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# STABILITY FOR FAMILIES OF NONLINEAR EQUATIONS

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**Abstract.** A stability principle concerning solutions of sequences of nonlinear equations for continuous operators on $\mathbb{R}^n$ is described that can be applied to a wide class of operators for which point-wise convergence already implies continuous convergence, in particular to sequences of monotone operators.

## §1. INTRODUCTION

The following theorem is of central significance for stability investigations :

**Theorem 1.1 (see [4]).** *Let $X$ and $Y$ be metric spaces and let $(f_n : X \longmapsto Y)$ be a sequence of continuous functions that converges point-wise to a function $f : X \longmapsto Y$. Then the following statements are equivalent :*

1. *$\{f_n\}$ is equicontinuous,*
2. *$f$ is continuous and $(f_n)$ converges continuously to $f$,*
3. *$(f_n)$ converges uniformly on compact subsets to $f$.*

If continuous convergence of a sequence has been established for a certain class of functions then this property is inherited by compositions (even though these compositions may not belong to the original class) in the following sense :

**Theorem 1.2.** *Let $X$,$Y$, and $Z$ be metric spaces, $(f_n : X \longmapsto Y)$ be a sequence of functions that converges continuously to a function $f : X \longmapsto Y$, and $(g_n : Y \longmapsto Z)$ be a sequence of functions that converges continuously to a function $g : Y \longmapsto Z$. Then $(g_n \circ f_n : X \longmapsto Z)$ converges continuously to $g \circ f$. In particular, if*

- *$f$ is continuous then $f \circ g_n \longmapsto f \circ g$ continuously,*
- *if $g$ continuous then $g \circ f_n \longmapsto g \circ f$ continuously.*

For sequences of convex functions, equicontinuity and hence continuous convergence already follows from pointwise convergence ([4], Satz 3, p. 219). Subsequently we will show that a similar statement holds for sequences of monotone operators. Continuous convergence in turn implies stability of solutions under certain conditions on the limiting problem.

Both, solutions of equations and optimization problems, can be treated in the framework of variational inequalities. In fact, sequences of variational inequalities show a similar stable behavior, where again continuous convergence is of central significance ([4], Satz 1, p. 245). We employ this scheme in the context of two-stage solutions (Section 4).

Stability questions for minimal solutions of point-wise convergent sequences of convex functions have been treated in a number of publications. It turns out that stability can be guaranteed if the set of minimal solutions of the limit problem is bounded (see [4]). As an application, ill posed optimization problems are replaced by sequences of (numerically) well posed problems (see [9, 10]). The question arises, whether a corresponding statement holds on the equation level for certain classes of mappings that are not necessarily potential operators. Questions of this type arise e.g. in the context of smooth projection methods for semi-infinite optimization (see [6]).

A general framework for treating stability questions involving nonlinear equations for sequences of continuous operators on $\mathbb{R}^n$ is given by the following scheme (see [6]) :

**Theorem 1.3.** *Let $U \subset \mathbb{R}^n$ and $A_k : U \longmapsto \mathbb{R}^n$ be a sequence of continuous operators that converges continuously on $U$ to a continuous operator $A : U \longmapsto \mathbb{R}^n$ with the property :*

*there exists a ball $\overline{K}(x_0, r) \subset U$, $r > 0$ such that*

$$\langle Ax, x - x_0 \rangle > 0 \tag{1}$$

*for all $x$ in the sphere $S(x_0, r)$.*

*Then there exists some $k_0 \in \mathbb{N}$ such that for any $k \geq k_0$ each equation $A_k x = 0$ has a solution $x_k$ in $K(x_0, r)$. Furthermore, every point of accumulation of $(x_k)$ is a solution of $Ax = 0$.*

The above theorem is a consequence of the following well known lemma.

**Lemma 1.4.** *Let $A : U \longmapsto \mathbb{R}^n$ be continuous. If there is $r > 0$ and a ball $\overline{K}(x_0, r) \subset U$ such that $\langle Ax, x - x_0 \rangle \geq 0$ for all $x \in S(x_0, r)$, then the nonlinear equation $Ax = 0$ has a solution in $\overline{K}(x_0, r)$.*

**Proof :** Otherwise Browder's fixed point theorem applied to the mapping

$$x \longmapsto g(x) = -r \left( \frac{Ax}{\|Ax\|} \right) + x_0$$

would lead to a contradiction.

