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RANDOM FLUCTUATIONS AT THE PLANCK LENGTH SCALE : A SOURCE OF QUANTUM RANDOMNESS

A. Khrennikov

International Center for Mathematical Modeling in Physics, Engineering, Economy and Cognitive Sciences, University of Växjö, S-35195, Sweden
E-mail : Andrei.Khrennikov@msi.vxu.se

Abstract. The mathematical formalism of quantum mechanics can be interpreted as a method for approximation of classical (measure-theoretic) averages of functions $f : L_2(\mathbb{R}^3) \to \mathbb{R}$. These are the classical physical variables in Prequantum Classical Statistical Field Theory (PCSFT), as we call our model with hidden variables. The present paper provides a simple stochastic picture of a quantum approximation procedure equivalent to an approximative method for computation of averages of random variables. Since in PCSFT the space of hidden variables is $L_2(\mathbb{R}^3)$, the role of a classical random variable is played by a random field. In PCSFT we consider Gaussian random fields representing random fluctuations at the prequantum length scale. Quantum mechanical expression for the average (given by the von Neumann trace formula) is obtained by moving from the prequantum to the quantum length scale (the scale that enables to perform measurements). The order of deviations of quantum (approximative) averages from the classical ones is given by the length scaling parameter, which is extremely small for quantum systems, e.g., $\kappa \sim 10^{-69}$ for an electron.

§0. INTRODUCTION

The problem of coupling of classical statistical mechanics and quantum mechanics has been the subject of stormy debates since the first days of quantum mechanics, before the first rigorous analysis was presented in the book of J. von Neumann [1]. His conclusion was rather supporting for the orthodox Copenhagen interpretation and is known as von Neumann's no-go theorem. It is also a rather common viewpoint that Heisenberg's uncertainty relations strongly support the Copenhagen interpretation,

see, however, Ballentine [2] and De Muynck [3] for the opposite viewpoint. Later there were proved various no-go theorems, the most popular nowadays is Bell's theorem [4]. In spite of all no-go theorems and other arguments in support of impossibility of a prequantum classical statistical model, a search for such a model never ceased (double solution approach of De Broglie, Bohmian mechanics, see, e.g., [5] , SED, see, e.g., [6], [7] Nelson's stochastic mechanics [8], [9] , Davidson's random field approach [10]). In a series of papers [11] the author developed a prequantum classical model where the role of hidden variables is played by classical fields. That Prequantum Classical Statistical Field theory (PCSFT) is a close relative both to SED and to Davidson's random field approach.

It was shown that the mathematical formalism of quantum mechanics can be interpreted as a method for approximation of classical (measure-theoretic) averages of functions $f : L_2(\mathbb{R}^3) \to \mathbb{R}$. In this paper we describe a simple stochastic quantum approximation procedure equivalent to an approximative method for computation of averages for functions of random variables. Since in PCSFT the space of hidden variables is $L_2(\mathbb{R}^3)$, the role of a classical random variable is played by a random field. In PCSFT we consider Gaussian random fields representing random fluctuations at the prequantum length scale.

Quantum mechanical expression for the average (given by the von Neumann trace formula) is obtained by moving from the prequantum to the quantum length scale that enables measurements. The order of deviations of quantum (approximative) averages from the classical ones is given by the length scaling parameter. If one considers the Planck scale as the prequantum length scale then that scaling parameter is extremely small for quantum systems, e.g., $\kappa \sim 10^{-69}$ for electron. But it increases as

$$\kappa \sim m^3,$$

where m is the mass of a system. This, on one hand, explains well why quantum mechanics provides excellent approximation for statistical behavior of ensembles of "quantum particles", electrons, neutrons and even for hypothetical gigantic particles as Higgs bosons. On the other hand, it becomes clear why quantum mechanics does not work for relatively heavy systems as compared with the Planck mass.

The Planck scaling of masses is a consequence of Gaussian random fluctuations at the Planck length scale. In contrast with the Planck length or time, the Planck mass is macroscopic – a disturbing fact for those who tried to couple this mass with micro-world. In PCSFT the Planck mass is simply a characteristic mass for our measurement devices which are used to investigate "quantum systems." The latter should be not too heavy as compared with the Planck mass.

