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STABILITY PRINCIPLES AND APPROXIMATION PROBLEMS IN VARIATIONAL CALCULUS

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The problem of pointwise minimization of the Lagrangian is approached by a simultaneous optimization with respect to both state and control variables. The Legendre-Riccati condition ensures the existence of an equivalent convex variational problem, making possible application of the corresponding stability principles. This approach also provides an elementary access to the fundamental theorems of variational calculus, without employing the theory of fields of extremals. Applications are in the problems of modular and parameter-free approximation of time-series data by monotone functions. We present a method, based on variational calculus, to determine a smooth monotone function that approximates a given time-series data in the least squares sense.

§ 1. INTRODUCTION

An important field for the application of stability principles of optimization theory is the calculus of variations. In the context of control theory, the problem of establishing the continuity of the control by using Pontrjagin's maximum principle - leading to a value of the control for fixed t - reduces to a stability question for finite-dimensional optimization. In our approach of pointwise minimization of the Lagrangian, we employ a simultaneous optimization with respect to both state and control variables. If in addition the Legendre-Riccati condition is satisfied, a condition that allows us to ensure the existence of an equivalent convex variational problem, we can apply corresponding stability principles, whose results carry over to the solution of the original problem. This approach also provides an elementary access to the fundamental theorems of variational calculus, without employing the theory of fields of extremals.

In the sequel we shall consider **variational problems** in the following setting : let $L : \mathbb{R}^{2n+1} \to \mathbb{R}$ be continuous. The restriction set S for given $\alpha, \beta \in \mathbb{R}^n$ is a set of functions that is described by

$$[S \subset \{x \in C^1[a,b]^n \mid x(a) = \alpha, x(b) = \beta\}.$$

The function $f : S \to \mathbb{R}$ to be minimized is defined by

$$f(x) = \int_a^b L(x(t), \dot{x}(t), t)dt.$$

The variational problem with fixed endpoints is then given by

$$\text{Minimize} \quad f \quad \text{on} \quad S.$$

The function f is also referred to as the variational functional.

In the subsequent discussion we introduce a supplement in integral form that is constant on the restriction set. This leads to new variational problem with a modified Lagrangian. The solutions of the original variational problem can now be found as minimal solutions of the modified variational functional. Because of the monotonicity of the integral, the variational problem is now solved by pointwise minimization of the Lagrangian with respect to the x- and $\dot{x}$-variables for every fixed t, employing the methods of finite-dimensional optimization.

This leads to sufficient conditions for a solution of the variational problem. This general approach does not even require differentiability of the integrand. Solutions of the pointwise minimization can even lie at the boundary of the restriction set so that the Euler-Lagrange equations do not have to be satisfied. For interior points the Euler-Lagrange equations will naturally appear by setting the partial derivatives to zero, using a linear supplement potential.

§2. EQUIVALENT VARIATIONAL PROBLEMS

We now attempt to describe an approach to variational problems that uses the idea of the Equivalent Problems of Caratheodory (see [5], also compare Krotov [15] and Klötzler [?]) employing suitable supplements to the original minimization problem. Caratheodory constructs equivalent problems by use of solutions of the Hamilton-Jacobi partial differential equations, in connection to the corresponding field of extremals, which are not needed in Klötzler's approach. In the context of Bellman's

Dynamic Programming (see [1]) the supplement can be interpreted as the so-called value function. The technique to modify the integrand of the variational problem already appears in the works of Legendre in the context of the second variation (accessory problem). For further reference also see [?].

The principal idea in this treatise is, to specify conditions that guarantee the existence of equivalent convex problems. As equivalent problems have identical extremals, the results obtained in the presence of convexity carry over to the solutions of the original problem. In this paper we shall demonstrate that explicitly given quadratic supplements are sufficient to yield the main results (in particular the Fundamental Theorems).

Definition 2.1. Let $F : [a,b] \times \mathbb{R}^n \to \mathbb{R}$ with $(t,x) \mapsto F(t,x)$ be continuously differentiable. and let F_{xx}, F_{xt}, F_{tx} exist and be continuous. Then we call F a **supplement potential**.

Lemma 2.2. *Let $F : [a,b] \times \mathbb{R}^n \to \mathbb{R}$ be a supplement potential. Then the integral over the supplement*

$$\int_a^b [\langle F_x(t,x(t)), \dot{x}(t)\rangle + F_t(t,x(t))]\, dt$$

is constant on S.