## §2. STABILITY FOR MONOTONE OPERATORS

A large class of operators can be treated using the above stability principle, where point-wise convergence already implies uniform convergence on compact subsets. Among them are the monotone operators according to :

**Definition 2.1.** Let $X$ be a normed space and let $U$ be a subset of $X$. A mapping $A : U \longmapsto X^*$ is called monotone on $U$ if for all $x, y \in U$,

$$\langle Ax - Ay, x - y \rangle \geq 0.$$

**Lemma 2.2 (see [13]).** *Let $U$ be an open subset of $\mathbb{R}^n$ and let $A_k : U \to \mathbb{R}^n$ be a sequence of continuous monotone operators that converges point-wise on $U$ to an operator $A : U \to \mathbb{R}^n$. Then for every sequence $(x_k) \subset U$ that converges in $U$ it follows that the sequence $(A_k x_k)$ is bounded.*

**Proof :** Assume that there is a sequence $(x_k)$ in $U$ with $\lim x_k = x_0 \in U$ such that $(A_k x_k)$ is unbounded. Then there is a subsequence $(A_{k_i} x_{k_i})$ with the property $||A_{k_i} x_{k_i}|| \geq i$ for all $i \in \mathbb{N}$. As $A_{k_i}$ is monotone we obtain for all $z \in U$ :

$$\langle A_{k_i} x_{k_i} - A_{k_i} z, x_{k_i} - z \rangle \geq 0.$$

If now $y_{k_i} = \frac{A_{k_i} x_{k_i}}{||A_{k_i} x_{k_i}||}$, then we can w.l.g. assume that the sequence $(y_{k_i})$ converges to some $y$ in the unit sphere. If the above inequality is divided by $||A_{k_i} x_{k_i}||$ we obtain for all $z \in U$ :

$$\left\langle y_{k_i} - \frac{A_{k_i} z}{||A_{k_i} x_{k_i}||}, x_{k_i} - z \right\rangle \geq 0$$

Point-wise convergence implies $A_{k_i} z \to Az$ and hence

$$\lim_{i \to \infty} \left\langle y_{k_i} - \frac{A_{k_i} z}{||A_{k_i} x_{k_i}||}, x_{k_i} - z \right\rangle = \langle y, x_0 - z \rangle \geq 0, \quad z \in U.$$

As $U$ is open, it follows that $y = 0$, a contradiction completing the proof.

The following theorem states that point-wise convergence of continuous monotone operators already implies continuous convergence.

**Theorem 2.3 (see [13]).** *Let $U$ be an open subset of $\mathbb{R}^n$ and let $A_k : U \longmapsto \mathbb{R}^n$ be a sequence of continuous monotone operators that converges point-wise on $U$ to a continuous operator $A : U \longmapsto \mathbb{R}^n$ then $(A_k)$ is equicontinuous on $U$.*

**Proof :** According to Theorem 1.1, it is sufficient to show the following : convergence of a sequence $(x_k)$ in $U$ to an element $x_0 \in U$ implies $\lim A_k x_k = Ax_0$.
Assume that there is a sequence $(x_k)$ in $U$ convergent to $x_0 \in U$ such that $(A_k x_k)$ does not converge to $Ax_0$, i.e. there is $\varepsilon > 0$ and a subsequence $(A_{k_i} x_{k_i})$ with the property

$$\|A_{k_i} x_{k_i} - Ax_0\| \geq \varepsilon$$

for all $i \in \mathbb{N}$. By Lemma 2.2 $(A_{k_i} x_{k_i})$ is bounded and w.l.g. we can assume that it converges to some $g \in \mathbb{R}^n$. Because of the previous inequality we have $\|g - Ax_0\| \geq \varepsilon$. On the other hand we obtain by the monotonicity of $A_{k_i}$ for all $u \in U$ :

$$\langle A_{k_i} x_{k_i} - A_{k_i} u, x_{k_i} - u \rangle \geq 0,$$

and hence, using point-wise convergence we get $\langle g - Au, x_0 - u \rangle \geq 0$ for all $u \in U$. By theorem 2.4 below it follows that $g = Ax_0$, a contradiction yielding the proof.

