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UNIVERSAL ENTIRE FUNCTIONS WITH ADDITIONAL PROPERTIES

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The aim of this note is the construction of universal entire functions of a new type. In particular, these functions are universal under translations and are bounded on each line and have zeros at certain prescribed points.

§1. OVERVIEW AND NOTATIONS

Throughout this paper we use the following abbreviations : $\mathfrak{M}$ is the collection of all compact subsets $K \subset \mathbb{C}$ with connected complement. By $A(K)$ we denote the set of all functions which are continuous on K and holomorphic in the interior of K. Finally let $H(\mathbb{C})$ be as usual the set of all entire functions.

It is our aim to construct entire functions which have a *universal* and also another *non-universal* property. The first results dealing with universal properties were proved by Birkhoff [2] and MacLane [7], which slightly modified state as follows.

Theorem 1 *(Birkhoff, 1929). Let $b = \{b_n\} \subset \mathbb{C}$ be a sequence with $b_n \to \infty$ $(n \to \infty)$. Then there exists an entire function φ with the property that for every compact set $K \in \mathfrak{M}$ and for every function $f \in A(K)$ there exists a sequence $\{n_k\} \subset \mathbb{N}$ with*

$$\varphi(z + b_{n_k}) \to f(z) \quad \textit{uniformly on} \quad K.$$

Theorem 2 *(MacLane, 1952). There exists an entire function φ with the property that for every compact set $K \in \mathfrak{M}$ and for every function $f \in A(K)$ there exists a sequence $\{n_k\} \subset \mathbb{N}$ with*

$$\varphi^{(n_k)}(z) \to f(z) \quad \textit{uniformly on} \quad K.$$

In the sequel we will call such functions "Birkhoff-universal" with respect to b and "MacLane-universal" respectively. As mentioned above we are going to construct similar functions which in addition are bounded on each line, tend to zero on each line, or have zeros at certain prescribed points.

§2. A BIRKHOFF-UNIVERSAL FUNCTION WHICH IS BOUNDED ON EACH LINE

Theorem 3. *There exists an entire function φ and a sequence $b = \{b_n\}$ of complex numbers with $b_n \to \infty$, such that the function φ and all its derivatives $\varphi^{(j)}$ $(j \in \mathbb{N})$*

1) are bounded on each line.

2) are Birkhoff-universal with respect to b.

Sketch of Proof. 1. We dispense with showing that the derivatives of φ have the same properties as φ. Let us consider the set

$$S = \{z : \operatorname{Re} z > 0, \quad \sqrt{\operatorname{Re} z} < \operatorname{Im} z < 2\sqrt{\operatorname{Re} z}\}^c$$

and pairwise disjoint circles B_n with center b_n, such that

$$B_n = \{z : |z - b_n| \le n\} \subset S^c.$$

Then

$$E := S \cup \bigcup_{n=1}^{\infty} B_n$$

is an Arakelian set.

Let $\{Q_n\}$ be any enumeration of all polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$. The functions

$$\delta(z) = \begin{cases} -\ln n & \text{if} \quad z \in B_n, \\ Q_n(z - b_n) & \text{if} \quad z \in S. \end{cases}$$

$$q(z) = \begin{cases} Q_n(z - b_n) & \text{if} \quad z \in B_n, \\ 0 & \text{if} \quad z \in S \end{cases}$$

are holomorphic on E. According to Arakelian's approximation theorem [1] we first can choose an entire function g with

$$|\delta(z) - g(z)| < 1 \quad (z \in E)$$

and then an entire function h with

$$\left| \frac{q(z)}{e^{g(z)-1}} - h(z) \right| < 1 \quad (z \in E).$$

The entire function $\varphi(z) = h(z) \cdot e^{g(z)-1}$ satisfies for all $z \in E$

$$|q(z) - \varphi(z)| < e^{Re(g(z)-1)} \leq e^{|g(z)-\delta(z)|-1+\delta(z)} < e^{\delta(z)}. \tag{2}$$

It follows that for all $z \in S$

$$|\varphi(z)| = |\varphi(z) - q(z)| < e^{\delta(z)} = 1.$$

The set S has the property that the intersection of any line with the complement of S is either empty or contained in a compact set. So, we get that φ is bounded on each line.
Furthermore, it follows from (2) that

$$\max_{|z-b_n|\leq n} |\varphi(z) - Q_n(z - b_n)| = \max_{z\in B_n} |\varphi(z) - q(z)| < \frac{1}{n},$$

which is equivalent to

$$\max_{|z|\leq n} |\varphi(z + b_n) - Q_n(z)| < \frac{1}{n}.$$

By the choice of Q_n and Mergelian's approximation theorem [8], we obtain that φ is universal under translations.

