Известия НАН Армении. Математика, 41, № 4, 2006, 3–6

COMPLETE HYPERSURFACES IN S^{n+1} WITH NONNEGATIVE WEYL TENSOR : NON-SYMMETRIC ROOTS IN THE UNIT DISC

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The paper presents a simple proof of a theorem by Q-M. Cheng on compact locally conformally flat Riemannian manifolds and shows that a complete connected oriented hypersurface in S^{n+1} with nonnegative Weyl tensor and Ricci curvature is isometric to a space form or a Riemannian product $S^{n-1} \times S^1$.

§1. INTRODUCTION

A well known result in Riemannian geometry states that every compact n-dimensional Riemannian manifold can be deformed to a Riemannian manifold with constant scalar curvature by a conformal transformation (see [3], [5], [6]). In [4], Tani proved that any compact connected oriented locally conformally flat n-dimensional Riemannian manifold M^n with constant scalar curvature is isometric to a space form if the Ricci curvature of M^n be positive. On the other hand, Cheng in [1] completely classified compact connected oriented locally conformally flat n-dimensional Riemannian manifolds with both the scalar curvature r constant and the Ricci curvature nonnegative and proved the following theorem.

Theorem (Cheng [1]). *Let M^n be a compact connected oriented locally conformally flat n-dimensional Riemannian manifold with constant scalar curvature. If the Ricci curvature of M^n is nonnegative, then M^n is isometric to a space form or a Riemannian product $S^{n-1}(c) \times S^1$.*

In this paper we present a simple proof of the above theorem. Further, we prove that if M^n is a complete connected oriented hypersurface in the sphere S^{n+1} with scalar curvature $r \geq n(n-1)$, nonnegative Ricci curvature and nonnegative Weyl tensor, then M^n is isometric to a space form or a Riemannian product $S^{n-1}(c) \times S^1$; that is, we obtain the following theorems.

Theorem 1. *Let M^n be a complete connected oriented locally conformally flat n-dimensional Riemannian manifold with constant scalar curvature. If the Ricci curvature of M^n is nonnegative, then M^n is isometric to a space form or a Riemannian product $S^{n-1}(c) \times S^1$.*

Theorem 2. *Let M^n be a complete connected oriented hypersurface in S^{n+1} with nonnegative Weyl tensor and scalar curvature. If Ricci curvature is nonnegative, then M^n is isometric to a space form or a Riemannian product $S^{n-1}(c) \times S^1$.*

§2. PRELIMINARIES

Let M^n be a complete hypersurface in the sphere S^{n+1}. We denote by $\overline{\nabla}$ and ∇ the Riemannian connections on S^{n+1} and M^n, respectively. Then we have

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \overline{\nabla}_X N = -AX \tag{1}$$

where $X, Y \in \chi(M)$, N is the normal vector field and h is the second fundamental form on M^n. Also A is a shape operator that satisfies

$$\langle AX, Y\rangle = \langle h(X,Y), N\rangle, \quad X, Y \in \chi(M).$$

For hypersurface M^n the *Gauss equation* is

$$R(X,Y)Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y + \langle AY, Z\rangle AX - \langle AX, Z\rangle AY \tag{2}$$

for $X, Y, Z \in \chi(M)$,where R is the curvature tensor corresponding to the connection ∇; that is, $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$. The Ricci curvature of M^n is defined by $Ric(X,Y) = \sum_i \langle R_{XE_i}Y, E_i\rangle$, where $\{E_1, E_2, ..., E_n\}$ is a local frame of orthonormal vector fields on M^n. The mean curvature and square norm of the second fundamental form of M^n are defined by $nH = \sum_i h_{ii}$, $S = \sum_{i,j} h_{ij}^2$, where h_{ij} denotes the components of the second fundamental form, i.e. $h_{ij} = \langle h(E_i, E_j), N\rangle$. From the above equation, we have

$$R_{ij} = (n-1)\delta_{ij} + nHh_{ij} - \sum_k h_{ik}h_{kj}, \tag{3}$$

$$r = n(n-1) + n^2H^2 - S, \tag{4}$$

where R_{ij} and r are components of the Ricci curvature tensor and the scalar curvature of M^n, respectively.