To simplify the presentation and to emphasize length scaling procedure, we cling to the case of the real Hilbert space. The complex quantum mechanics can be developed in the same way as in [11].

§2. AN APPROXIMATE METHOD FOR CALCULATION OF MEAN VALUES IN CLASSICAL PROBABILITY THEORY

Let $y = f(x)$, where f is not linear but differs not too much from a line on some interval $[m_\eta - \delta, m_\eta + \delta]$, where $\eta = \eta(\omega)$ is a random variable,

$$m_\eta \equiv E = \int \eta(\omega)\, d\mathbf{P}(\omega)$$

is its average and $\delta > 0$ is sufficiently small. Writing the first order Taylor expansion at the point m_η,

$$y(\omega) \approx f(m_\eta) + f'(m_\eta)(\eta(\omega) - m_\eta), \tag{1}$$

and taking the average of both sides one obtains :

$$m_y \approx f(m_\eta). \tag{2}$$

The crucial point is that the linear term $f'(m_\eta)(\eta(\omega) - m_\eta)$ does not give any contribution. We remark that the approximative formula (2) was first discovered by Gauss and in the probabilistic literature it is sometimes called the Gaussian formula for averages.

Now we take the first three terms in the expansion of f into the Taylor series at the point m_x :

$$y(\omega) \approx f(m_\eta) + f'(m_\eta)(\eta(\omega) - m_\eta) + \frac{1}{2} f''(m_\eta)(\eta(\omega) - m_\eta)^2. \tag{3}$$

Hence

$$m_y \approx f(m_\eta) + \frac{\sigma_\eta^2}{2} f''(m_\eta), \tag{4}$$

where

$$\sigma_\eta^2 = E\,(\eta - m_\eta)^2 = \int (\eta(\omega) - m_\eta)^2\, d\mathbf{P}(\omega)$$

is the variance of the random variable η.

Let us consider the special case of symmetric fluctuations $m_\eta = 0$, and assume that $f(0) = 0$. Then we obtain the following special form of (4) :

$$m_y \approx \frac{\sigma_\eta^2}{2} f''(0). \tag{5}$$

Thus at some level of approximation we can calculate averages not by using the Lebesgue integral (as we do in classical probability theory), but by finding the second derivative. Such a "calculus of probability" would match well with experiment. Probably the reader has already found analogy with the quantum calculus of probabilities. This analogy is better seen in the multi-dimensional case. Let

$$\eta = (\eta_1, ..., \eta_n),$$

so we consider a system of n random variables. We consider the vector average

$$m_\eta = (m_{\eta_1}, ..., m_{\eta_n})$$

and the covariance matrix

$$B_\eta = (B_\eta^{ij}),\ B_\eta^{ij} = E\,(\eta_i - m_{\eta_i})\,(\eta_j - m_{\eta_j}).$$

For the random variable $y(\omega) = f(\eta_1(\omega), \ldots, \eta_n(\omega))$ we write the second order Taylor expansion :

$$y(\omega) \approx f(m_{\eta_1}, ..., m_{\eta_n}) + \sum_{i=1}^{n} \frac{\partial f}{\partial \eta_i}(m_{\eta_1}, ..., m_{\eta_n})(\eta_i(\omega) - m_{\eta_i}) +$$

$$+ \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(m_{\eta_1}, ..., m_{\eta_n})(\eta_i(\omega) - m_{\eta_i})(\eta_j(\omega) - m_{\eta_j}), \qquad (6)$$

and hence

$$m_y \approx f(m_{\eta_1}, ..., m_{\eta_1}) + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(m_{\eta_1}, ..., m_{\eta_1}) B_\eta^{ij}. \qquad (7)$$

In vector notations

$$y(\omega) \approx f(m_\eta) + (f'(m_\eta), \eta(\omega) - m_\eta) + \frac{1}{2}(f''(m_\eta)(\eta(\omega) - m_\eta), \eta(\omega) - m_\eta). \qquad (8)$$

and

$$m_y \approx f(m_\eta) + \frac{1}{2}\mathrm{Tr}\ B_\eta f''(m_\eta). \qquad (9)$$

