will be shown that $\mathcal{F}^K$ is the Laplace transform of the probability density $t \to \frac{2}{3}\,t\,\mathbf{1}(1 < t < 2)$.

Lemma 13. The survival function $\mathcal{F}^K : (0,\infty) \to [0,1]$ *of the length of the typical K-segment given by (25) can be represented as*

$$\mathcal{F}^K(x) = \frac{2}{3}\int_1^2 t\,\mathrm{e}^{-xt}\,\mathrm{d}t \quad \textit{for} \quad x > 0.$$

Proof : Analogously to the proof of Theorem 2, where $\mathcal{F}$ was related to $\mathcal{H}$, one can show that

$$\mathcal{F}^K = -\frac{2}{3}\left(\mathcal{H}^K\right)'. \tag{26}$$

Thus Lemma 12 implies the assertion. (Of course, the statement can also be verified by a direct calculation.) The proof is complete.

The following result is a consequence of Corollary 1 and (26).

Lemma 14. Let P_λ be the distribution of a stationary Poisson point process with intensity λ on the real line $\mathbb{R}$ and let Φ be a mixed Poisson process with the distribution $\int_1^2 \mathrm{d}\lambda\, P_\lambda$. Then the survival function of the length of the typical interval of Φ is equal to $\mathcal{F}^K$.

Lemma 15. Let η and ξ be independent random variables, η exponentially distributed with parameter 1 and ξ uniformly distributed on the interval $(0,1)$. Then the survival function of the random variable $\eta/\sqrt{1+3\,\xi}$ is equal to $\mathcal{F}^K$.

Proof : For $x > 0$ we have

$$\mathcal{P}(\eta/\sqrt{1+3\,\xi} \geq x) = \mathcal{P}(\eta \geq x\sqrt{1+3\,\xi}) = \int_0^1 \exp(-x\sqrt{1+3t})\,\mathrm{d}t = \frac{2}{3}\int_1^2 s\,\mathrm{e}^{-xs}\,\mathrm{d}s.$$

According to Lemma 13, this last expression is equal to $\mathcal{F}^K$. Lemma 15 is proved.

The results of this section can also be translated into the language of distributions.

Theorem 5. Denote by D_{RI}, D_I, D_{RK}, D_K the distributions of the lengths of the remaining I-segment, the typical I-segment, the remaining K-segment and the typical K-segment, respectively. Further let E_t be the exponential distribution with parameter t, i.e. with mean value $1/t$. (All these distributions are regarded as probability measures on the positive real half-line.) Then

$$D_{RI} = \int_0^1 \mathrm{d}t\, E_t, \qquad D_{RK} = \int_1^2 \mathrm{d}t\, E_t,$$

$$D_I = 2\int_0^1 \mathrm{d}t\,(t \cdot E_t), \qquad D_K = \frac{2}{3}\int_1^2 \mathrm{d}t\,(t \cdot E_t).$$

3.8. Comparison of the length distributions of the typical I-, J- and K-segments.

For a stationary and isotropic STIT tessellation with length intensity $L_A = \frac{\pi}{2}$ the lengths of the typical I-, J- and K-segments have the cumulative distribution functions for $x > 0$

$$F^I(x) = 1 - \frac{2}{x^2}\left(1 - (1+x)\mathrm{e}^{-x}\right),$$

$$F^J(x) = 1 - \mathrm{e}^{-x},$$

$$F^K(x) = 1 - \frac{2}{3x^2}\left((1+x)\mathrm{e}^{-x} - (1+2x)\mathrm{e}^{-2x}\right).$$

Differentiation yields the corresponding density functions

$$p^I(x) = \frac{4}{x^3}\left(1 - \left(1 + x + \frac{x^2}{2}\right)\mathrm{e}^{-x}\right),$$

$$p^J(x) = \mathrm{e}^{-x},$$

$$p^K(x) = \frac{4}{3x^3}\left(\left(1 + x + \frac{x^2}{2}\right)\mathrm{e}^{-x} - (1 + 2x + 2x^2)\mathrm{e}^{-2x}\right).$$

The calculation of the moments, based on these densities yields the mean values

$$L^I = 2, \quad L^J = 1, \quad L^K = \frac{2}{3}.$$

These mean values coincide with those ones given in Lemma 4 which was shown with other methods.