Proof : It suffices to see that

$$\int_a^b [\langle F_x(t,x(t)), \dot{x}(t)\rangle + F_t(t,x(t))]\, dt = F(b,\beta) - F(a,\alpha).$$

An **equivalent problem** is then given through the supplemented Lagrangian $\tilde{L}$:

$$\tilde{L} := L - \langle F_x, \dot{x}\rangle - F_t$$

§ 3. PRINCIPLE OF POINTWISE MINIMIZATION

The aim is to develop sufficient criteria for minimal solutions of the variational problem by replacing the minimization of the variational functional on subsets of a function space by finite dimensional minimization. This can be accomplished by point-wise minimization of an explicitly given supplemented integrand for fixed t using the monotonicity of the integral (application of this method to general control problems was treated in [14]) The minimization is done simultaneously, with respect to the x- and the $\dot{x}$-variables in $\mathbb{R}^{2n}$. This is the main difference as compared to Hamilton/Pontrjagin where minimization is done solely with respect to the $\dot{x}$-variables, methods, which lead to necessary conditions in the first place.

Theorem 3.1 *(Principle of Pointwise Minimization). Let a variational problem with Lagrangian L and restriction set S be given. If for an equivalent variational problem*

$$\text{Minimize} \quad g(x) := \int_a^b \tilde{L}(x(t), \dot{x}(t), t)\, dt,$$

where

$$\tilde{L} = L - \langle F_x, \dot{x} \rangle - F_t,$$

an $x^* \in S$ can be found, such that for all $t \in [a, b]$ the point $(p_t, q_t) := (x^*(t), \dot{x}^*(t))$ is a minimal solution of the function $(p, q) \mapsto \tilde{L}(p, q, t) =: \varphi_t(p, q)$ on $\mathbb{R}^{2n}$.
Then x^* is a solution of the original variational problem.

Proof : According to Lemma 1.5, the integral over the supplement is constant.
It turns out (see below) that the straight forward approach of a linear (with respect to x) supplement already leads to the Euler-Lagrange equation by setting the partial derivatives of $\tilde{L}$ (with respect to p and q) to zero.

§ 4. LINEAR AND QUADRATIC SUPPLEMENTS

Certain results of the classical theory are related to a linear supplement, where the supplement potential F has the structure

$$(t, x) \mapsto F(t, x) = \langle \lambda(t), x \rangle \tag{1}$$

and $\lambda \in C^1[a, b]^n$ is a function that has to be determined in a suitable way.
As $F_x(t, x) = \lambda(t)$ and $F_t(t, x) = \langle \dot{\lambda}(t), x \rangle$ we obtain for the equivalent problem :

$$\text{Minimize} \quad g(x) = \int_a^b L(x(t), \dot{x}(t), t) - \langle \lambda(t), \dot{x}(t) \rangle - \langle \dot{\lambda}(t), x(t) \rangle dt \quad \text{on} \quad S \tag{2}$$

Let $L : \mathbb{R}^{3n+1} \to \mathbb{R}$ be continuous and continuously partially differentiable with respect to p and q. If for fixed $t \in [a, b]$ a point $(p_t, q_t) \in \mathbb{R}^{2n}$ is a corresponding minimal solution, then the partial derivatives of ℓ_t have to be equal to zero at this point. This leads to the equations :

$$L_p(p_t, q_t, t) = \dot{\lambda}(t) \tag{3}$$

$$L_q(p_t, q_t, t) = \lambda(t). \tag{4}$$

The pointwise minimum (p_t, q_t) yields a function $t \mapsto (p_t, q_t)$. It is our aim to show that this pair provides a solution x^* of the variational problem where $x^*(t) := p_t$ and $\dot{x}^*(t) = q_t$. In the spirit of the supplement method this means that the global minimum is an element of the restriction set S. The freedom of choosing a suitable function λ is exploited to achieve this goal.

Definition 4.1. A function $x^* \in C^1[a,b]^n$ is called an **extremal**, if it satisfies the Euler-Lagrange equation :

$$L_x(x(\tau), \dot{x}(\tau), \tau) = \frac{d}{dt} L_{\dot{x}}(x(t), \dot{x}(t), t) \quad \forall t \in (a, b]$$

An extremal x^* is called admissible if $x^* \in S$.
It is our primary aim to specify, under what conditions an extremal is a solution of the variational problem. This is the case for convex problems, i.e. for Lagrangians where $L(\cdot, \cdot, t)$ is convex for all $t \in [a, b]$. Obviously, then also linearly supplemented problems are convex, and vanishing of the partial derivatives of the Lagrangian is sufficient for a minimum. As equivalent problems have identical extremals it suffices for a positive answer to the above question to identify an equivalent convex problem.