**Theorem 2.4 (Browder and Minty [12]).** *Let $E$ be a Banach space, $U$ an open subset of $E$ and $A : U \longmapsto E^*$ a semi-continuous operator. If for a pair $u_0 \in U$ and $v_0 \in E^*$ and for all $u \in U$ the inequality :*

$$\langle Au - v_0, u - u_0 \rangle \geq 0$$

*holds, then $v_0 = Au_0$.*
An immediate consequence of the theorem of Browder and Minty is the following characterization theorem for solutions of the equation $Ax = 0$, if $A$ is a monotone operator.

**Theorem 2.5 (see [12]).** *Let $E$ be a Banach space, $U$ an open subset of $E$ and $A : U \longmapsto E^*$ a continuous and monotone operator. Then $Au_0 = 0$ for $u_0 \in U$ if and only if for all $u \in U$*

$$\langle Au, u - u_0 \rangle \geq 0.$$

**Proof :** the if-part follows from Theorem 2.4 for $v_0 = 0$. Let now $Au_0 = 0$ then, from the monotonicity of $A$, we obtain :

$$0 \leq \langle Au - Au_0, u - u_0 \rangle = \langle Au, u - u_0 \rangle.$$

**Remark 2.6.** If $U$ is convex then the above theorem directly implies that the set $S_A := \{x \in U | Ax = 0\}$ is convex.
For monotone operators we obtain the following existence theorem which in a way is a stronger version of Lemma 1.4.

**Theorem 2.7.** *Let $U \subset \mathbb{R}^n$ be convex and $A : U \longmapsto \mathbb{R}^n$ be a continuous monotone operator. If there exists a ball $\overline{K}(x_0, r) \subset U$ and $r > 0$ such that $\langle Ax, x - x_0 \rangle > 0$ for all $x \in S(x_0, r)$, then $\emptyset \neq S_A \subset K(x_0, r)$, where $S_A$ is the set of solutions of the nonlinear equation $Ax = 0$.*

**Proof :** The first part follows from Lemma 1.4. For the second part let $\lambda, \mu \in \mathbb{R}$ with $\lambda > \mu$ and let $x \in S(x_0, r)$. Then monotonicity of $A$ yields

$$\langle A(\lambda(x - x_0) + x_0) - A(\mu(x - x_0) + x_0), (\lambda - \mu)(x - x_0) \rangle \geq 0.$$

Let $I$ be the intersect of $U$ with the straight line passing through $x$ and $x_0$. From the above inequality it follows that $g : I \longmapsto \mathbb{R}$ with $g(\lambda) := \langle A(\lambda(x - x_0) + x_0), x - x_0 \rangle$ is an increasing function. In particular $g(1) = \langle Ax, x - x_0 \rangle > 0$. Suppose there is a $1 < \lambda_* \in I$ such that $A(\lambda_*(x - x_0) + x_0) = 0$ then $g(\lambda_*) = 0$, a contradiction completing the proof.
We are now in the position to present a stronger version of Theorem 1.3 for sequences of monotone operators.

**Theorem 2.8.** *Let $U \subset \mathbb{R}^n$ be open and convex and $A_k : U \to \mathbb{R}^n$ be a sequence of continuous monotone operators that converges point-wise on $U$ to a continuous operator $A : U \longmapsto \mathbb{R}^n$ with the property :*

*there exists a ball $\overline{K}(x_0, r) \subset U, r > 0$ such that $\langle Ax, x - x_0 \rangle > 0$ for all $x$ on the sphere $S(x_0, r)$.*

*Then there exists some $k_0 \in \mathbb{N}$ such that the set of the solutions of the equation $A_k x = 0$ is nonempty $\forall k \geq k_0$ and contained in $K(x_0, r)$. Furthermore, if $x_k \in \{x \in U | A_k x = 0\}$, $k \geq k_0$, then every point of accumulation of $(x_k)$ is a solution of $Ax = 0$.*
Property (1) is satisfied by various classes of operators, among them the derivatives of convex functions.

**Lemma 2.9.** *If a monotone operator $A$ defined on $\mathbb{R}^n$ has a convex potential $f$ with a bounded set of minimal solutions $M(f, \mathbb{R}^n)$ (which, of course, coincides with the set of solutions of $Ax = 0$), then $A$ satisfies the property (1).*

**Proof :** Apparently for each $x_0 \in M(f, \mathbb{R}^n)$ there is a sphere $S(x_0, r)$ such that $f(x) - f(x_0) > 0$ for all $x \in S(x_0, r)$. As $A = f'$, the subgradient-inequality on that sphere yields :

$$0 < f(x) - f(x_0) \leq \langle Ax, x - x_0 \rangle$$

The proof is complete.