For the second moments of the lengths the result of the calculation is

$$\mathbb{E}|s_I|^2 = \infty, \quad \mathbb{E}|s_J|^2 = 2, \quad \mathbb{E}|s_K|^2 = \frac{\ln 16}{3} \approx 0.924$$

where $|s_I|$, $|s_J|$, $|s_K|$ denote the lengths of the typical I-, J- and K-segment, respectively.

Also the survival functions as a whole can be compared. In this section denote $\mathcal{F}^I = \mathcal{F}$, $\mathcal{H}^I = \mathcal{H}$ and by $\mathcal{F}^J$ and $\mathcal{H}^J$ the respective survival functions of the lengths of the typical J-segments and of the remaining J-segment. The definitions of the different types of segments immediately yield

$$\mathcal{F}^I \geq \mathcal{F}^J \geq \mathcal{F}^K.$$

This can also be shown by a calculation based on the explicit formulas given above.

Furthermore, also the remaining segments can be included into the comparison with respect to a stochastic order relation.

Theorem 6. The survival functions satisfy the following chain of inequalities

$$\mathcal{H}^I \geq \mathcal{F}^I \geq \mathcal{H}^J = \mathcal{F}^J \geq \mathcal{H}^K \geq \mathcal{F}^K.$$

Proof : Let η and ξ be independent random variables, η exponentially distributed with parameter 1 and ξ uniformly distributed on the interval $(0,1)$. Then, according to Lemma 8, Lemma 10, Lemma 1, Corollary 2 and Lemma 15 we have for $x>0$

$$\mathcal{H}^I(x)=\mathcal{P}(\eta/\xi\geq x),\quad \mathcal{F}^I(x)=\mathcal{P}(\eta/\sqrt{\xi}\geq x),$$

$$\mathcal{H}^J(x)=\mathcal{F}^J(x)=\mathcal{P}(\eta\geq x),$$

$$\mathcal{H}^K(x)=\mathcal{P}(\eta/(1+\xi)\geq x),\quad \mathcal{F}^K(x)=\mathcal{P}(\eta/\sqrt{1+3\,\xi}\geq x).$$

Since $0\leq\xi\leq 1$, we find

$$\xi^2\leq\xi\leq 1=1\leq 1+2\,\xi+\xi^2\leq 1+3\,\xi$$

or

$$\xi\leq\sqrt{\xi}\leq 1=1\leq 1+\xi\leq\sqrt{1+3\,\xi}$$

and finally

$$\frac{\eta}{\xi}\geq\frac{\eta}{\sqrt{\xi}}\geq\eta=\eta\geq\frac{\eta}{1+\xi}\geq\frac{\eta}{\sqrt{1+3\,\xi}}$$

Theorem 6 is proved.

3.9. Remarks on an alternative proof for the length distribution of the typical I-segment based on a differential equation

The derivation of a functional equation for the survival function of the length of the 'remaining I-segment' as it is given above uses the Palm distribution with respect to the length measure on the edges, i.e. the length weighted distribution of the typical segment.

An alternative method uses the Palm distribution of the typical I-segment, i.e. the 'number-weighted' distribution. It is much more laborious, but nevertheless it can probably be a fruitful method to solve also other problems. This method is now sketched briefly.

For $\varepsilon>0$ we consider the rescaled iteration $\mathcal{I}(\varepsilon)=\mathcal{I}((1+\varepsilon)Y,\frac{1+\varepsilon}{\varepsilon}\mathcal{Y})$. The rescaling factors are chosen such that the parameter L_A remains constant. For $\varepsilon=1$ we have the equal homothetic factor 2 for both the 'frame' Y and the nested sequence $\mathcal{Y}$.