Theorem 4.2. *Every extremal for the Lagrangian L is an extremal for the supplemented Lagrangian*

$$\tilde{L} := L - \langle F_x, \dot{x} \rangle - F_t$$

and vice versa, where F is a supplement potential.

Proof : We have :

$$\tilde{L}_{\dot{x}} = L_{\dot{x}} - F_x \quad \text{and} \quad \tilde{L}_x = L_x - \dot{x}^T F_{xx} - F_{tx}.$$

Moreover,

$$\frac{d}{dt} F_x(t, x(t)) = F_{xt}(t, x(t)) + \dot{x}(t)^T F_{xx}(t, x(t)).$$

If x satisfies the Euler-Lagrange equation in integral form with respect to L, i.e.

$$L_{\dot{x}} = \int_a^t L_x d\tau + c,$$

then there is a constant $\tilde{c}$ such that

$$\tilde{L}_{\dot{x}} = \int_a^t \tilde{L}_x d\tau + \tilde{c},$$

which we show using the continuity of F_x :

$$\int_a^t \tilde{L}_x d\tau = \int_a^t (L_x - \dot{x}^T F_{xx} - F_{tx}) d\tau = L_{\dot{x}} - c - \int_a^t (\dot{x}^T F_{xx} + F_{xt}) d\tau$$
$$= L_{\dot{x}} - F_x + F_x(x(a), a) - c = \tilde{L}_{\dot{x}} - \tilde{c}$$

Historically speaking, one would be satisfied to be able to answer the above fundamental question at least in a local sense. It turns out that such a convexification can already be realized through quadratic supplements in order to obtain the classical fundamental theorems of variational calculus.

Definition 4.3. A point $x^* \in S$ is called a strong local minimal solution if there is an $\epsilon > 0$ such that for all $x \in S$ with $||x(t) - x^*(t)|| < \epsilon$ for all $t \in [a, b]$ we have for the variational functional $f(x^*) \leq f(x)$.
A point $x^* \in S$ is called a weak local minimal solution if there is an $\epsilon > 0$ such that for all $x \in S$ with $||x(t) - x^*(t)|| + ||\dot{x}(t) - \dot{x}^*(t)|| < \epsilon$ for all $t \in [a, b]$ we have for the variational functional $f(x^*) \leq f(x)$.
For $T = [a, b]$ the subsequent Lemma leads to conditions on the Lagrangian for obtaining a weak local minimum.

Lemma 4.4. *Let T be a compact subset of $\mathbb{R}^m$ and let $L : \mathbb{R}^n \times \mathbb{R}^n \times T \to \mathbb{R}$ and let $\phi_t := L(\cdot, \cdot, t)$ be twice continuously differentiable, and let $\zeta : T \to \mathbb{R}^{2n}$ with $t \mapsto (p_t, q_t)$ be continuous, where (p_t, q_t) are such that for all $t \in T$*

1. *$L_p(p_t, q_t, t) = L_q(p_t, q_t, t) = 0$*
2. *the Hessian $\phi_t''(p_t, q_t)$ positive definite*

Then there is a $\delta > 0$ such that

1. *(p_t, q_t) is minimum of $L(\cdot, \cdot, t)$ on $K_\delta(t) := (p_t, q_t) + K_\delta(0, 0)$ for all $t \in T$*
2. *L is uniformly strongly convex on $K := \bigcup_{t \in T} \{(p, q, t) | (p, q) \in K_\delta(t)\}$.*

Proof : As the set $S_1 := \{(p, q) \in \mathbb{R}^{2n} | ||p||^2 + ||q||^2 = 1\}$ is compact and as $t \mapsto \phi_t''(p_t, q_t)$ is continuous on T there is a positive $c \in \mathbb{R}$ such that for all $t \in T$

$$\begin{pmatrix} p \\ q \end{pmatrix}^T \phi_t''(p_t, q_t) \begin{pmatrix} p \\ q \end{pmatrix} \geq c \tag{5}$$

on S_1, i.e. $t \mapsto \phi_t''(p_t, q_t)$ is uniformly positive definite on T.
Let $\rho > 0$, then on the compact set in $\mathbb{R}^{2n+1}$:

