Известия НАН Армении. Математика, 41, № 4, 2006, 43–60

CONORMAL SYMBOLS OF MIXED ELLIPTIC PROBLEMS WITH SINGULAR INTERFACES

Gohar Harutyunyan and B.-W. Schulze

Carl von Ossietzky University, Oldenburg, Germany
E-mail : gohar@mathematik.uni-oldenburg.de
Universität Potsdam, Potsdam, Germany
E-mail : schulze@math.uni-potsdam.de

Mixed elliptic problems are characterized by conditions that have a discontinuity on the boundary interface of codimension 1. Earlier additional interface conditions were studied in the case of smooth interface. The present paper studies a structure used in the case of interfaces with conical singularities, namely, corner conormal symbols of the operators. The main focus is on the second order operators and additional interface conditions that are holomorphic in an extra parameter, in particular on the Zaremba problem.

§1. INTRODUCTION

This paper studies the conormal symbolic structure of mixed elliptic problems when the interface has conical singularities. Let X be the closure of an open bounded set $G \subset \mathbb{R}^d$ with boundary Y subdivided into subsets $Y_\pm$, such that $Y = Y_- \cup Y_+$ and $Y_\pm \subset Y$ are closed, where $Z := Y_+ \cap Y_-$ is a submanifold of Y with conical singularity v and $Z_{reg} := Z \setminus \{v\}$ is of the codimension 1 in Y. Assuming that A is an elliptic differential second order operator in $G = \operatorname{int} X$ with coefficients smooth up to the boundary, we consider the mixed boundary value problem

$$Au = f \quad \text{in} \quad G, \quad T_\pm u = g_\pm \quad \text{on} \quad \operatorname{int} Y_\pm, \tag{1}$$

with boundary operators $T_\pm = r_\pm B_\pm$, where the differential operators $B_\pm$ of the orders $\mu_\pm$ are given in a neighborhood of $Y_\pm$ in $\mathbb{R}^d$ and satisfy the Shapiro-Lopatinskij condition uniformly up to Z from the respective sides. Here $r_\pm$ denote the operators of restriction to $\operatorname{int} Y_\pm$.

A well known case is the so called Zaremba problem for the Laplacian $A = \Delta$, possessing Dirichlet/Neumann conditions on the minus/plus side of the boundary. Here the problem is to understand the regularity of solutions and parametrices of the operator $\mathcal{A} = {}^t(A\ T_-\ T_+)$ in suitable weighted Sobolev spaces, both near Z_{reg} and v. This problem has been studied in [5], based on earlier papers [1] and [3] for the case of smooth Z with regularity in standard and weighted edge Sobolev spaces.

In [1], the interface Z is regarded as a smooth edge on the boundary of X. In [5], the regular part Z_{reg} of Z is a smooth edge, v plays the role of a corner point and the elliptic regularity of solutions in weighted corner Sobolev spaces was established. In [5], the operators are described by a principal symbolic hierarchy $\sigma = (\sigma_\psi, \sigma_\partial, \sigma_\wedge, \sigma_{\mathrm{C}})$, where σ_ψ is the standard homogeneous principal symbol of $\mathcal{A}$, σ_∂ is the pair of boundary symbols on the $\pm$ sides of the boundary, $\sigma_\wedge$ is the edge of the symbol on Z_{reg} and σ_{C} is the corner conormal symbol.

In the corner Sobolev spaces there are two weights $(\gamma, \delta) \in \mathbb{R}^2$, the $\sigma_\wedge$ ellipticity refers to the "cone weight" γ, while the σ_{C} ellipticity refers to the corner weight δ. In the Zaremba case the authors proved that for a suitable set of admissible weights γ the ellipticity with respect to σ_{C} is satisfied for all δ, except for a discrete set of exceptional weights. The complete answer contains extra interface conditions on Z_{reg}, which depend on γ, which quantity is computed.

The present paper is aimed at a more detailed study of the meromorphic corner conormal symbolic structure.

§2. MIXED PROBLEMS IN AN INFINITE CYLINDER

In contrast to the notation in Introduction, we now slightly change the context and consider mixed elliptic problems in an infinite cylinder.

Let $N = \overline{\Omega}$ be the closure of a smooth, bounded domain in $\mathbb{R}^m$ and let $M = 2N$ denote the double of N obtained by gluing together two copies $N_\pm$ of N along the common boundary to a closed compact C^∞ manifold (we identify N with N_+). Further, let $H^s(\mathbb{R} \times M)$ denote the cylindrical Sobolev space on $\mathbb{R} \times M$ of smoothness $s \in \mathbb{R}$.

Let us briefly recall corresponding definitions. The space $L^\mu_{cl}(M; \mathbb{R}^l_\lambda)$ of parameter-dependent pseudo-differential operators on M of order μ contains an element $R^\mu(\lambda)$ which is parameter-dependent elliptic und induces isomorphisms. $R^\mu(\lambda) : H^s(M) \longrightarrow H^{s-\mu}(M)$ for all $s \in \mathbb{R}$ and parameter values $\lambda \in \mathbb{R}^l$. The space $H^s(\mathbb{R} \times M)$ is defined to be the closure of $C_0^\infty(\mathbb{R}, C^\infty(M))$ with respect to the norm

$$\left\{ \int_{\mathbb{R}} \|R^s(\tau)(Fu)(\tau)\|^2_{L^2(M)}\, d\tau \right\}^{1/2},$$

where $R^s(\tau) \in L^s_{cl}(M; \mathbb{R}_\tau)$ is an order reducing element and F is the Fourier transform in $\mathbb{R}$. The space $L^2(M)$ is defined by a fixed Riemannian metric on M. Moreover, let

$$H^s(\mathbb{R} \times \text{int } N) := \left\{ u|_{\mathbb{R}\times\text{int } N} : u \in H^s(\mathbb{R} \times 2N) \right\}.$$

In order to investigate conormal symbols in a corner situation, we study mixed elliptic problems in an infinite cylinder. To avoid too complicated notation, we assume that X, $Y = \partial X$, $Y_\pm$ and Z are as before, but Z is a C^∞ submanifold of Y of the codimension 1, $Y_- \cup Y_+ = Y$ and $Y_- \cap Y_+ = Z$. Accordingly, we obtain the cylindrical Sobolev spaces $H^s(\mathbb{R} \times \text{int } X)$ and $H^s(\mathbb{R} \times \text{int } Y_\pm), H^s(\mathbb{R} \times Z)$. We set

$$H^{s,\delta}(\mathbb{R} \times \text{int } X) := e^{-t\delta} H^s(\mathbb{R} \times \text{int } X),$$

$$H^{s,\delta}(\mathbb{R} \times \text{int } Y_\pm) = e^{-t\delta} H^s(\mathbb{R} \times \text{int } Y_\pm),$$

and assume that

$$A = \sum_{|\alpha| \leq 2} a_\alpha(t,x) D^\alpha_{t,x}$$

is an elliptic differential operator of the second order, with coefficients $a_\alpha \in C^\infty(\mathbb{R} \times X)$. Besides, $T_\pm := \mathrm{r}_\pm B_\pm$ are the boundary operators, $\mathrm{r}_\pm$ are the restrictions to $\mathbb{R} \times \text{int } Y_\pm$ of the differential operators

$$B_\pm = \sum_{|\beta| \leq \mu_\pm} b_{\beta,\pm}(t,x) D^\beta_{t,x}$$

where the coefficients $b_{\beta,\pm} \in C^\infty(\mathbb{R} \times U_\pm)$ for some open neighbourhoods $U_\pm$ of $Y_\pm$. We assume that the boundary operators $T_\pm$ are elliptic on $Y_\pm$ with respect to A, i.e. they satisfy the Shapiro-Lopatinskij condition uniformly up to Z from the respective $\pm$ sides. Under suitable assumptions on the coefficients for $|t| \longrightarrow \infty$, for any fixed δ and for all $s \in \mathbb{R}, s > \max\{\mu_\pm + \frac{1}{2}\}$ we obtain continuous operators

$$\mathcal{A} = \begin{pmatrix} A \\ T_- \\ T_+ \end{pmatrix} : H^{s,\delta}(\mathbb{R} \times \text{int } X) \longmapsto \begin{pmatrix} H^{s-2,\delta-2}(\mathbb{R} \times \text{int } X) \\ \oplus \\ H^{s-\mu_- - \frac{1}{2},\delta-\mu_-}(\mathbb{R} \times \text{int } Y_-) \\ \oplus \\ H^{s-\mu_+ - \frac{1}{2},\delta-\mu_+}(\mathbb{R} \times \text{int } Y_+) \end{pmatrix}. \qquad (2)$$