For general monotone operators such a statement is not available, i.e. the property (1) does not follow from the boundedness of the solutions of $Ax = 0$, as the following example shows.

**Example 2.10.** Let $A : \mathbb{R}^2 \longmapsto \mathbb{R}^2$ be a linear operator that represents a $\frac{\pi}{2}$-rotation. $A$ is monotone as

$$\langle Ax - Ay, x - y\rangle = \langle A(x-y), x-y\rangle = 0,$$

but on any sphere around the origin we have $\langle Ax, x\rangle = 0$. Obviously, $\{x|Ax = 0\} = \{0\}$.

An important class of operators in this context are the Fejér-contractions.

**Definition 2.11.** An operator $P : \mathbb{R}^n \longmapsto \mathbb{R}^n$ is called **Fejér-contraction** w.r.t. $x_0$ (see [1]) or strictly quasi-non-expansive (see [3]) if $x_0$ is a fixed point of $P$ and there is an $r > 0$ such that $||P(x) - x_0|| < ||x - x_0||$ for all $x \notin K(x_0, r)$.

**Remark 2.12.** The above definition of a Fejér-contraction differs somewhat from that given in [1].

**Remark 2.13.** It follows immediately from the definition that the set of fixed points of a Fejér-contraction w.r.t. $x_0$ is bounded.

If $P$ is a Fejér-contraction w.r.t. $x_0$ then the operator $A := I - P$ has property (1) as the following lemma shows.

**Lemma 2.14.** *Let $P : \mathbb{R}^n \longmapsto \mathbb{R}^n$ be a Fejér-contraction w.r.t. $x_0$. Then for $A := I - P$*

$$\langle Ax, x - x_0\rangle > 0$$

*for all $x \notin K(x_0, r)$.*

**Proof :** If $x \notin K(x_0, r)$, then we obtain :

$$\langle Ax, x - x_0\rangle = \langle x - x_0 - (P(x) - x_0), x - x_0\rangle = ||x - x_0||^2 - \langle P(x) - x_0, x - x_0\rangle \geq$$

$$\geq ||x - x_0||^2 - ||P(x) - x_0|| ||x - x_0|| > 0.$$

**Remark 2.15.** If $P$ is also non-expansive on $\mathbb{R}^n$ then $A = I - P$ is apparently monotone and continuous.

It is easily seen that a projection $P$ onto a bounded convex set is a non-expansive Fejér-contraction. It can be shown that the same is true for certain compositions of projections (see [6]).

## §3. STABILITY FOR WIDER CLASSES OF OPERATORS

A large class of operators can be treated using the above stability principle, where point-wise convergence already implies continuous convergence. To illustrate this, consider a continuous operator $A : \mathbb{R}^n \longmapsto \mathbb{R}^n$ satisfying property (1). Then it is easily seen that in the following situations continuous convergence, and hence stability of the solutions follows from Theorem 1.1 :

1. *$A_k = \alpha_k P + A$, where $(\alpha_k)$ be a sequence in $\mathbb{R}_+$ tending to 0, while $P : \mathbb{R}^n \longmapsto \mathbb{R}^n$ be continuous.*

   **Proof :** let $x_k \to x_0$ then the sequence $(P(x_k))$ is bounded, hence $\alpha_k P(x_k) \to 0$ and $A_k(x_k) \to A(x_0)$ because of the continuity of $A$.

2. *$A_k : \mathbb{R}^n \longmapsto \mathbb{R}^n$ is continuous and $A_k \longmapsto A$ component-wise monotone, i.e. for $A_k(x) = (f_k^{(i)}(x))_{i=1}^n$ and $A(x) = (f^{(i)}(x))_{i=1}^n$ one has point-wise monotone convergence of $f_k^{(i)} \to f^{(i)}$ for $i = 1, \ldots, n$ on $\mathbb{R}^n$.*

   **Proof :** follows from the Theorem of Dini (see [4]) applied to the components of $A_k$ and $A$ respectively.

3. *$A_k : \mathbb{R}^n \longmapsto \mathbb{R}^n$ is continuous, $A_k \longmapsto A$ point-wise on $\mathbb{R}^n$ and $A_k - A$ is monotone for all $k \in \mathbb{N}$.*