$$\overline{\bigcup_{t \in T} K_\rho(p_t, q_t, t)}$$

we have uniform continuity of $(p,q,t) \mapsto \phi_t''(p,q,t)$. Hence there is a $\delta > 0$ such that for all $(u,v) \in K_\delta(t)$ we have

$$\|\phi_t''(u,v) - \phi_t''(p_t,q_t)\| \le \frac{c}{2}$$

and hence on that set

$$\begin{pmatrix} p \\ q \end{pmatrix}^T \phi_t''(u,v) \begin{pmatrix} p \\ q \end{pmatrix} \ge \frac{c}{2}.$$

Thus we obtain that $(p,q) \mapsto L(p,q,t)$ is uniformly strongly convex on K for all $t \in T$, i.e.

$$L(\frac{p+u}{2}, \frac{q+v}{2}, t) \le \frac{1}{2}L(p,q,t) + \frac{1}{2}L(u,v,t) - \frac{c}{8}(\|p-u\|^2 + \|q-v\|^2)$$

for all $(p,q),(u,v) \in K$ and all $t \in T$ (see [9], p. 39, Satz 4).

The above lemma assumes the perspective of pointwise minimization. If $L(\cdot,\cdot,t)$ is convex and the minima (p_t,q_t) exist for every $t \in T$ and are unique, then, according to stability considerations, the mapping ζ is continuous (provided that L is continuous) :

Theorem 4.5. *Let the Lagrangian L be continuous, let $L(\cdot,\cdot,t)$ be convex for all $t \in [a,b]$, and let (p_t,q_t) be the unique pointwise minimum for all $t \in [a,b]$. Then $\zeta : T \to \mathbb{R}^{2n}$ with $t \mapsto (p_t,q_t)$ is continuous.*

Proof : This is an immediate application of Stabilitätssatz 2 [8], p. 225.

Remark 4.6. If, in addition, the Lagrangian is twice continuously differentiable and the Hessian (for fixed t) is positive definite, then the implicit function theorem yields that ζ is already continuously differentiable.

The following example shows that the convexity condition in the above theorem cannot be omitted :

Example 4.7. Let $z : [0,1] \times \mathbb{R} \to \mathbb{R}$ with

$$z(t,s) := \begin{cases} \left(\frac{1}{t}\right)^2 \left(s - \frac{1}{t}\right)^2 - \left(\frac{1}{t}\right)^2 & \text{for } \ t > 0 \ \text{ and } \ \left|s - \frac{1}{t}\right| \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Apparently, z is continuous, in particular for $t = 0$: let $\delta > 0$ and $s_0 \in \mathbb{R}$. Let $|s - s_0| < \delta$ and $0 < t < \frac{1}{1+|s_0|+\delta}$, then $|s - \frac{1}{t}| > 1$ and hence $z(t,s) = 0 = z(0,s_0)$. Let further $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ with $g(t,s) := s^2 + z(t,s)$. The function g is differentiable for $t > 0$ and $|s - \frac{1}{t}| < 1$ and we obtain as a necessary condition for a minimum of the convex function $g(t, \cdot)$ restricted to the interval $I_t := (\frac{1}{t} - 1, \frac{1}{t} + 1)$:

$$g_s(t, s) = 2s + 2(\frac{1}{t})^2(s - \frac{1}{t}) = 0.$$

We obtain for the minima $s(t) = \frac{1}{1+t^2} \cdot \frac{1}{t} \in (\frac{1}{t} - 1, \frac{1}{t} + 1)$ for $t > 0$ and $s(0) = 0$. For the function value we obtain for $t > 0$:

$$g(t, s(t)) = -\frac{2}{(1 + t^2)^2}$$

and for $t = 0$ we have $g(0, s(0)) = 0$. As $g(t, \cdot)$ is nonnegative outside of I_t, the minimal solution $s(t)$ is also a global solution of $g(t, \cdot)$ for $t > 0$. Apparently neither the minimal solutions nor the minimal values converge to those of $g(0, \cdot)$.
If we define the Lagrangian in the following way : $L(p, q, t) := g(p, t) + g(q, t)$ then $p_t = s(t)$ and $q_t = s(t)$ for $t \in [0, 1]$ and are hence discontinuous at $t = 0$.