We compare (2) with the mixed boundary value problem on the infinite, stretched cone $(\text{int } X)^\wedge = \mathbb{R}_+ \times \text{int } X \ni (r,x)$ obtained by substituting the diffeomorphism $\mathbb{R} \longmapsto \mathbb{R}_+$, $t \longrightarrow e^{-t} = r$. Assuming that M is a closed, compact C^∞ manifold of the dimension d, we write

$$\mathcal{H}^{s,\gamma}(M^\wedge), \quad s,\gamma \in \mathbb{R}, \quad M^\wedge := \mathbb{R}_+ \times M \ni (r,x),$$

for the completion of the space $C_0^\infty(\mathbb{R}_+, C^\infty(M))$ with respect to the norm

$$\left\{ \frac{1}{2\pi i} \int_{\Gamma_{\frac{d+1}{2}-\gamma}} \|R^s(\Im \mathrm{w}) M_{r\to \mathrm{w}} u(\mathrm{w})\|^2_{L^2(M)} \, d\mathrm{w} \right\}^{1/2},$$

where $M_{r\to\mathrm{w}}$ is the Mellin transform

$$M_{r\to\mathrm{w}} u(\mathrm{w}) = \int_0^\infty r^{\mathrm{w}-1} u(r) dr$$

on $u(r) \in C_0^\infty(\mathbb{R}_+, C^\infty(M))$ (which function is holomorphic in w), $\Gamma_\beta := \{\mathrm{w} \in \mathbb{C} : \Re \mathrm{w} = \beta\}$, and $R^s(\tau) \in L^s_{cl}(M; \mathbb{R}_\tau)$ is an order reducing family of order s. Further, assuming that X is a compact C^∞ manifold with C^∞ boundary ∂X, we define

$$\mathcal{H}^{s,\gamma}((\mathrm{int}\ X)^\wedge) := \left\{ u|_{(\mathrm{int}\ X)^\wedge} : u \in \mathcal{H}^{s,\gamma}((2X)^\wedge) \right\},$$

where $2X$ is the double of X, obtained by gluing two copies $X_\pm$ along the common boundary ∂X, where X_+ is identified with X. Then the mapping $v(t,x) \longmapsto u(r,x)$ as defined by $u(e^{-t},x) = v(t,x)$ induces an isomorphism $H^{s,\delta}(\mathbb{R} \times \mathrm{int}\ X) \longmapsto \mathcal{H}^{s,\gamma}((\mathrm{int}\ X)^\wedge)$, where $\gamma = \delta + \frac{d+1}{2}$ and $d = \dim X$. Consequently, the operator A takes the form

$$\mathbf{A} := r^{-2} \sum_{j=0}^{2} \mathbf{a}_j(r)(-r\partial_r)^j,$$

i.e. turns to be a Fuchs type differential operator on the infinite, stretched cone $(\mathrm{int}\ X)^\wedge = \mathbb{R}_+ \times \mathrm{int}\ X$, possessing coefficients $\mathbf{a}_j \in C^\infty(\overline{\mathbb{R}}_+ \times \mathrm{Diff}^{2-j}(X))$. Besides, the boundary operators are transformed into

$$\mathbf{T}_\pm := \mathrm{r}_\pm \mathbf{B}_\pm, \quad \text{where} \quad \mathbf{B}_\pm := r^{-\mu_\pm} \sum_{k=0}^{\mu_\pm} \mathbf{b}_{k,\pm}(r)(-r\partial_r)^k,$$

with coefficients $\mathbf{b}_{k,\pm} \in C^\infty(\overline{\mathbb{R}}_+, \mathrm{Diff}^{\mu_\pm - k}(U_\pm))$, where $U_\pm$ is as above. Assuming that the coefficients $\mathbf{a}_j$ and $\mathbf{b}_{k,\pm}$ are independent of r for r large enough, we obtain the continuous operators

$$\mathfrak{A} := \begin{pmatrix} \mathbf{A} \\ \mathbf{T}_- \\ \mathbf{T}_+ \end{pmatrix} : \mathcal{H}^{s,\gamma}((\mathrm{int}\ X)^\wedge) \longmapsto \begin{matrix} \mathcal{H}^{s-2,\gamma-2}((\mathrm{int}\ X)^\wedge) \\ \oplus \\ \mathcal{H}^{s-\mu_- - \frac{1}{2}, \gamma - \mu_- - \frac{1}{2}}((\mathrm{int}\ Y_-)^\wedge) \\ \oplus \\ \mathcal{H}^{s-\mu_+ - \frac{1}{2}, \gamma - \mu_+ - \frac{1}{2}}((\mathrm{int}\ Y_+)^\wedge) \end{matrix}, \tag{3}$$

$$s \in \mathbb{R}, \quad s > \max\left\{\mu_\pm + \frac{1}{2}\right\}.$$

The operator (3) represents a mixed boundary value problem in a cone $(\text{int } X)^\wedge$ with a subdivision of $\partial X^\wedge$ into $Y_\pm^\wedge$, where the interface $Z^\wedge$ has conical singularities at $r = 0$. According to the operator calculus on manifolds with conical points, the operator (3) has a conormal symbol defined as the holomorphic in $\mathbf{w} \in \mathbf{C}$ operator family

$$\sigma_{\mathrm{C}}(\mathfrak{A})(\mathbf{w}) := \begin{pmatrix} \sigma_{\mathrm{C}}(\mathbf{A}) \\ \sigma_{\mathrm{C}}(\mathbf{T}_-) \\ \sigma_{\mathrm{C}}(\mathbf{T}_+) \end{pmatrix}(\mathbf{w}) : H^s(\text{int } X) \longmapsto \begin{array}{c} H^{s-2}(\text{int } X) \\ \oplus \\ H^{s-\mu_- -\frac{1}{2}}(\text{int } Y_-) \\ \oplus \\ H^{s-\mu_+ -\frac{1}{2}}(\text{int } Y_+) \end{array}$$

where

$$\sigma_{\mathrm{C}}(\mathbf{A})(\mathbf{w}) = \sum_{j=0}^{2} \mathbf{a}_j(0)\mathbf{w}^j \quad \text{and} \quad \sigma_{\mathrm{C}}(\mathbf{T}_\pm)(\mathbf{w}) = \mathrm{r}_\pm \sum_{k=0}^{\mu_\pm} \mathbf{b}_{k,\pm}(0)\mathbf{w}^k.$$

§2. REDUCTION TO THE BOUNDARY

Now consider another boundary operator :

$$\mathbf{T} = \mathrm{r}\mathbf{B}, \quad \text{where} \quad \mathbf{B} = r^{-\mu}\sum_{k=0}^{\mu} \mathbf{b}_k(r)(-r\partial_r)^k,$$

where the coefficients $\mathbf{b}_k \in C^\infty(\overline{\mathbb{R}}_+ \times \mathrm{Diff}^{\mu-k}(U))$ for a neighbourhood U of Y. Assuming as above that $\mathbf{a}_j, \mathbf{b}_k$ are independent of r for r large enough, we conclude that the operator

$$\mathbb{D} := \begin{pmatrix} \sum_{j=0}^{2} \mathbf{a}_j(r)(-r\partial_r)^j \\ \mathrm{r}\sum_{k=0}^{\mu} \mathbf{b}_k(r)(-r\partial_r)^k \end{pmatrix} : \mathcal{H}^{s,\gamma}((\text{int } X)^\wedge) \longmapsto \begin{array}{c} \mathcal{H}^{s-2,\gamma}((\text{int } X)^\wedge) \\ \oplus \\ \mathcal{H}^{s-\mu-\frac{1}{2},\gamma}(Y^\wedge) \end{array} \tag{4}$$

represents a boundary value problem on $X^\wedge$ for any $\gamma, s \in \mathbb{R}$, $s > \mu + \frac{1}{2}$.