   **Proof :** we have $A_k - A \to 0$ point-wise on $\mathbb{R}^n$ and from Theorem 2.3 continuous convergence follows.

4. *Compositions of continuously convergent sequences of functions preserves continuous convergence,*

   i.e. if $g_k : \mathbb{R}^n \longmapsto \mathbb{R}^n$ is continuously convergent to $g : \mathbb{R}^n \longmapsto \mathbb{R}^n$ and $f_k : \mathbb{R}^n \longmapsto \mathbb{R}^n$ is continuously convergent to $f : \mathbb{R}^n \longmapsto \mathbb{R}^n$ then $f_k \circ g_k$ converges continuously to $f \circ g$.

A special case is obtained if either $(f_k)$ or $(g_k)$ is a constant sequence of a continuous function, e.g. if $B : \mathbb{R}^n \longmapsto \mathbb{R}^n$ is linear, $A_k : \mathbb{R}^n \longmapsto \mathbb{R}^n$ is continuous and monotone, and $A_k \to A$ point-wise on $\mathbb{R}^n$, then $B \circ A_k$ converges continuously to $B \circ A$.

## §4. TWO–STAGE SOLUTIONS

Two-stage solutions have been studied in [4, 8, 9, 10] in particular for sequences of convex functions. The following theorem (see [4], p. 246) gives a framework for sequences of nonlinear equations, where the second stage is described in terms of a variational inequality.

**Theorem 4.1.** *Let $X, Y$ be a normed spaces, $(A : X \to Y)$ continuous and for the sequence of continuous operators $(A_k : X \to Y)$ let $L = \overline{\lim}_{k\to\infty}\{x | A_k x = 0\} \subset \{x | Ax = 0\} =: S_A$. Let further $(a_k)$ be a sequence of positive numbers and $B : X \longmapsto X^*$ a continuous mapping with $B(0) = 0$ such that*

1. *$B \circ A : X \longmapsto X^*$ is monotone,*
2. *$a_k(B \circ A_k - B \circ A)$ converges continuously to a mapping $D : X \longmapsto X^*$.*

*If $\overline{x} \in L$ then for all $x \in S_A$ the inequality $\langle D\overline{x}, x - \overline{x}\rangle \geq 0$ holds.*

**Proof :** Let $x_k \epsilon \{x | A_k x = 0\}$ such that $(x_k)$ converges to an $\overline{x} \in X$, i.e. $\overline{x} \in L$. Let $x \in S_A$, i.e. $Ax = 0$. Since $B \circ A$ monotone and continuous it follows that

$$a_k \langle (B \circ A_k - B \circ A) x_k, x - x_k \rangle = a_k \langle B \circ A x_k, x_k - x \rangle \geq 0.$$

Since $a_k(B \circ A_k - B \circ A)$ converges continuously to $D$ it follows that $a_k(B \circ A_k - B \circ A)x_k$ converges to $D\overline{x}$ in the norm, and hence inequality $\langle D\overline{x}, x - \overline{x} \rangle \geq 0$ follows.

**Example 4.2.** Let $A_k = \alpha_k P + A$, where $A$ is monotone, $P$ is a positive definite linear operator, and $(\alpha_k)$ is a sequence of positive numbers tending to 0 (compare with class 1. of previous section). Then, choosing $a_k = \frac{1}{\alpha_k}$, $D = P$ and the inequality $\langle P\overline{x}, x - \overline{x} \rangle \geq 0$ for all $x \in S_A$ is the characterization of a minimal solution of the strictly convex functional $x \mapsto \langle Px, x \rangle$ on the convex set $S_A$ (compare with Remark 2.6). In this case, convergence of $(x_k)$ to $\overline{x}$ follows.