Remark 4.8. Below we will discuss the question, under what conditions a C^1-solution x^* of the Euler-Lagrange equations is a weak (or strong) local minimum of the variational problem. In that context the continuity of $t \mapsto (x^*(t), \dot{x}^*(t))$ is already implied. The positive definiteness of the Hessian will be guaranteed by the Legendre-Riccati condition. Finally, the necessary condition (gradient equal to zero) for a pointwise minimum of the Lagrangian, equipped with a linear supplement, leads to the Euler-Lagrange equation.

Definition 4.9. Let $x^* \in C^1[a, b]^n$ be an extremal, and let

$$L^0_{\dot{x}\dot{x}}(t) := L_{\dot{x}\dot{x}}(x^*(t), \dot{x}^*(t), t)$$

satisfy the **strong Legendre-Clebsch condition**, i.e. $L^0_{\dot{x}\dot{x}}$ is positive definite on $[a, b]$, then x^* is called a **regular extremal**.

Remark 4.10. The Legendre-Clebsch condition, i.e. $L^0_{\dot{x}\dot{x}}$ positive semi-definite on $[a, b]$, is a classical necessary condition for a minimal solution of the variational problem (see [6]).
The subsequent Lemma provides a tool to relate the Legendre–Riccati condition (see below) to the positive definiteness of the Hessian of the Lagrangian (compare with Lemma 4.4).

Lemma 4.11. *Let $M \in L(\mathbb{R}^{2n})$ be a matrix of the structure*

$$M = \begin{pmatrix} A & C^T \\ C & D \end{pmatrix},$$

where $A, C, D \in L(\mathbb{R}^n)$ and D positive definite and symmetric. Then M is positive (semi-) definite if and only if $A - C^T D^{-1} C$ is positive (semi-)definite.

Proof : Let $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(p, q) := \begin{pmatrix} p \\ q \end{pmatrix}^T \begin{pmatrix} A & C^T \\ C & D \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = p^T Ap + 2q^T Cp + q^T Dq.$$

Minimization of the convex function $f(p, \cdot)$ yields : $2Dq = -2Cp$ and hence

$$q(p) = -D^{-1}Cp.$$

By inserting this result into f we obtain :

$$f(p, q(p)) = p^T Ap - 2p^T C^T D^{-1} Cp + p^T C^T D^{-1} Cp = p^T (A - C^T D^{-1} C)p.$$

Using our assumption $A - C^T D^{-1} C$ positive (semi-) definite it follows that $f(p, q) \geq f(p, q(p)) > 0$ for $p \neq 0$ ($f(p, q(p)) \geq 0$ in the semi-definite case). For $p = 0$ and $q \neq 0$ obviously $f(p, q) > 0$.
On the other hand, let M be positive (semi-)definite. Then $(0, 0)$ is the only (a) minimal solution of f. Hence the function $p \mapsto f(p, q(p))$ has 0 as the (a) minimal solution, i.e. $A - C^T D^{-1} C$ is positive (semi-)definite.

Definition 4.12. We say that the **Legendre–Riccati condition** is satisfied, if there exists a continuously differentiable symmetrical matrix-function $W : [a, b] \to L(\mathbb{R}^n)$ such that for all $t \in [a, b]$ the expression

$$L^0_{xx} + \dot{W} - (L^0_{x\dot{x}} + W)(L^0_{\dot{x}\dot{x}})^{-1}(L^0_{\dot{x}x} + W) \tag{6}$$

is positive definite.
If the Legendre-Riccati condition is satisfied, we introduce a **quadratic supplement potential**

$$F : [a, b] \times \mathbb{R}^n \to \mathbb{R}$$

with $F(t, p) = -\frac{1}{2} p^T W(t) p$ based on the corresponding matrix W such that the supplemented Lagrangian is strictly convex (see [13]).

Theorem 4.13 *(Fundamental Theorem). Let $L : \mathbb{R}^n \times \mathbb{R}^n \times [a, b] \to \mathbb{R}$ be continuous and $L(\cdot, \cdot, t)$ twice continuously differentiable. Then an admissible regular extremal x^* is a weak local minimal solution of the given variational problem if the Legendre-Riccati condition is satisfied.*