Let $\mathbf{T}$ satisfy the Shapiro-Lopatinskij condition with respect to $\mathbf{A}$ (in the Fuchs type sense, cf. [6]). Then the conormal symbol

$$\sigma_{\mathrm{C}}(\mathbb{D})(\mathbf{w}) := \begin{pmatrix} \sum\limits_{j=0}^{2} \mathbf{a}_j(0)\mathbf{w}^j \\ \mathrm{r}\sum\limits_{k=0}^{\mu} \mathbf{b}_k(0)\mathbf{w}^k \end{pmatrix} \tag{5}$$

is an operator-valued function holomorphic in $\mathbf{w} \in \mathbf{C}$, which defines a parameter-dependent elliptic family of boundary value problems on X

with the parameter $\tau = \Im \mathrm{w}$. Hence, for every $c \leq c'$ there is a countable set $\mathbf{D} \subset \mathbf{C}$ with finite intersection $\mathbf{D} \cap \{\mathrm{w} \in \mathbf{C} : c \leq \Re \mathrm{w} \leq c'\}$, such that the operators (5) define the isomorphisms

$$\sigma_{\mathrm{C}}(\mathbb{D})(\mathrm{w}) : H^{s}(\mathrm{int}\ X) \longmapsto \begin{array}{c} H^{s-2}(\mathrm{int}\ X) \\ \oplus \\ H^{s-\mu-\frac{1}{2}}(Y) \end{array}$$

for all $\mathrm{w} \in \mathbf{C} \setminus \mathbf{D}$ and all sufficiently large $s \in \mathbb{R}$. The main purpose of the present study is to determine admissible corner weights of the mixed problems, therefore, for simplicity we assume that the coefficients a_j and b_k do not depend on r. Hence for all sufficiently large $s \in \mathbb{R}$ and $\gamma \in \mathbb{R}$,

$$\mathbb{D} = \mathrm{op}_M^{\gamma-\frac{d}{2}}(\sigma_{\mathrm{C}}(\mathbb{D}))$$

(4) reduces to isomorphisms, such that $\Gamma_{\frac{d+1}{2}-\gamma} \cap \mathbf{D} = \emptyset$ with $\Gamma_\beta := \{\mathrm{w} \in \mathbf{C} : \mathrm{w} = \beta + i\tau, \tau \in \mathbb{R}\}$. We have

$$\mathrm{op}_M^{\gamma-\frac{d}{2}}(\sigma_{\mathrm{C}}(\mathbb{D})) = \frac{1}{2\pi i} \int_{\Gamma_{\frac{d+1}{2}-\gamma}} \int_0^\infty \left(\frac{r}{r'}\right)^{-\mathrm{w}} \sigma_{\mathrm{C}}(\mathbb{D})(\mathrm{w}) u(r') \frac{dr'}{r'} d\mathrm{w} =$$

$$= \frac{1}{2\pi} r^{-\frac{d+1}{2}+\gamma} \int_{-\infty}^{\infty} \int_0^\infty \left(\frac{r}{r'}\right)^{-i\tau} \sigma_{\mathrm{C}}(\mathbb{D}) \left(\frac{d+1}{2} - \gamma + i\tau\right) (r')^{\frac{d+1}{2}-\gamma} u(r') \frac{dr'}{r'} d\tau.$$

As noted above, the transformation $u(r,x) \longmapsto u(e^{-t},x)$ induces an isomorphism $\mathcal{H}^{s,\frac{d+1}{2}}(X^\wedge) \longmapsto H^s(\mathbb{R} \times \mathrm{int}\ X)$ for all $s \in \mathbb{R}$. Hence we come to an operator

$$\mathcal{D} := \mathrm{op}^\delta(|\mathfrak{d}) = F^{-1}\mathfrak{d}(w)F : H^{s,\delta}(\mathbb{R} \times \mathrm{int}\ X) \longmapsto \begin{array}{c} H^{s-2,\delta}(\mathbb{R} \times \mathrm{int}\ X) \\ \oplus \\ H^{s-\mu-\frac{1}{2},\delta}(\mathbb{R} \times Y), \end{array} \tag{6}$$

where $\mathfrak{d}(w) := {}^{t}(\mathfrak{e}(w)\,\mathfrak{t}(w)) = \sigma_{\mathrm{C}}(\mathcal{D})(\mathrm{w}), \mathrm{w} = iw$ is an isomorphism for all $\delta \in \mathbb{R}$ such that $I_\delta \cap D = \emptyset$ for $I_\beta := \{w \in \mathbb{C} : \Im w = \beta\}$, $\beta \in \mathbb{R}$ and $D = \{w \in \mathbb{C} : iw \in \mathbf{D}\}$.

For a compact C^∞ manifold M, by $L^\mu_{cl}(M;\mathbb{R}^l)$ we denote the space of all classical pseudo-differential operators of order $\mu \in \mathbb{R}$ on M depending on a parameter $\lambda \in \mathbb{R}$. Moreover, for a Fréchet space F and an open set $U \subseteq \mathbb{C}$ by $\mathcal{A}(U, F)$ we denote the space of all holomorphic functions in U with values in F.

We now employ the fact that for every constants $c \leq c'$ there exists a holomorphic operator function

$$\mathrm{r}(w) \in \mathcal{A}(\mathbb{C}, L^{s-\mu-\frac{1}{2}}_{cl}(Y)),$$

such that for every $\beta \in \mathbb{R}$

$$\mathrm{r}(\tau + i\beta) \in L_{cl}^{s-\mu-\frac{1}{2}}(Y;\mathbb{R}_\tau)$$

uniformly in compact β-intervals and for any fixed $s \in \mathbb{R}$, all $\tau \in \mathbb{R}$ and all $c \leq \beta \leq c'$

$$\mathrm{r}(\tau + i\beta) : H^{s-\mu-\frac{1}{2}}(Y) \longmapsto L^2(Y)$$

is a family of isomorphisms. Using this, one can in particular obtain an isomorphism

$$\mathrm{op}^\delta(\mathrm{r}) : H^{s-\mu-\frac{1}{2},\delta}(\mathbb{R} \times Y) \longmapsto H^{0,\delta}(\mathbb{R} \times Y), \quad \delta \in \mathbb{R}.$$

We choose $\mathrm{r}(w)$ as follows. Let $\alpha \in \mathbb{R}$ (α plays the role of $s - \mu - \frac{1}{2}$) and fix a collar neighbourhood $\cong [-1,1] \times Z$ of the interface $Z \subset Y$. Choose local coordinates $(n, z) \in [-1,1] \times U$ for an open set $U \subset \mathbb{R}^{d-2}$ with covariables $(\nu, \zeta) \in \mathbb{R}^{d-1}$ and form a symbol

$$p_-^\alpha(n,\nu,\zeta,\lambda) := \left(f\left(\frac{\nu}{C\langle\zeta,\lambda\rangle}\right)\langle\zeta,\lambda\rangle - i\nu\right)^{\alpha\omega(n)} \langle\nu,\zeta,\lambda\rangle^{\alpha(1-\omega(n))}, \tag{7}$$

where $\omega \in C_0^\infty(-1,1)$ is a real-valued function, $0 \leq \omega \leq 1$, that equals 1 in a neighbourhood of the origin, $\lambda \in \mathbb{R}^l$ and $f(\nu) \in \mathcal{S}(\mathbb{R})$ is a function such that $f(0) = 1$ and $\operatorname{supp} F^{-1}f \subset \mathbb{R}_-$ (with Fourier transform on n-axis). Then we get $p_-^\alpha(n,\nu,\zeta,\lambda) \in S_{cl}^\alpha(\mathbb{R} \times \mathbb{R}^{d-1+l}_{\nu,\zeta,\lambda})$, where p_-^α is elliptic with respect to the covariable (ν, ζ, λ) for $C > 0$ sufficiently large. After that, on Y we define a parameter-dependent elliptic operator $\mathrm{p}_-^\alpha(\lambda) \in L_{cl}^\alpha(Y;\mathbb{R}^l)$ by taking $p_-^\alpha(n,\nu,\zeta,\lambda)$ to be the local amplitude functions in the collar neighbourhood of Z and $\langle\eta,\lambda\rangle^\alpha$ outside that neighbourhood and assuming that η is the covariable on Y.