For the special case $m_\eta = 0$ and $f(0) = 0$ we have

$$m_y \approx \frac{1}{2}\mathrm{Tr}\ B_\eta f''(0). \qquad (10)$$

The Hessian $f''(0)$ is **always a symmetric operator.** Let us now represent f by its second derivative at zero :

$$f \to A = \frac{1}{2} f''(0).$$

So at some level of approximation, instead of Lebesgue integrals, one can use linear algebra :

$$m_y \approx \mathrm{Tr}\ B_\eta A. \tag{11}$$

§3. GAUSSIAN RANDOM FIELDS ON PREQUANTUM AND QUANTUM LENGTH SCALES

Let us consider a prequantum length scale l_{pq} and the corresponding system of coordinates $y = (y_1, y_2, y_3)$. The problem of the correct choice of l_{pq} is complicated. We shall come back to this problem in §5.

Let $\xi^\rho(y, \omega)$ $(y \in \mathbb{R}^3)$ be a Gaussian random field with zero mean value ("symmetric fluctuations of vacuum") :

$$E\left(\int_{\mathbb{R}^3} \psi(y)\xi^\rho(y,\omega)d^3y\right) = 0,$$

which is determined by its covariance operator ρ.

$$E\left(\int_{\mathbb{R}^3} \psi_1(y)\xi^\rho(y,\omega)d^3y\right)\left(\int_{\mathbb{R}^3} \psi_2(u)\xi^\rho(u,\omega)d^3u\right) = \int_{\mathbb{R}^3} \rho(y,u)\psi_1(y)\psi_2(u)d^3yd^3u.$$

Here $\rho(y, u)$ is the kernel of the covariance operator ρ. We shall assume that the prequantum fluctuations are normalized, i.e.

$$\mathrm{Tr}\ \rho = \int_{\mathbb{R}^3} \rho(y,y)d^3y = E\left(\int_{\mathbb{R}^3} |\xi^\rho(y,\omega)|^2 d^3y\right) = 1.$$

Such a random field can be considered as a Gaussian random variable taking values in the Hilbert space $H = L_2(\mathbb{R}^3)$.

We now consider the characteristic length scale of quantum mechanics l_q and the corresponding system of coordinates $x = (x_1, x_2, x_3)$. The problem of the correct choice of l_q will be discussed in §5. Our basic assumption is that the prequantum and quantum length scales are coupled through a linear scaling, i.e.

$$x = \gamma y.$$

The prequantum interval of the unit length $r_{pq} = \sqrt{y_1^2 + y_2^2 + y_3^2} = 1$ corresponds to the quantum interval of the length $r_q = \sqrt{x_1^2 + x_2^2 + x_3^2} = \gamma$. If $\gamma \to 0$, then

$r_q \to 0$. Thus a "quantum point" has an extended prequantum spatial structure. On the other hand, the quantum interval of the unit length $r_q = \sqrt{x_1^2 + x_2^2 + x_3^2} = 1$ can be considered as the image (available from the measurement devices) of the huge prequantum interval $r_{pq} = \sqrt{y_1^2 + y_2^2 + y_3^2} = 1/\gamma \to \infty$ as $\gamma \to 0$. Thus, systems which are commonly considered as point-like particles have huge spatial structures in the prequantum space (prespace).