Remark 4. If we replace the function δ on the set S in the proof of Theorem 3 in (2) by a suitable holomorphic branch of a logarithm, we obtain an entire function φ with the same universal properties as in Theorem 3, which in addition tends to zero on each line.
Next we discuss two subsequent questions

- Under which conditions on the sequence b may such an entire function φ as in Theorem 3 be constructed ?
- How "big" is the set $\mathfrak{U}_b(\mathbb{C})$ of all the entire functions φ with the above explained properties ?

For this we define

$$B_n^R(b) := \{z : |z - b_n| \leq R\}.$$

We say that the sequence b *has the property* **(G)** if there exists a subsequence $\{b_{n_k}\}$ of $\{b_n\}$, such that for every $R > 0$ and every line Γ in $\mathbb{C}$

$$\Gamma \cap B_{n_k}^R(b) \neq \emptyset \qquad \textbf{only for finitely many} \quad k \in \mathbb{N}.$$

With these notations the following result holds.

Theorem 5. *Suppose that $H(\mathbb{C})$ is endowed with the compact-open topology.*

1. *If the sequence b has not the property (G), then $\mathfrak{U}_b(\mathbb{C})$ is empty.*
2. *If the sequence b has the property (G), then there exists a dense linear subspace L of $H(\mathbb{C})$, such that*
$$L \setminus \{0\} \subset \mathfrak{U}_b(\mathbb{C}).$$

However $\mathfrak{U}_b(\mathbb{C})$ is **not** *a residual subset of $H(\mathbb{C})$.*

Proof. We concentrate on showing, that $\mathfrak{U}_b(\mathbb{C})$ is **not** a residual subset of $H(\mathbb{C})$. It is well-known (see, for instance, Grosse–Erdmann [5]) that the set

$$\mathfrak{T}(\mathbb{C}) := \{\varphi \in H(\mathbb{C}) : \quad \textbf{For every} \quad K \in \mathfrak{M}, \quad \textbf{every} \quad f \in A(K), \\ \textbf{there exists a sequence} \quad \{n_k\} \subset \mathbb{N} \\ \textbf{with} \quad \varphi(z + n_k) \to f(z) \quad \textbf{uniformly on} \quad K(k \to \infty)\}.$$

(in other words :

$$\mathfrak{T}(\mathbb{C}) = \{\varphi \in H(\mathbb{C}) : \varphi \quad \textbf{is hypercyclic for the translation operator } T_1 \textbf{ by } 1\})$$

is a dense G_δ-subset in $H(\mathbb{C})$.
Suppose that the dense set $\mathfrak{U}_b(\mathbb{C})$ contains a dense and G_δ-subset in $H(\mathbb{C})$. By Baire's category theorem it follows that the intersection $\mathfrak{T}(\mathbb{C}) \cap \mathfrak{U}_b(\mathbb{C})$ is also dense in $H(\mathbb{C})$, in particular not empty.
Assume $\varphi_0 \in \mathfrak{T}(\mathbb{C}) \cap \mathfrak{U}_b(\mathbb{C})$, i.e. $|\varphi_0(z)|$ is bounded on $z_0 + \mathbb{R}$ for some $z_0 \in \mathbb{C}$ by a suitable constant c_0,

$$\sup_{z \in \mathbb{R}} |\varphi_0(z_0 + z)| \le c_0. \tag{3}$$

Let $K = \{z_0\}$ and $f(z) \equiv c_0 + 1$. Then there exists a subsequence $\{n_k\} \subset \mathbb{N}$ with

$$\varphi_0(z_0 + n_k) \to f(z) \equiv c_0 + 1 \quad (k \to \infty)$$

in contradiction to (3). So φ_0 is not bounded on each line, i. e. $\varphi_0 \notin \mathfrak{U}_b(\mathbb{C})$, and the proof is complete.
One should notice that so far only very few cases of a universality have come up in the literature where the set of universal elements turned out to be non-residual, cf. Grosse–Erdmann [5].