One can find in [3] the precise standard construction in terms of open coverings of Y by charts, subordinate partitions of unity, etc. Similar to that, starting from $p_+^\alpha(n,\nu,\zeta,\lambda)$ defined as the complex conjugate of (7), we obtain a family $\mathrm{p}_+^\alpha(\lambda) \in L_{cl}^\alpha(Y;\mathbb{R}^l)$. By virtue of the specific properties of the symbol (7) in a neighbourhood of Z we come to the following theorems, where $\mathrm{e}_+^s : H^s(\text{int } Y_+) \longmapsto H^s(Y)$ denotes a continuous operator such that $\mathrm{r}_+\mathrm{e}_+^s = \mathrm{id}_{H^s(\text{int } Y_+)}$, and for $s > -\frac{1}{2}$ $\mathrm{e}_+ : H^s(\text{int } Y_+) \longmapsto H^{\min(s,0)}(Y)$ is the operator of extension by 0 to the opposite side of Y.

Theorem 2.1 *There is a constant $M > 0$ such that the operators*

$$\mathrm{p}_-^\alpha(\lambda) : H^s(\text{int } Y) \longmapsto H^{s-\alpha}(Y)$$

and

$$r_+ p_-^\alpha(\lambda) e_+^s : H^s(\text{int } Y_+) \longmapsto H^{s-\alpha}(\text{int } Y_+)$$

are isomorphisms for all $\lambda \in \mathbb{R}^l$, $|\lambda| \geq M$. *For* $s > -\frac{1}{2}$ *and for all* $\lambda \in \mathbb{R}^l$, $|\lambda| \geq M$,

$$r_+ p_-^\alpha(\lambda) e_+ : H^s(\text{int } Y_+) \longmapsto H^{s-\alpha}(\text{int } Y_+)$$

is a family of isomorphisms. Besides, $(r_+ p_-^\alpha(\lambda) e_+)^{-1} = r_+ (p_-^\alpha)^{-1}(\lambda) e_+$. *The similar statements hold for* $p_+(\lambda)$ *by replacing the signes* $+$ *and* $-$.

Denoting $L_{cl}^\alpha(Y; \mathbb{C} \times \mathbb{R}^l)$ the space of all $h(w, \lambda) \in \mathcal{A}(\mathbb{C}, L_{cl}^\alpha(Y; \mathbb{R}_\lambda^l))$ such that for all $\beta \in \mathbb{R}$

$$h(\tau + i\beta, \lambda) \in L_{cl}^\alpha(Y; \mathbb{R}_{\tau,\lambda}^{1+l})$$

uniformly in compact β-intervals, we replace the parameter λ by $(\tau, \lambda) \in \mathbb{R}^{1+l}$ and consider the corresponding families $p_\pm^\alpha(\tau, \lambda)$. Then we choose some $\psi(b) \in C_0^\infty(\mathbb{R})$ such that $\psi(b) \equiv 1$ in a neighbourhood of $b = 0$ and setting

$$r_\pm^\alpha(w, \lambda) := \int_{\mathbb{R}} e^{-iwb} \left\{ \int_{\mathbb{R}} \psi(b) e^{i\tau b} p_\pm^\alpha(\tau, \lambda) đ\tau \right\} db \tag{8}$$

we obtain an operator function in $L_{cl}^\alpha(Y; \mathbb{C} \times \mathbb{R}^l)$. The following theorem is proved in [4].

Theorem 2.2. *For every two constants* $c \leq c'$ *there exists an* $M > 0$ *such that*

$$r_\pm^\alpha(w, \lambda) : H^s(Y) \longmapsto H^{s-\alpha}(Y),$$

and

$$r_+ r_-^\alpha(w, \lambda) e_+^s : H^s(\text{int } Y_+) \longmapsto H^{s-\alpha}(\text{int } Y_+),$$
$$r_- r_+^\alpha(w, \lambda) e_-^s : H^s(\text{int } Y_-) \longmapsto H^{s-\alpha}(\text{int } Y_-,)$$

are isomorphisms for all $c \leq \Im w \leq c'$ *and all* $\lambda \in \mathbb{R}^l$, $|\lambda| \geq M$. *Besides, for* $s > -\frac{1}{2}$

$$r_+ r_-^\alpha(w, \lambda) e_+ : H^s(\text{int } Y_+) \longmapsto H^{s-\alpha}(\text{int } Y_+),$$
$$r_- r_+^\alpha(w, \lambda) e_- : H^s(\text{int } Y_-) \longmapsto H^{s-\alpha}(\text{int } Y_-)$$

are families of isomorphisms for the mentioned values of w *and* λ.

We use the notation $e_\pm^s$ and $e_\pm$ for the corresponding extension operators

$$H^{s,\delta}(\mathbb{R} \times (\text{int } Y_\pm)) \longmapsto H^{s,\delta}(\mathbb{R} \times Y), \quad s \in \mathbb{R}$$

and

$$H^{s,\delta}(\mathbb{R} \times (\text{int } Y_\pm)) \longmapsto H^{\min(s,0),\delta}(\mathbb{R} \times Y), \quad s \in \mathbb{R}, \quad s > -1/2$$

respectively, where $\delta \in \mathbb{R}$ is arbitrary. Then Theorem 2.2 implies the following corollary.

Corollary 2.3. Let $\mathrm{r}_-(w,\lambda)$ denote the operator function as in Theorem 2.2. Then for any $\delta, s \in \mathbb{R}$,

$$\mathrm{op}^\delta(\mathrm{r}_-^\alpha)(\lambda) : H^{s,\delta}(\mathbb{R}\times Y) \longmapsto H^{s-\alpha,\delta}(\mathbb{R}\times Y),$$

$$\mathrm{r}_+\mathrm{op}^\delta(\mathrm{r}_-^\alpha)(\lambda)\mathrm{e}_+^s : H^{s,\delta}(\mathbb{R}\times \mathrm{int}\ Y_+) \longmapsto H^{s-\alpha,\delta}(\mathbb{R}\times \mathrm{int}\ Y_+),$$

and for any $\delta, s \in \mathbb{R}$, $s > -1/2$, the operators

$$\mathrm{r}_+\mathrm{op}^\delta(\mathrm{r}_-^\alpha)(\lambda)\mathrm{e}_+ : H^{s,\delta}(\mathbb{R}\times \mathrm{int}\ Y_+) \longmapsto H^{s-\alpha,\delta}(\mathbb{R}\times \mathrm{int}\ Y_+)$$

are isomorphisms for all $|\lambda| \geq M$ and suitably chosen $M > 0$. Similar relations obtained by an interchange of the signes + and − are also valid.

We fix some $\lambda_1 \in \mathbb{R}^l$ $(|\lambda_1| > M)$ and set $\mathrm{r}_\pm^\alpha(w) := \mathrm{r}_\pm^\alpha(w,\lambda_1)$. It is known that there exists a meromorphic inverse $(\mathrm{r}_\pm^\alpha)^{-1}(w)$ and

$$\mathrm{op}^\delta((\mathrm{r}_\pm^\alpha)^{-1}) = (\mathrm{op}^\delta(\mathrm{r}_\pm^\alpha))^{-1}.$$

Similarly, the operators

$$\mathrm{r}_+\mathrm{op}^\delta(\mathrm{r}_-^\alpha)\mathrm{e}_+^s, \quad \mathrm{r}_+\mathrm{op}^\delta(\mathrm{r}_-^\alpha)\mathrm{e}_+, \quad \mathrm{r}_-\mathrm{op}^\delta(\mathrm{r}_+^\alpha)\mathrm{e}_-^s, \quad \mathrm{r}_-\mathrm{op}^\delta(\mathrm{r}_+^\alpha)\mathrm{e}_-$$

can be inverted.