We now consider the Gaussian random field $\eta^\rho(x,\omega)$ $(x \in \mathbb{R}^3)$ that corresponds to a prequantum Gaussian random field $\xi^\rho(y,\omega)$ $(y \in \mathbb{R}^3)$ through transition from the prequantum length scale to the quantum one :

$$\eta^\rho(x,\omega) = \xi^\rho\left(\frac{x}{\gamma},\omega\right).$$

The mean value of the quantum-scale random field $\eta^\rho(x,\omega)$ is zero and the variance

$$E\left(\int_{\mathbb{R}^3} |\eta^\rho(x,\omega)|^2 d^3x\right) = E\left(\int_{\mathbb{R}^3} \left|\xi^\rho\left(\frac{x}{\gamma},\omega\right)\right|^2 d^3x\right) =$$

$$= \gamma^3 E\left(\int_{\mathbb{R}^3} |\xi^\rho(y,\omega)|^2 d^3y\right) = \gamma^3.$$

Thus, transition from the prequantum space-scale to the quantum induces a rescaling of a prequantum Gaussian random field :

$$\text{Prob. distr. } \{\eta^\rho(x,\omega) : x \in \mathbb{R}^3\} = \text{Prob. distr. } \{\gamma^{3/2}\xi^\rho(y,\omega) : y \in \mathbb{R}^3\}.$$

Consider a map $f : L_2(\mathbb{R}^3) \to \mathbb{R}$, classical physical variable – a functional of classical fields $\psi \in L_2(\mathbb{R}^3)$. We are interested in the average of the function $f(\eta^\rho(x,\omega))$ of the random field $\eta^\rho(x,\omega)$. To find $Ef(\eta^\rho(x,\omega))$ precisely is a complicated problem. Therefore, we use the approximation method of functions of random variables based on the Taylor formula (see §2). The only difference is that we now consider $f = f(\psi)$ to be a function of a Hilbert vector. However, there is a well developed differential calculus on Hilbert spaces, as well as general normed spaces. So we obtain the asymptotic expansion (see [11])

$$Ef(\eta^\rho(x,\omega)) = Ef(\sqrt{\kappa}\xi^\rho(y,\omega)) = \frac{\kappa}{2}\mathrm{Tr}\ \rho\, f''(0) + O(\kappa^2), \qquad (12)$$

where

$$\kappa = \gamma^3.$$

To produce observable effects, the classical physical variable f should be strongly amplified :

$$f \to f_\kappa(\psi) \equiv \frac{2}{\kappa} f(\psi).$$

For the mean value of such an amplification we obtain the asymptotic expansion :

$$Ef_\kappa(\eta^\rho(x,\omega)) = \mathrm{Tr}\ \rho\, f''(0) + O(\kappa). \tag{13}$$

Thus, the average with respect to fluctuations at the prequantum length scale can be well approximated by the von Neumann trace-average basic for quantum mechanics.

From the point of view of PCSFT, quantum mechanics is an approximative statistical theory providing first order approximation with respect to random Gaussian fluctuations at the prequantum space-scale.

§4. HIDDEN VARIABLES : SPACE OF CLASSICAL FIELDS

For the space of hidden variables we choose

$$H = L_2(\mathbb{R}^3).$$

We repeat that we consider the real theory and generalization to a complex theory can be obtained by methods developed in [11]. The scalar product is given by the formula

$$(\psi_1, \psi_2) = \int_{\mathbb{R}^3} \psi_1(y)\psi_2(y)d^3y,$$

and the norm is

$$\|\psi\|^2 = (\psi, \psi) = \int_{\mathbb{R}^3} \psi^2(y)d^3y.$$

Let us consider the functional space $\mathcal{V}(H)$ of those functions $f : H \to \mathbb{R}$ for which

a) the state of vacuum is preserved, i.e.

$$f(0) = 0,$$

b) f is four times continuously differentiable as a functional (generally nonlinear) on the Hilbert space H,

c) the fourth derivative of f is of exponential growth, i.e. $|||f^{(4)}(\psi)||| \leq c_f e^{r_f\|\psi\|}$ for some $c_f, r_f \geq 0$ and for all $\psi \in H$.

The last two conditions have purely mathematical significance. They are necessary for rigorous treatment of expansions of the averages by small parameters (see (13), details can be found in [11]).