The key observation is that for a STIT tessellation Y

$$Y\stackrel{D}{=}\mathcal{I}((1+\varepsilon)Y,\frac{1+\varepsilon}{\varepsilon}\mathcal{Y})\quad\text{for all}\quad\varepsilon>0. \tag{27}$$

This relation can easily be derived from the equations (2) and (3).

The transform $Y \to (1+\varepsilon)Y$ means that the I-segments are stretched with the factor $(1+\varepsilon)$ and their mean number per unit area changes as $N^I \to (1+\varepsilon)^{-2}N^I$.

For small $\varepsilon > 0$ the transform $\frac{1+\varepsilon}{\varepsilon}\mathcal{Y}$ provides tessellations with huge cells and low densities of edges. Thus, intuitively, the nesting of such tessellation into the frame $(1+\varepsilon)Y$ yields either no new edges in the frame cells or exactly one new edge, which is a new I-segment. With a modification of the theorems by Korolyuk and Dobrushin (for point processes on the line) it can be shown strictly that the probability of the remaining cases (i.e. two or more new edges in a frame cell) is of order $o(\varepsilon)$. A version of such an assertion was already shown in [9].

Thus from (27) and the fact that the interior of the typical cell of $(1+\varepsilon)Y$ is a Poisson typical cell, we obtain for the survival function of the length of the typical I-segment

$$\mathcal{F}(x) = \frac{1}{(1+\varepsilon)^2}\mathcal{F}\left(\frac{1}{1+\varepsilon}x\right) + 2\frac{\varepsilon}{(1+\varepsilon)^2}\mathrm{e}^{-x} + o(\varepsilon) \quad \text{for all} \quad \varepsilon > 0,\ x > 0.$$

This yields immediately

$$\lim_{\varepsilon\to 0}\frac{1+\varepsilon}{\varepsilon}\left(\mathcal{F}(x) - \frac{1}{(1+\varepsilon)^2}\mathcal{F}\left(\frac{1}{1+\varepsilon}x\right)\right) = 2\mathrm{e}^{-x}.$$

On the other hand, if $\mathcal{F}$ is differentiable, we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{1+\varepsilon}{\varepsilon}\left(\mathcal{F}(x) - \frac{1}{(1+\varepsilon)^2}\mathcal{F}\left(\frac{1}{1+\varepsilon}x\right)\right) = \\
&= \lim_{\varepsilon\to 0}\frac{1+\varepsilon}{\varepsilon}\left(\mathcal{F}(x) - \mathcal{F}\left(\frac{1}{1+\varepsilon}x\right) + \left(1 - \frac{1}{(1+\varepsilon)^2}\right)\mathcal{F}\left(\frac{1}{1+\varepsilon}x\right)\right) = \\
&= x\cdot\lim_{\varepsilon\to 0}\left(\frac{1+\varepsilon}{\varepsilon x}\left(\mathcal{F}(x) - \mathcal{F}\left(x - \frac{\varepsilon}{1+\varepsilon}x\right)\right) + \frac{2+\varepsilon}{(1+\varepsilon)x}\mathcal{F}\left(\frac{1}{1+\varepsilon}x\right)\right) = \\
&= x\cdot\mathcal{F}'(x) + 2\mathcal{F}(x).
\end{aligned}$$

Hence

$$\mathcal{F}'(x) = -\frac{2}{x}\mathcal{F}(x) + \frac{2}{x}\mathrm{e}^{-x} \quad \text{for} \quad x > 0.$$

The solution of this differential equation with the condition $\mathcal{F}(0+) = 1$ is the same function as given in Theorem 2.

Резюме. В статье изучаются стационарные и изотропные случайные мозаики на евклидовой плоскости, обладающие свойством устойчивости относительно итераций (или гнездования), для краткости УСИТ. Так как стороны ячеек не соприкасаются, то образуются три различных вида прямолинейных отрезков. Для всех трёх видов найдено распределение длины типичного отрезка.

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