Using the operator (6) we pass to reduction of orders to 0 on the boundary. As above, we write $\partial(w) := {}^{\mathrm{t}}(e(w)\ \mathrm{t}(w))$, $\alpha = s - \mu - \frac{1}{2}$ and to be concrete, form

$$\mathrm{diag}\,(1, \mathrm{op}^\delta(\mathrm{r}_+^\alpha))\mathrm{op}^\delta(\partial) = \mathrm{op}^\delta\begin{pmatrix} e \\ \mathrm{r}_+^\alpha \mathrm{t}\end{pmatrix} : H^{s,\delta}(\mathbb{R}\times \mathrm{int}\ X) \longmapsto \begin{matrix} H^{s-2,\delta}(\mathbb{R}\times \mathrm{int}\ X) \\ \oplus \\ L^{2,\delta}(\mathbb{R}\times Y). \end{matrix}$$

Along with the restriction operators $\mathrm{r}_\pm$ to $\mathbb{R}\times \mathrm{int}\ Y_\pm$ and the extensions $\mathrm{e}_\pm$ by zero to $\mathbb{R}\times \mathrm{int}\ Y$ we have an isomorphism

$$(\mathrm{e}_- \quad \mathrm{e}_+) : \begin{matrix} L^{2,\delta}(\mathbb{R}\times Y_-) \\ \oplus \\ L^{2,\delta}(\mathbb{R}\times Y_+) \end{matrix} \longmapsto L^{2,\delta}(\mathbb{R}\times Y)$$

with the inverse ${}^{\mathrm{t}}(\mathrm{r}_- \quad \mathrm{r}_+)$. Similarly as in the calculus of pseudo-differential boundary value problems, the operator function $\mathfrak{d}(w)$ has a meromorphic inverse $\mathfrak{d}^{-1}(w) =: (\mathrm{g}(w)\ \mathrm{k}(w))$.

Remark 2.4. It is known that the Laurent coefficients of $\mathfrak{d}^{-1}$ are smoothing operators of finite rank, more precisely, smoothing in the calculus of boundary value problems on X with the transmission property at Y.

Let us now form the operator

$$\mathcal{L} := (\mathrm{op}^\delta(\mathrm{g})\ \mathrm{op}^\delta(\mathrm{k}(\mathrm{r}_+^\alpha)^{-1})\mathrm{e}_-\ \mathrm{op}^\delta(\mathrm{k}(\mathrm{r}_+^\alpha)^{-1})\mathrm{e}_+) : \begin{array}{c} H^{s-2,\delta}(\mathbb{R}\times \operatorname{int} X) \\ \oplus \\ L^{2,\delta}(\mathbb{R}\times Y_-) \\ \oplus \\ L^{2,\delta}(\mathbb{R}\times Y_+) \end{array} \longmapsto H^{s,\delta}(\mathbb{R}\times \operatorname{int} X),$$

which is an isomorphism (recall that $\alpha = s - \mu - 1/2$). It is evident that multiplying $\mathcal{L}$ by $\mathcal{A}$ from the left (cf. formula (2)) we obtain the operator

$$\mathcal{AL} : \begin{array}{c} H^{s-2,\delta}(\mathbb{R}\times \operatorname{int} X) \\ \oplus \\ L^{2,\delta}(\mathbb{R}\times Y_-) \\ \oplus \\ L^{2,\delta}(\mathbb{R}\times Y_+) \end{array} \longmapsto \begin{array}{c} H^{s-2,\delta}(\mathbb{R}\times \operatorname{int} X) \\ \oplus \\ H^{s-\mu_- -\frac{1}{2},\delta}(\mathbb{R}\times \operatorname{int} Y_-) \\ \oplus \\ H^{s-\mu_+ -\frac{1}{2},\delta}(\mathbb{R}\times \operatorname{int} Y_+) \end{array} \tag{9}$$

By virtue of $\mathcal{D}\mathcal{D}^{-1} = \operatorname{diag}(1,1)$ we obtain the operator $\mathcal{AL}$ in the form

$$\mathcal{AL} = \begin{pmatrix} 1 & 0 & 0 \\ T_-G & T_-KR^{-1}\mathrm{e}_- & T_-KR^{-1}\mathrm{e}_+ \\ T_+G & T_+KR^{-1}\mathrm{e}_- & T_+KR^{-1}\mathrm{e}_+ \end{pmatrix},$$

where we employ the abbreviation $G := \mathrm{op}^\delta(\mathrm{g})$, $K := \mathrm{op}^\delta(\mathrm{k})$, $R := \mathrm{op}^\delta(\mathrm{r}_+^\alpha), \alpha = s - \mu - \frac{1}{2}$. We also want to reduce the Sobolev spaces on the $\mathbb{R}\times \operatorname{int} Y_\mp$ on the right of (9) to zero. To this end we take the elements $\mathrm{r}_\pm^{\alpha_\mp}(w), \alpha_\mp = s - \mu_\mp - \frac{1}{2}$ and for $s > \max\{\mu_+, \mu_-\}$ set

$$R_- := \mathrm{r}_-\mathrm{op}^\delta(\mathrm{r}_+^{\alpha-})\mathrm{e}_-, \quad R_+ := \mathrm{r}_+\mathrm{op}^\delta(\mathrm{r}_-^{\alpha+})\mathrm{e}_+.$$

Setting $\mathcal{R} := \operatorname{diag}(1, R_-, R_+)$, and multiplying (9) from the left by $\mathcal{R}$ we get an operator

$$\mathcal{A}_0 := \mathcal{RAL} = \begin{pmatrix} 1 & 0 & 0 \\ R_-T_-G & R_-T_-KR^{-1}\mathrm{e}_- & R_-T_-KR^{-1}\mathrm{e}_+ \\ R_+T_+G & R_+T_+KR^{-1}\mathrm{e}_- & R_+T_+KR^{-1}\mathrm{e}_+ \end{pmatrix}$$

with the 2×2 lower right corner

$$\begin{pmatrix} R_-T_-KR^{-1}\mathrm{e}_- & R_-T_-KR^{-1}\mathrm{e}_+ \\ R_+T_+KR^{-1}\mathrm{e}_- & R_+T_+KR^{-1}\mathrm{e}_+ \end{pmatrix} : \begin{array}{c} L^{2,\delta}(\mathbb{R}\times Y_-) \\ \oplus \\ L^{2,\delta}(\mathbb{R}\times Y_+) \end{array} \longmapsto \begin{array}{c} L^{2,\delta}(\mathbb{R}\times Y_-) \\ \oplus \\ L^{2,\delta}(\mathbb{R}\times Y_+) \end{array}. \tag{10}$$

The latter operator represents the reduction of our mixed problem to the boundary, combined with suitable reductions of orders.

§3. ELLIPTICITY WITH INTERFACE CONDITIONS

We assume that the boundary condition T_- is the restriction of T to int Y_-, that means $\mu = \mu_-$ or $\alpha = \alpha_-$. In that case, since the order reducing operators R and R_- are connected by the relation $R_- = \mathrm{r}_- R\mathrm{e}_-$, we obtain

$$R_- T_- K R^{-1} \mathrm{e}_- = \mathrm{id}_{L^{2,\delta}(\mathbb{R}\times Y_-)} \tag{11}$$

and

$$R_- T_- K R^{-1} \mathrm{e}_+ = 0. \tag{12}$$

In fact, from $T = \mathrm{r}B$ for a differential operator B in a neighbourhood of Y it follows that $T_- = \mathrm{r}_- \mathrm{r}B$. This implies that $\mathrm{r}BK = 1$ and

$$R_- T_- K R^{-1} \mathrm{e}_- = \mathrm{r}_- R \mathrm{e}_- \mathrm{r}_- R^{-1} \mathrm{e}_-,$$

i.e. we obtain (11). Moreover, (12) is equal to $\mathrm{r}_- R\mathrm{e}_- \mathrm{r}_- R^{-1}\mathrm{e}_+ = 0$, because

$$\mathrm{r}_- R^{-1}\mathrm{e}_+ = \mathrm{r}_- \mathrm{op}^\delta(\mathrm{r}_+^{-\alpha})\mathrm{e}_+ = 0.$$

Thus the operator (10) is a triangular matrix with the lower right corner

$$F :\doteq R_+ T_+ K R^{-1} \mathrm{e}_+ : L^{2,\delta}(\mathbb{R}\times Y_+) \longmapsto L^{2,\delta}(\mathbb{R}\times Y_+). \tag{13}$$

The operator (13) can be written in the form $F = \mathrm{op}^\delta(\mathrm{f})$ for a meromorphic operator family

$$\mathrm{f}(w) = \mathrm{r}_+ \mathrm{r}_-^{\alpha_+}(w)\mathrm{e}_+ \mathrm{t}_+(w)\mathrm{k}(w)\mathrm{r}_+^{-\alpha_-}(w)\mathrm{e}_+ : L^2(Y_+) \longmapsto L^2(Y_+).$$