The space of all Gaussian random fields $\xi^\rho(y,\omega)$ of the type considered in §3 we denote by $S(H)$. We consider a classical statistical model on the space H with physical variables of the class $\mathcal{V}(H)$ and the statistical states given by Gaussian random fields of the class $S(H)$ – **prequantum classical statistical field theory**, PCSFT :

$$M_{PCFT} = (S(H), \mathcal{V}(H)).$$

We recall that the conventional quantum statistical model with the real Hilbert state space $H = L_2(\mathbb{R}^3)$ is described in the following way :

a) physical observables are represented by operators $A : H \longmapsto H$ from the class $\mathcal{L}_s \equiv \mathcal{L}_s(H)$ of continuous self-adjoint operators,

b) statistical states are represented by von Neumann density operators (the class of such operators is denoted by $\mathcal{D} \equiv \mathcal{D}(H)$),

c) the average of a physical observable (which is represented by the operator $A \in \mathcal{L}_s(H)$) with respect to a statistical state (represented by a density operator $\in \mathcal{D}(H)$) is given by von Neumann's formula

$$\langle A \rangle_\rho \equiv \mathrm{Tr}\ \rho A. \tag{14}$$

The **quantum statistical model** is the pair $M_{quant} = (\mathcal{D}, \mathcal{L}_s)$.

We define T to be maps from the classical model into the quantum model :

$$T : S(H) \longmapsto \mathcal{D}(H), \quad T(\xi^\rho) = \rho, \tag{15}$$

$$T : \mathcal{V} \longmapsto \mathcal{L}_s(H), \quad T(f) = f''(0). \tag{16}$$

Theorem 1. *Both maps (15) and (16) are mappings onto the corresponding spaces. The map (15) is even and one-to-one. The map (16) is not one-to-one and is linear. The asymptotic equality (13) between the classical and the quantum averages is valid.*

§5. THE MAGNITUDE OF LENGTH–SCALING

The small parameter of our model

$$\kappa = \gamma^3,$$

we depends on the choice of the quantum and the prequantum length scales. We now choose the **atom length**-scale in QM .and the **Planck length**-scale in the prequantum classical theory. We start with the Planck scale based on the Planck length

$$l_{prq} = l_P = \sqrt{\frac{\hbar G}{c^3}} \approx 1.616 \times 10^{-33} sm \tag{17}$$

The Planck length l_P can be expressed as

$$l_P = \frac{\hbar}{m_P c}, \tag{18}$$

where

$$m_P = \sqrt{\frac{\hbar c}{G}} \approx 2.176 \times 10^{-8} kg$$

is the Planck mass. We apply the generalization of the formula

$$l_m = \frac{\hbar}{mc} \tag{19}$$

to a system with an arbitrary mass m and find the corresponding length scale by formula (19). To obtain the atom time-scale, we choose the electron mass scale $m_e \approx 9.109 \times 10^{-31} kg$. This mass scale induces the length scale which is in the limits of the atom length scale :

$$l_q = l_e = \frac{\hbar}{m_e c} \approx 3.86 \times 10^{-11} sm. \tag{20}$$

We recall that the Bohr radius is

$$a_0 = \frac{\hbar}{m_e \alpha c} \approx 5.292 \times 10^{-9} sm,$$

where α is the fine structure constant. Therefore,

$$\gamma = \frac{l_{prq}}{l_q} = \frac{l_P}{l_e} \approx 4.186 \times 10^{-23}. \tag{21}$$

We also remark that

$$\gamma = \frac{m_e}{m_P}. \tag{22}$$

Thus, our length-scaling parameter has the magnitude

$$\gamma \sim 10^{-23}.$$

Under such choice of the prequantum scale, the difference between statistical predictions of PCSFT and QM (given by (13)) is of the order

$$\kappa \sim 10^{-69}.$$

For instance, for the quantum observable A is given by (16) and the classical physical variable

$$f(\psi) = \frac{1}{2}\langle A\psi, \psi\rangle + \frac{1}{4}\langle A\psi, \psi\rangle^2, \quad A \in \mathcal{L}_s$$

the difference between the quantum prediction for the average of its quantum image $\langle A\rangle_\rho = \mathrm{Tr}\ \rho A$ and the PCSFT-prediction for the average of $f(\psi)$ should be of the order 10^{-69} (under the assumption that the Planck length l_P really provides the correct prequantum time-scale, for this we do not have any internal justification of such a choice inside PCSFT).