§3. BIRKHOFF–UNIVERSAL FUNCTIONS WITH PRESCRIBED ZEROS

Next we construct a Birkhoff-universal function φ, which in addition has zeros at certain prescribed points w_n of order ρ_n. Moreover, all the other zeros of φ are known.

Theorem 6. *Let $\{w_n\}$ be a sequence of prescribed complex numbers and let $\{\rho_n\}$ be a sequence of natural numbers, where $\{w_n\}$ is supposed to have the following property :*

The set $\{w_n : n \in \mathbb{N}\}$ does not have any cluster point in $\mathbb{C}$ and there exists an unbounded sequence $\{z_n\}$ of complex numbers with :

1. *The closed discs $F_n := \{z : |z - z_n| \leq n\}$ satisfy $F_n \cap F_m = \emptyset$ for $n \neq m$. Choose (arbitrarily small but fixed) open sets $\tilde{F}_n \supset F_n$ such that $\tilde{F}_n \cap \tilde{F}_m = \emptyset$ for $n \neq m$.*
2. *The set $F := \bigcup_{n=0}^{\infty} F_n$ satisfies : $w_n \notin F$ for all $n \in \mathbb{N}$.*

$$\text{Let } \{Q_n\} \text{ be any enumeration of all the polynomials with coefficients in } \mathbb{Q} + i\mathbb{Q}. \tag{4}$$

Then there exists an entire function φ with the following properties :

1. *$\varphi(w_n) = 0$ of order ρ_n.*
2. *For all $n \in \mathbb{N}$ the zeros of $Q_n(\cdot - z_n)$ on the set $\tilde{F}_n$ are also zeros of φ with the same order.*
3. *φ has no other zeros.*
4. *φ is a Birkhoff-universal function with respect to $\{z_n\}$.*

Remark 7. • It is possible to choose only a finite number of points w_n.

- Instead of the set of all polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$ we may use in (4) any other countable dense subset of the set $H(\mathbb{C})$, endowed with the compact-open topology.
- The additional zeros which are mentioned in the second statement of Theorem 6 arise from the construction in the proof of this theorem, in which we once again apply Arakelian's approximation theorem. But they are also necessary in a sense. Let φ be a Birkhoff-universal function and let $\{z_n\}$ be the sequence of translation points. By a simple application of Rouche's theorem, we obtain that for every $z_0 \in \mathbb{C}$ and every neighbourhood $U(z_0)$ of z_0, there exists a subsequence $\{n_k\}$ in $\mathbb{N}$, such that $\varphi(\zeta_0 + z_{n_k}) = 0$ $(k \in \mathbb{N})$ with $\zeta_0 \in U(z_0)$.

By an application of a slighty modified version of Theorem 6, it can be easily shown that there exist Birkhoff-universal functions, which have a regular distribution of its zeros. Therefore we denote by $n_\varphi(r)$ the number of zeros (counted according to multiplicity) of the function φ on the set $\{z : |z| \leq r\}$.

Theorem 8. *Suppose that there are given $c, q \in (0, \infty)$. Then there exists an entire function φ with the following properties :*

1. $\lim_{r \to \infty} \frac{n_\varphi(r)}{r^q} = c$.
2. φ is a Birkhoff-universal function with respect to a prescribed sequence $\{z_n\}$.

§4. MACLANE–UNIVERSAL FUNCTIONS WITH PRESCRIBED ZEROS

Now we discuss the same problem of prescribing the zeros of MacLane-universal functions. Herzog [6] showed the existence of a zero-free entire function, which is universal in the sense of MacLane. Moreover, it is obvious that there exist MacLane-universal functions with finitely many prescribed zeros.

Theorem 9. *Let $\{z_n\}$ be any sequence in $\mathbb{C}$, which has no cluster point in $\mathbb{C}$. Then there exists a MacLane-universal function φ with $\varphi(z_n) = 0$ for all $n \in \mathbb{N}$.*

Proofs, that are not given here, are already published or will be published elsewhere soon. For proving the last theorem a power series needs to be constructed. Furthermore, we need interpolation by Lagrange polynomials, basic properties of the Weierstrass elementary factor and Walsh's theorem on simultaneous approximation and interpolation to get the zeros as desired.

Резюме. Цель данной заметки – построение универсальных целых функций нового типа. В частности, эти функции универсальны относительно переносов, ограничены на каждой прямой и имеют нули в заранее определенных точках.

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