The operators $\mathrm{f}(w)$ are parameter-dependent elliptic of order zero, with the parameter $\Re w = \tau \in \mathbb{R}$. The homogeneous principal boundary symbol $\sigma_\partial(\mathrm{f})(z,\tau,\zeta)$ is a family of continuous operators

$$\sigma_\partial(\mathrm{f})(z,\tau,\zeta) : L^2(\mathbb{R}_+) \longmapsto L^2(\mathbb{R}_+), \tag{14}$$

independent of the choice of δ and homogeneous in the sense

$$\sigma_\partial(\mathrm{f})(z,\lambda\tau,\lambda\zeta) = \sigma_\partial(\mathrm{f})(z,\tau,\zeta)$$

for every $\lambda \in \mathbb{R}_+$, $(\tau,\zeta) \neq 0$. By construction, the operator family $f(w)$ depends on $s \in \mathbb{R}$. We now assume that $s \in \mathbb{R}$ is chosen in such a way that (14) is a family of Fredholm operators for all $(\tau,\zeta) \neq 0$. The sufficient property, that the subordinate conormal symbol has no zeros on the line $\Gamma_{\frac{1}{2}}$, will be checked in a concrete example below. In the case of the Fredholm property we have a K-theoretic index element

$$\operatorname{ind}_{S^*Z}\sigma_\partial(\mathbf{f}) \in K(S^*Z),$$

where S^*Z is defined to be the compact space $\{(z,\tau,\zeta) \in \mathbb{R}\times T^*Z : |\tau,\zeta| = 1\}$ with the canonical projection $\pi_1 : S^*Z \longmapsto Z$. Another condition we impose

$$\operatorname{ind}_{S^*Z}\sigma_\partial(\mathbf{f}) \in \pi_1^* K(Z).$$

For suitable $J_\pm \in \operatorname{Vect}(Z)$ there is a block matrix family of isomorphisms

$$\begin{pmatrix} \sigma_\partial(\mathbf{f})(z,\tau,\zeta) & \sigma_\partial(\mathbf{k})(z,\tau,\zeta) \\ \sigma_\partial(\mathbf{t})(z,\tau,\zeta) & \sigma_\partial(\mathbf{q})(z,\tau,\zeta) \end{pmatrix} : \pi_1^* \begin{pmatrix} L^2(\mathbb{R}_+) \\ \oplus \\ J_- \end{pmatrix} \longmapsto \pi_1^* \begin{pmatrix} L^2(\mathbb{R}_+) \\ \oplus \\ J_+ \end{pmatrix}$$

between the corresponding pull backs with respect to π_1.

We now choose a system of charts $\chi_j : U_j \longmapsto \mathbb{R}^{d-2}$ on Z for an open covering $(U_j)_{j=1,\dots,N}$ of Z. Let $(\varphi_j)_{j=1,\dots,N}$ be a subordinate partition of unity and $(\psi_j)_{j=1,\dots,N}$ a system of functions $\psi_j \in C_0^\infty(U_j)$ such that $\psi_j \equiv 1$ on $\operatorname{supp}\varphi_j$ for all j. Moreover, let $\sigma, \tilde\sigma \in C^\infty(Y_+)$ be supported in a collar neighbourhood of Z, $\tilde\sigma \equiv 1$ in a neighbourhood of $\operatorname{supp}\sigma$, and $\sigma \equiv 1$ in a neighbourhood of Z. We then define the operator family

$$\sum_{j=1}^N \begin{pmatrix} \sigma\varphi_j(\chi_j^* \times \mathrm{id}) & 0 \\ 0 & \varphi_j\chi_j^* \end{pmatrix} \operatorname{Op}_z(g_j)(\tau) \begin{pmatrix} (\chi_j^* \times \mathrm{id})^{-1}\tilde\sigma\psi_j & 0 \\ 0 & (\chi_j^*)^{-1}\psi_j \end{pmatrix} \tag{15}$$

where $g_j(z,\tau,\zeta)$ is given by

$$\chi(\tau,\zeta)\begin{pmatrix} 0 & \sigma_\partial(\mathbf{k}) \\ \sigma_\partial(\mathbf{t}) & \sigma_\partial(\mathbf{q}) \end{pmatrix}(z,\tau,\zeta)$$

in local coordinates with respect to the charts $\chi_j : U_j \longmapsto \mathbb{R}^{d-2}$ and $\chi_j \times \mathrm{id} : U_j \times [0,1) \longmapsto \mathbb{R}^{d-2} \times \overline{\mathbb{R}}_+$ on Z and in a collar neighbourhood of Z with the normal variable in $[0,1)$.

Now (15) is a family of block matrix operators

$$g(\tau) := \begin{pmatrix} 0 & g_{12} \\ g_{21} & g_{22} \end{pmatrix}(\tau) : \begin{matrix} L^2(Y_+) \\ \oplus \\ L^2(Z,J_-) \end{matrix} \longmapsto \begin{matrix} L^2(Y_+) \\ \oplus \\ L^2(Z,J_+) \end{matrix}.$$

Given some constant $C > 0$, for $|\tau| > C$,

$$\begin{pmatrix} \mathbf{f}(\tau + i\delta) & g_{12}(\tau) \\ g_{21}(\tau) & g_{22}(\tau) \end{pmatrix} \tag{16}$$

is a family of Fredholm operators defining isomorphisms. Similarly as (8) we now pass to a holomorphic operator function

$$\mathrm{g}(w) := \int_{\mathbb{R}} e^{-iwb} \left\{ \int \psi(b) e^{-i\tau b} g(\tau) đ\tau \right\} db$$

for a $\psi \in C_0^\infty(\mathbb{R})$ that is equal to 1 near the origin. We then have $\mathrm{g}(w) = (\mathrm{g}_{ij}(w))_{i,j=1,2}$ with $\mathrm{g}_{11}(w) = 0$. This gives us a family of operators

$$\begin{pmatrix} \mathrm{f}(w) & \mathrm{g}_{12}(w) \\ \mathrm{g}_{21}(w) & \mathrm{g}_{22}(w) \end{pmatrix} : \begin{matrix} L^2(Y_+) \\ \oplus \\ L^2(Z, J_-) \end{matrix} \longmapsto \begin{matrix} L^2(Y_+) \\ \oplus \\ L^2(Z, J_+) \end{matrix} \tag{17}$$

meromorphic in $w \in \mathbb{C}$.

Proposition 3.1. *There exists a discrete set $M \subset \mathbb{R}$ such that (17) is a family of isomorphisms for all $w = \tau + i\delta, \tau \in \mathbb{R}, \delta \in \mathbb{R} \setminus M$.*

Proof : The family (17) is parameter-dependent elliptic in the class of boundary value problems on Y_+ (of order zero and without the transmission property at Z), cf. [2], [8], with parameter $\tau = \Re\, w$. The meromorphy is clear by construction, in fact $\mathrm{g}_{ij}(w)$ are even holomorphic for all i, j. Let us assume for the moment that $\mathrm{f}(w)$ is holomorphic in the complex plane. The operators (16) are parameter-dependent elliptic and the principal parameter-dependent interior and boundary symbols are independent of δ. The same is true for (17), i.e., (17) is Fredholm for every $w \in \mathbb{C}$ and a holomorphic operator function. Moreover, there is a constant $c > 0$ such that (16) are isomorphisms for all $|\tau| > c$. Thus our operator function satisfies a well known condition on holomorphic Fredholm families which are isomorphic for at least one value of the complex parameter. This gives us the invertibility for all w with $\Im\, w$ outside some discrete set.

In the case where $\mathrm{f}(w)$ is meromorphic we can argue in a similar manner taking into account that the Laurent coefficients are smoothing and of finite rank, see Remark 2.4. This completes the proof.