If we choose the characteristic atom length scale to be the Bohr radius, then the deviation would be 10^{-6} times smaller.

As mentioned above, we cannot guarantee that the Planck length really provides the right prequantum scale. The main problem induced by such a choice is that there is a huge gap between the atomic and the Planck scales. A scale between the Planck and atomic scales can be more natural. In that case κ could be larger, simplifying the experimental verification of PCSFT. On the other hand, choosing the Planck scale and the corresponding $\kappa \equiv \kappa_e \sim 10^{-23}$ clarifies why predictions of QM have not yet been violated since the deviation is really negligibly small.

One of the reasons in favor of the Planck scale as the scale of prequantum fluctuations is that the Planck mass has macroscopic magnitude.

Let a system have a mass m. Then choosing the corresponding time scale $l_m = \frac{\hbar}{mc}$ we obtain $\gamma = \frac{m}{m_P}$. Therefore QM should be violated for systems of macroscopic mass (as compared with the Planck mass). In principle, one may expect that it would be easier to produce deviations from QM for heavy elementary particles, e.g., muons. Let us take $m = m_{muon}$ and the corresponding time scale

$$\gamma_{muon} = \gamma_e \frac{m_{muon}}{m_e} \approx 207\gamma_e.$$

Then statistical deviations for muons become essentially larger than for electrons, but they are still very small $\kappa_{muon} \sim 10^{-63}$.

For the neutron, i.e. for a quite heavy quantum system we have

$$l_n = \frac{\hbar}{m_n c} \sim 10^{-14} sm.$$

We remark that the experimentally defined radius of neutron is about $r_n \approx 8 \times 10^{-14} sm$. Thus

$$\gamma_n \sim 10^{-19}, \ \kappa_n \sim 10^{-57}.$$

PCSFT predicts that for neutrons QM works 10^6 worse than for muons, but this is still a negligibly small deviation.

Let us now consider the hypothetical particles such as e.g. Higgs bosons. Some models with supersymmetries predicts

$$m_{Higgs} \sim 120 GeV.$$

Here

$$\gamma_{Higgs} \sim 10^{-17}, \quad \kappa_{Higgs} \sim 10^{-51}.$$

Thus even possible discovery of Higgs bosons would not induce visible violations of laws of quantum mechanics. On the other hand, decreasing of the mass strongly increases the precision of the quantum approximation. For electron neutrino and antineutrino

$$\gamma_{e-neutrino} = \gamma_e \frac{m_{e-neutrino}}{m_e} < 4.31 \times 10^{-7} \gamma_e \sim 10^{-30}.$$

Here $\kappa_{e-neutrino} \sim 10^{-90}$.

It is impossible to interpolate directly our theory to photons, since we considered nonrelativistic QM. By a direct interpolation $\gamma_{photon} = 0$. Thus it would imply that the QM model is precise for photons. However, as already mentioned, such an interpolation can be too straightforward.

Резюме. Математический формализм квантовой механики можно интерпретировать как метод для приближений классических усреднений (по мере) функций $f : L_2(\mathbb{R}^3) \to \mathbb{R}$. Они являются классическими физическими переменными в Предквантовой классической статистической теории поля (ПКСТП). Так мы называем нашу модель со скрытыми переменными. Настоящая статья даёт простую стохастическую картину процедуры квантового приближения эквивалентную аппроксимативному методу вычисления средних для случайных переменных. Поскольку в ПКСТП пространством скрытых переменных является $L_2(\mathbb{R}^3)$, роль классической случайной величины играет случайное поле. В ПКСТП мы рассматриваем гауссовские случайные поля, представляющие случайные колебания на шкале предквантовой длины. Кванто–механическое выражение для среднего (задаваемого формулой следа фон Неймана) получена движением из предквантовой к квантовой шкале длины (шкала, которая даёт возможность проводить измерения). Порядок отклонений квантовых (аппроксимативных) средних в классическом случае задаётся параметром определяющим масштаб длины, который очень мал для квантовых систем, например, для электрона равен $\kappa \sim 10^{-69}$.

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