**Example 4.3.** Let $A, C : \mathbb{R}^n \longmapsto \mathbb{R}^m$ be linear operators, $b \in \mathbb{R}^m$ and let $S_A := \{x \in \mathbb{R}^n | Ax = b\}$ be nonempty. Let further $(\alpha_k)$ be a sequence in $\mathbb{R}_+$ tending to zero and let $A_k := \alpha_k C + A$. Then for $a_k = \frac{1}{\alpha_k}$ and $B = A^T$ it follows that

$$a_k(BA_k - BA) = \frac{1}{\alpha_k}(\alpha_k A^T C + A^T A - A^T A) = A^T C =: D$$

and hence $\langle A^T C\bar{x}, x - \bar{x} \rangle \geq 0$ for all $x$ in the affine subspace $S_A$, in other words : $A^T C\bar{x}$ is orthogonal to kernel of $A$.

**Example 4.4** (see [6]). $LP$–problem : for $c, x \in \mathbb{R}^n$, $A \in L(\mathbb{R}^n, \mathbb{R}^m)$, $b \in \mathbb{R}^m$ we consider the following problem

$$\min\{\langle c, x \rangle | Ax = b, x \geq 0\}$$

with bounded and nonempty set of solutions. We chose the mapping $P = P_m \circ P_{m-1} \circ \ldots \circ P_1$ of the successive projections $P_i$ onto the hyperplanes

$$H_i = \{s \in \mathbb{R}^n | \langle a_i, s \rangle = b_i\} \quad i \in \{1, \ldots, m\},$$

where $a_i$ denotes the $i$-th row of $A$, and the projection $P_K$ onto the positive cone $\mathbb{R}^n_{\geq 0}$, given by

$$P_K(x) = ((x_1)_+, \ldots, (x_n)_+)).$$

As $P_K$ is non differentiable, a smoothing of the projection $P_K$ is obtained by replacing $s \longmapsto (s)_+$ by a smooth function $\varphi_\alpha : \mathbb{R} \longmapsto \mathbb{R}$ $(\alpha > 0)$ that approximates $(\cdot)_+$.

The projection $P_K$ is then replaced by $P_\alpha = (\varphi_\alpha(x_1), \dots, \varphi_\alpha(x_n))$. By use of the Newton-method the nonlinear equation

$$F_\alpha(x) := x - P_\alpha \circ P(x) + \alpha c = 0$$

can be solved very efficiently.

It can be shown that $P_K \circ P$ is a non-expansive Fejér-contraction w.r.t. any $\hat{x} \in S := \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$ and that $P_\alpha \circ P$ is non-expansive. Let $(\alpha_k)$ be a positive sequence tending to 0. Stability then follows from Theorem 2.8 for the sequence of monotone operators $A_k = F_{\alpha_k}$ converging point-wise to the monotone operator $A = I - P_K \circ P$ satisfying (1).

Application of Theorem 4.1 yields a condition upon $\varphi_{\alpha_k}$ that enforces continuous convergence. We have for $a_k = \frac{1}{\alpha_k}$ :

$$a_k(A_k - A) = a_k(-P_{\alpha_k} \circ P + \alpha_k c + P_K \circ P) = \frac{1}{\alpha_k}(P_K \circ P - P_{\alpha_k} \circ P) + c.$$

It follows that if $\frac{1}{\alpha_k}(\varphi_{\alpha_k} - (\cdot)_+) \to 0$ uniformly on compact subsets of $\mathbb{R}$, then $a_k(A_k - A)$ converges continuously to $c$. If $\overline{x}$ is any limit point of the sequence of solutions of the equations $(A_k x = 0)$ then for all $x \in S$ we obtain $\langle c, x - \overline{x} \rangle \geq 0$.

**Remark 4.5.** Convex optimization problems with linear constraints can be treated in the same manner. For convex and differentiable $f : \mathbb{R}^n \longmapsto \mathbb{R}^n$ consider

$$\min\{f(x) : Ax = b,\ x \geq 0\}.$$

The (monotone) operator $F_\alpha$ becomes

$$F_\alpha(x) := x - P_\alpha \circ P(x) + \alpha f'(x)$$

implying $\langle f'(\overline{x}), x - \overline{x} \rangle \geq 0$ for every point of accumulation $\overline{x}$ and all $x \in S$. According to the Characterization Theorem of convex optimization, $\overline{x}$ is a minimal solution of $f$ on $S$.

**Резюме.** В работе описан принцип стабильности, касающийся решений последовательностей нелинейных уравнений для непрерывных операторов на $\mathbb{R}^n$, который можно применить к широкому классу операторов, для которых из поточечной сходимости всегда следует непрерывная сходимость. В частности, этот принцип применяется к последовательностям монотонных операторов.

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