So (17) is a meromorphic operator function invertible for all $w \in \mathbb{C} \setminus N$, where $N \subset \mathbb{C}$ is a discrete set such that $N \cap \{c \leq \Im\, w \leq c'\}$ is finite for every $c \leq c'$. Now we pass from the symbol $\mathrm{a}(w) : H^s(\mathrm{int}\, X) \longmapsto \tilde{H}^{s-2}(\mathrm{int}\, X)$ for $\tilde{H}^{s-2}(\mathrm{int}\, X) := H^{s-2}(\mathrm{int}\, X) \oplus H^{s-\mu_- -\frac{1}{2}}(\mathrm{int}\, Y_-) \oplus H^{s-\mu_+ -\frac{1}{2}}(\mathrm{int}\, Y_+)$ to an operator function $\tilde{\mathrm{a}}(w)$ by adding extra entries of trace and potential type such that $\tilde{\mathrm{a}}(w) : H^s(\mathrm{int}\, X) \oplus L^2(Z, J_-) \longmapsto \tilde{H}^{s-2}(\mathrm{int}\, X) \oplus L^2(Z, J_+)$ are meromorphic and invertible. To this end we form the block matrix operator family

$$\begin{pmatrix} 1 & 0 & 0 \\ \mathfrak{m}(w) & \mathfrak{f}(w) & \mathfrak{g}_{12}(w) \\ 0 & \mathfrak{g}_{21}(w) & \mathfrak{g}_{22}(w) \end{pmatrix} : \begin{matrix} L^2(Y_-) \\ \oplus \\ L^2(Y_+) \\ \oplus \\ L^2(Z, J_-) \end{matrix} \longmapsto \begin{matrix} L^2(Y_-) \\ \oplus \\ L^2(Y_+) \\ \oplus \\ L^2(Z, J_+) \end{matrix} \tag{18}$$

for the meromorphic operator function

$$\mathfrak{m}(w) := \mathfrak{r}_+ \mathfrak{r}_-^{\alpha_+}(w) \mathfrak{e}_+ \mathfrak{t}_+(w) \mathfrak{k}(w) \mathfrak{r}_+^{-\alpha_-}(w) \mathfrak{e}_-,$$

which has the property

$$\mathrm{op}^\delta(\mathfrak{m}) = R_+ T_+ K R^{-1} \mathfrak{e}_-.$$

Moreover, for

$$\mathfrak{n}_\pm(w) := \mathfrak{r}_\pm \mathfrak{r}_\mp^{\alpha_\pm}(w) \mathfrak{e}_\pm \mathfrak{t}_\pm(w) \mathfrak{g}(w)$$

we have $\mathrm{op}^\delta(\mathfrak{n}_\pm) = R_\pm T_\pm G$.
Setting

$$\tilde{\mathfrak{a}}_0(w) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \mathfrak{n}_-(w) & 1 & 0 & 0 \\ \mathfrak{n}_+(w) & \mathfrak{m}(w) & \mathfrak{f}(w) & \mathfrak{g}_{12}(w) \\ 0 & 0 & \mathfrak{g}_{21}(w) & \mathfrak{g}_{22}(w) \end{pmatrix} : \begin{matrix} H^{s-2}(\mathrm{int}\, X) \\ \oplus \\ L^2(Y_-) \\ \oplus \\ L^2(Y_+) \\ \oplus \\ L^2(Z, J_-) \end{matrix} \longmapsto \begin{matrix} H^{s-2}(\mathrm{int}\, X) \\ \oplus \\ L^2(Y_-) \\ \oplus \\ L^2(Y_+) \\ \oplus \\ L^2(Z, J_+) \end{matrix}$$

yields an operator $\tilde{\mathcal{A}}_0 = \mathrm{op}^\delta(\tilde{\mathfrak{a}}_0)$ that has $\mathcal{A}_0$ at the upper left corner.
Setting

$$\mathfrak{l}(w) := \mathrm{diag}\,(\mathfrak{g}(w), \mathfrak{k}(w)(\mathfrak{r}_+^\alpha(w))^{-1}\mathfrak{e}_-, \mathfrak{k}(w)(\mathfrak{r}_-^\alpha(w))^{-1}\mathfrak{e}_+),$$

$$\mathfrak{r}(w) := \mathrm{diag}\,(1, \mathfrak{r}_- \mathfrak{r}_+^{\alpha_-}(w)\mathfrak{e}_-, \mathfrak{r}_+ \mathfrak{r}_-^{\alpha_+}(w)\mathfrak{e}_+),$$

we have $\mathcal{L} = \mathrm{op}^\delta(\mathfrak{l})$ and $\mathcal{R} = \mathrm{op}^\delta(\mathfrak{r})$. Moreover, let

$$\tilde{\mathfrak{l}}(w) := \mathrm{diag}\,(\mathfrak{l}(w), \mathrm{id}_{L^2(Z,J_-)}),$$

$$\tilde{\mathfrak{r}}(w) = \mathrm{diag}\,(\mathfrak{r}(w), \mathrm{id}_{L^2(Z,J_+)}).$$

We then obtain an operator function

$$\tilde{\mathrm{a}}(w) := \tilde{\mathrm{r}}^{-1}(w)\tilde{\mathrm{a}}_0(w)\tilde{\mathrm{l}}^{-1}(w) : \begin{array}{c} H^s(\mathrm{int}\, X) \\ \oplus \\ L^2(Z, J_-) \end{array} \longmapsto \begin{array}{c} H^{s-2}(\mathrm{int}\, X) \\ \oplus \\ H^{s-\mu_- - \frac{1}{2}}(\mathrm{int}\, Y_-) \\ \oplus \\ H^{s-\mu_+ - \frac{1}{2}}(\mathrm{int}\, Y_+) \\ \oplus \\ L^2(Z, J_+) \end{array} \tag{19}$$

Remark 3.2. We have

$$\tilde{\mathrm{a}}(w) = \begin{pmatrix} \mathrm{a}(w) & \mathrm{k}_Z(w) \\ \mathrm{t}_Z(w) & \mathrm{q}_Z(w) \end{pmatrix},$$

where $\mathrm{a}(w)$ is the symbol of the original mixed problem (2). The other entries play the role of trace, potential, etc., symbols with respect to the interface Z.

Theorem 3.3. *There exists a discrete set $N \subset \mathbb{C}, N \cap \{c \leq \Im w \leq c'\}$ finite for every $c \leq c'$, such that $\tilde{\mathrm{a}}(w)$ is a family of isomorphisms for all $w \in \mathbb{C} \setminus N$.*
Proof : Since from Proposition 3.1 we have an operator function of the asserted kind, (18) as well as $\tilde{\mathrm{a}}_0(w)$ also have that property. It remains to note, that the factors at $\tilde{\mathrm{a}}_0(w)$ on the left hand side of (19) preserve this structure.

Corollary 3.4. The operator

$$\tilde{\mathcal{A}} := \mathrm{op}^\delta(\tilde{\mathrm{a}}) : \begin{array}{c} H^s(\mathbb{R} \times \mathrm{int}\, X) \\ \oplus \\ L^2(\mathbb{R} \times Z, J_-) \end{array} \longmapsto \begin{array}{c} H^{s-2}(\mathbb{R} \times \mathrm{int}\, X) \\ \oplus \\ H^{s-\mu_- - \frac{1}{2}}(\mathbb{R} \times \mathrm{int}\, Y_-) \\ \oplus \\ H^{s-\mu_+ - \frac{1}{2}}(\mathbb{R} \times \mathrm{int}\, Y_+) \\ \oplus \\ L^2(\mathbb{R} \times Z, J_+) \end{array}$$

is an isomorphism for all $\delta \in \mathbb{R}$ such that $I_\delta \cap N = \emptyset$.

§4. THE ZAREMBA PROBLEM

Let us consider the Zaremba problem

$$\mathcal{A} = \begin{pmatrix} \Delta \\ T_- \\ T_+ \end{pmatrix} : H^{s,\delta}(\mathbb{R} \times \mathrm{int}\, X) \longmapsto \begin{array}{c} H^{s-2,\delta-2}(\mathbb{R} \times \mathrm{int}\, X) \\ \oplus \\ H^{s-\frac{1}{2},\delta}(\mathbb{R} \times \mathrm{int}\, Y_-) \\ \oplus \\ H^{s-\frac{3}{2},\delta-1}(\mathbb{R} \times \mathrm{int}\, Y_+) \end{array}$$

on a cylinder $\mathbb{R} \times X$ for the Laplace operator Δ with Dirichlet and Neumann conditions on Y_- and Y_+, respectively, where $X := \{x = (x_1, x_2) \in \mathbb{R}^2 : |x| \leq 1\}$. We identify $\mathbb{R}^2$ with the complex plane, $Y_- := \{x = e^{i\phi} : 0 \leq \phi \leq \alpha\}, Y_+ := \{x = e^{i\phi} : \alpha \leq \phi \leq 2\pi\}$ for some $0 < \alpha < 2\pi$. We have