A Riemannian manifold M^n is called *locally conformally flat*, if every point on M^n admits a coordinate neighborhood with coordinates $x_1, \ldots, x_n$ in which the Riemannian metric can be expressed as

$$ds^2 = \sum_{i=1}^{n} \lambda^2(x) dx_i^2,$$

where $\lambda(x)$ is a positive function defined in the coordinate neighborhood. We choose a local frame of orthonormal vector fields $\{E_1, E_2, \ldots, E_n\}$ adapted to the Riemannian metric of M^n. Let C_{ijkl} denote the components of the Weyl tensor of M^n ; that is

$$C_{ijkl} = R_{ijkl} - \frac{1}{n-2}(R_{ik}\delta_{jl} - R_{il}\delta_{jk} + R_{jl}\delta_{ik} - R_{jk}\delta_{il}) + \frac{r}{(n-1)(n-2)}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}).$$

Also, the Beck tensor with the components C_{ijk} is defined by

$$C_{ijk} = \frac{1}{n-2}(R_{ij,k} - R_{ik,j}) - \frac{1}{2(n-1)(n-2)}(\delta_{ij} r_k - \delta_{ik} r_j),$$

where $R_{ij,k} = \nabla_{E_k} R_{ij}$ and $r_k = \nabla_{E_k} r$. For $n \geq 4$, M^n is know to be a locally conformally flat Riemannian manifold if and only if $C_{ijkl} = 0$, and that $C_{ijk} = 0$ on M^n in this case. When $n = 3$, we always have $C_{ijkl} = 0$. Hence, if M^n is a locally conformally flat n-dimensional Riemannian manifold, then

$$R_{ijkl} = \frac{1}{n-2}(R_{ik}\delta_{jl} - R_{il}\delta_{jk}) - \frac{r}{(n-1)(n-2)}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \tag{5}$$

and

$$\left(R_{ij,k} - \frac{1}{2(n-1)}\delta_{ij} r_k\right) - \left(R_{ik,j} - \frac{1}{2(n-1)}\delta_{ik} r_j\right) = 0. \tag{6}$$

Moreover, if the scalar curvature r is constant, then we obtain

$$R_{ij,k} = R_{ik,j}. \tag{7}$$

§3. PROOFS OF THEOREMS

Proof of Theorem 1. We select a local frame of orthonormal vector fields, $E_1, E_2, \ldots, E_n$ adapted to the Riemannian metric of M^n, such that $R_{ij} = \lambda_i \delta_{ij}$. The values of λ_i for $i = 1, 2, \ldots, n$ are the eigenvalues of the Ricci curvature tensor (R_{ij}) of M^n. Since scalar curvature r of M^n is constant, from equation (7), with choice $j = i$ and $k \neq i$, we have

$$\lambda_{i,k} = 0, \quad k \neq i. \tag{8}$$

Since $r = \sum_k R_{kk} = \sum_k \lambda_k$ is constant, we have

$$0 = r_i = \sum_k \lambda_{k,i}. \tag{9}$$

The equations (8) and (9) imply that λ_i for $i = 1, 2, \ldots, n$ are constant. Therefore, $\sum_{i,j} R_{ij}^2$ is constant. As shown by Goldberg [2], Theorem 3, a compact M^n is isometric to a space form or isometric to $S^{n-1}(c) \times S^1$.

Proof of Theorem 2. Suppose $E_1, E_2, \ldots, E_n$ be a local frame of orthonormal vector fields of M^n such that $R_{ij} = \lambda_i \delta_{ij}$. From equation (2), we have $R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk})$. Hence, we infer that

$$R_{ijij} = 1 + h_{ii}h_{jj} - h_{ij}^2, \quad i \neq j. \tag{10}$$

Moreover,

$$C_{ijij} = R_{ijij} - \frac{1}{n-2}(R_{ii} + R_{jj}) + \frac{r}{(n-1)(n-2)}, \quad i \neq j. \tag{11}$$

From equations (4), (10) and (11), we obtain

$$\sum_{i \neq j} C_{ijij} = r - \frac{2n(n-1)}{n-2} r + \frac{n}{(n-2)} r = \frac{-2(n-1)^2}{n-2} r. \tag{12}$$

Since r and Weyl tensor is nonnegative, (12) implies that $r = 0$ is constant and $C_{ijij} = 0$ for $i, j = 1, 2, \ldots, n$ and $i \neq j$. Since C_{ijkl} have symmetries of curvature tensor R_{ijkl} and $C_{ijkl} + C_{jkil} + C_{kijl} = 0$, we have $C_{ijkl} = 0$. Thus M^n is locally conformally flat manifold. By Theorem 1 the proof is complete.

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Поступила 23 февраля 2006