$$\Delta v = e^{2t}\{\partial_t^2 v - \partial_t v + \Delta_X v\},$$

$$T_- v = v(t, e^{i\phi})|_{0 \leq \phi \leq \alpha}, \tag{20}$$

$$T_+ v = \rho^{-1}\partial_\phi v(t, e^{i\phi})|_{\alpha \leq \phi \leq 2\pi}, \quad v(t,x) \in H^{s,\delta}(\mathbb{R} \times \operatorname{int} X),$$

where ρ is the exterior normal direction to Y, and

$$\mathcal{A} = \begin{pmatrix} A \\ T_- \\ T_+ \end{pmatrix} : H^{s,\delta}(\mathbb{R} \times \operatorname{int} X) \longmapsto \begin{matrix} H^{s-2,\delta-2}(\mathbb{R} \times \operatorname{int} X) \\ \oplus \\ H^{s-\frac{1}{2},\delta}(\mathbb{R} \times \operatorname{int} Y_-) \\ \oplus \\ H^{s-\frac{1}{2},\delta}(\mathbb{R} \times \operatorname{int} Y_+) \end{matrix}$$

for every fixed δ and all $s \in \mathbb{R}, s > \frac{3}{2}$. After the diffeomorphism

$$H^{s,\delta}(\mathbb{R} \times X) \longmapsto \mathcal{H}^{s,\gamma}(X^\wedge), \quad \mathbb{R} \longmapsto \mathbb{R}_+, \quad e^{-t} \longmapsto r,$$

$\gamma = \delta + \frac{3}{2}$, the operators in (20) take the form

$$\underline{\Delta} u = r^{-2}\{(r\partial_r)^2 u + r\partial_r u + \Delta_X u\},$$

$$\mathbf{T}_- u = u(r, e^{i\phi})|_{0 \leq \phi \leq \alpha}, \quad \mathbf{T}_+ u = \rho^{-1}\partial_\phi u(r, e^{i\phi})|_{\alpha \leq \phi \leq 2\pi},$$

and we get the continuous operators

$$\mathbf{A} = \begin{pmatrix} \underline{\Delta} \\ \mathbf{T}_- \\ \mathbf{T}_+ \end{pmatrix} : \mathcal{H}^{s,\gamma}((\operatorname{int} X)^\wedge) \longmapsto \begin{matrix} \mathcal{H}^{s-2,\gamma-2}((\operatorname{int} X)^\wedge) \\ \oplus \\ \mathcal{H}^{s-\frac{1}{2},\gamma}((\operatorname{int} Y_-)^\wedge) \\ \oplus \\ \mathcal{H}^{s-\frac{3}{2},\gamma}((\operatorname{int} Y_+)^\wedge) \end{matrix}$$

$v(t,x) = u(e^{-t}, x)$, for every fixed γ and for all $s \in \mathbb{R}, s > \frac{3}{2}$. The corresponding conormal symbols have the form

$$\sigma_{\mathrm{c}}(\underline{\Delta})(\mathrm{w})u = \mathrm{w}^2 u - \mathrm{w}u + \Delta_X u, \quad \sigma_{\mathrm{c}}(\mathbf{T}_-)u = u(e^{i\varphi})|_{0 \leq \varphi \leq \alpha},$$

$$\sigma_{\mathrm{c}}(\mathbf{T}_+)u = \rho^{-1}\partial_\phi u|_{\alpha \leq \varphi \leq 2\pi}, \quad u \in H^s(X).$$

Let us take as another boundary operator $\mathrm{r}u := \mathrm{T}u = u(r, e^{i\phi})$ which represents the Dirichlet condition on Y. Then

$$\mathcal{D} = \begin{pmatrix} (r\partial_r)^2 + r\partial_r + \Delta_X \\ \mathrm{r} \end{pmatrix} : \mathcal{H}^{s,\gamma}((\mathrm{int}\ X)^\wedge) \longmapsto \begin{matrix} \mathcal{H}^{s-2,\gamma-2}((\mathrm{int}\ X)^\wedge) \\ \oplus \\ \mathcal{H}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(Y^\wedge) \end{matrix}$$

for all $s, \gamma \in \mathbb{R}, s > \frac{1}{2}$, and

$$\sigma_{\mathrm{C}}(\mathcal{D})(\mathrm{w})u = \begin{pmatrix} \mathrm{w}^2 u - \mathrm{w}u + \Delta_X u \\ \mathrm{r}u \end{pmatrix}, \quad u \in H^s(X).$$

We have

$$\mathbb{D} = \mathrm{op}_M^{\gamma-1}(\sigma_{\mathrm{C}}(\mathbb{D}))$$

and

$$\mathcal{D} = \mathrm{op}^\delta(\mathfrak{d}), \quad \mathfrak{d}(w) = {}^{\mathrm{t}}(\mathfrak{e}(w)\ \mathfrak{t}(w)),$$

where

$$\mathfrak{e}(w) = -w^2 - iw + \Delta_X, \quad \mathfrak{t}(w) = \mathrm{r}.$$

According to [7] (Section 11.1), the symbol $\mathfrak{d}(w)$ defines the isomorphisms

$$H^s(\mathrm{int}\ X) \longmapsto \begin{matrix} H^{s-2}(\mathrm{int}\ X) \\ \oplus \\ H^{s-\frac{1}{2}}(Y) \end{matrix}, \quad w = \tau + i\delta, \quad \delta \in [-1, 0].$$

Given such a δ, we have $\alpha = \alpha_- = s - \frac{1}{2}$, $\alpha_+ = s - \frac{3}{2}$, and as order reduction operators we take

$$\mathrm{r}_-^{s-\frac{3}{2}}(w) = (f(\frac{\nu}{C\langle\tau\rangle})\langle\tau\rangle - i\nu)^{s-\frac{3}{2}},$$

$$\mathrm{r}_+^{-s+\frac{1}{2}}(w) = \overline{(f(\frac{\nu}{C\langle\tau\rangle})\langle\tau\rangle - i\nu)^{-s+\frac{1}{2}}}, \quad w = \tau + i\delta.$$

The corresponding family (14) is a family of Fredholm operators for all $(\tau, \zeta) \neq 0$ if $s \notin \mathbb{Z} + \frac{1}{2}$, cf. Proposition 3.1 in [3].
In our example we have

$$\sigma_\partial(f)(\tau) = \mathrm{r}^+\mathrm{op}(b)(\tau)\mathrm{e}^+, \tag{21}$$

where

$$b(\nu, \tau) = (f(\frac{\nu}{C|\tau|})|\tau| - i\nu)^{s-\frac{3}{2}}|\nu, \tau|\overline{(f(\frac{\nu}{C|\tau|})|\tau| - i\nu)^{-s+\frac{1}{2}}}.$$

According to the result from Section 3.1 of [3], the operator (21) is

(i) bijective for $\frac{1}{2} < s < \frac{1}{2}$,
(ii) surjective for $\frac{1}{2} < s + j < \frac{3}{2}$, $j \in \mathbb{N}$, where $\dim \ker \sigma_\theta(f)(\tau) = j$,
(iii) injective for $\frac{1}{2} < s + j < \frac{3}{2}, -j \in \mathbb{N}$, where $\dim \operatorname{coker} \sigma_\theta(f)(\tau) = -j$.
Hence there is a family of isomorphisms

$$(\sigma_\theta(\mathbf{f})(\tau) \quad \sigma_\theta(\mathbf{k})(\tau)) : \begin{array}{c} L^2(\mathbb{R}_+) \\ \oplus \\ Z \times \mathbb{C}^{j_-} \end{array} \longmapsto L^2(\mathbb{R}_+), \quad j_- := [s - \frac{1}{2}].$$

Remark 4.1. In problems of Zaremba type, given as meromorphic families of conormal symbols there is a parameter-dependent case, where w is replaced by (w, λ) and $(\Re w, \lambda) \in \mathbb{R}^{1+l}$ is the parameter. The extra conditions (of potential type) can also depend on λ. Then for every weight δ there is a λ such that $\bar{\mathbf{a}}(w, \lambda)$ (the parameter-dependent version of $\bar{\mathbf{a}}(w)$) is a family of isomorphisms (19) for all $\Re w = \delta$, whenever $|\lambda|$ is chosen sufficiently large.

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Поступила 3 марта 2006