Proof : Let $F(t,p) = -\frac{1}{2}p^T W(t)p$ and let

$$\tilde{L}(p,q,t) := L(p,q,t) - \langle q, F_p(t,p)\rangle - F_t(t,p) = L(p,q,t) - \langle q, Wp\rangle - \frac{1}{2}\langle p, \dot{W}p\rangle$$

and let $\hat{L}(p,q,t) := \tilde{L}(p,q,t) - \langle \dot{\lambda}, p\rangle - \langle \lambda, p\rangle$. Setting the partial derivatives of $\hat{L}(\cdot,\cdot,t)$ to zero, yields the Euler-Lagrange equations (see (4) and (5)) for $\tilde{L}$. If x^* is an admissible extremal for L then also for $\tilde{L}$ and vice versa (Theorem 4.2). Because of the Legendre–Riccati condition and Lemma 4.11 the Hessian of $\hat{L}$ (and hence also of $\tilde{L}$) is positive definite. Lemma 4.4 then implies that x^* is a weak local minimum of the variational functional for $\tilde{L}$ and hence also for that of L, as the supplement is constant on S (see Lemma 2.2).
Using the subsequent

Lemma 4.14. *Let $A : U \to \mathbb{R}^n$ be continuous. If there is $r > 0$ and a ball $\overline{K}(x_0,r) \subset U$ such that $\langle Ax, x - x_0\rangle \geq 0$ for all $x \in S(x_0,r)$, then the nonlinear equation $Ax = 0$ has a solution in $\overline{K}(x_0,r)$.*

Proof : Otherwise Brouwer's fixed point theorem applied to the mapping

$$x \mapsto g(x) := -r\left(\frac{Ax}{\|Ax\|}\right) + x_0$$

would lead to a contradiction.
we will be able to establish that a weak local minimum is already a strong local minimum if the Lagrangian is uniformly convex with respect to the q-variable. This is due to

Lemma 4.15. *Let T be a compact subset of $\mathbb{R}^m$ and let $L : \mathbb{R}^n \times \mathbb{R}^n \times T \to \mathbb{R}$ and let L be continuously partially differentiable with respect to q.*
Let $\zeta : T \to \mathbb{R}^{2n}$ with $t \mapsto (p_t, q_t, t)$ be continuous with the following properties : there exists a $\delta > 0$ such that for all $t \in T$

1. *(p_t, q_t) is a minimum of $L(\cdot,\cdot,t)$ on $K_\delta(t) := (p_t, q_t) + K_\delta(0,0)$,*
2. (a) *$L(p,\cdot,t)$ is convex for $\|p - p_t\| < \delta$*

   (b) *$L(p_t,\cdot,t)$ is locally uniformly convex for $\|q - q_t\| < \delta$ in the following sense there is a module-function, independent of t such that*

$$L_q(p_t,q,t) - L_q(p_t,q_t,t) \geq \tau(\|q - q_t\|).$$

Then there is a $d > 0$ such that for all $p \in \mathbb{R}^n$ with $\|p - p_t\| < d$, for all $q \in \mathbb{R}^n$, and for all $t \in T$ we have

$$L(p_t,q_t,t) < L(p,q,t).$$

Proof : As the set $K := \bigcup_{t\in T}\{(p,q,t)|t \in K_\delta(t)\}$ is bounded in $\mathbb{R}^{2n+m}$, L_q is uniformly continuous on $\overline{K}$, i.e. there is a $0 < d < \delta$ such that for all $\|p - p_t\| < d$ and all $(p,q) \in K_\delta$ we have :

$$|L_q(p_t,q,t) - L_q(p,q,t)| < \epsilon < \frac{\tau(\delta/2)}{\delta/2}$$

Let $\rho := \delta/2$ and $S_\rho := \{q \in \mathbb{R}^n \,|\, \|q - q_t\| = \rho\}$. Then for all $p \in K(p_t, d)$ and $q \in S_\rho$ we obtain :

$$\begin{aligned}
&\langle L_q(p,q,t), q - q_t\rangle = \langle L_q(p,q,t) - L_q(p_t,q,t), q - q_t\rangle + \langle L_q(p_t,q,t), q - q_t\rangle \geq \\
&\geq \tau(\|q - q_t\|) - |\langle L_q(p,q,t) - L_q(p_t,q,t), q - q_t\rangle| \geq \\
&\geq \tau(\|q - q_t\|) - \|L_q(p,q,t) - L_q(p_t,q,t)\|\|q - q_t\| > \tau(\|q - q_t\|) - \epsilon\|q - q_t\| = \\
&= \tau(\rho) - \epsilon\cdot\rho > 0.
\end{aligned}$$

Then, according to Lemma 4.14 there is a $q(p,t) \in K(q_t,\rho)$ with the property $L_q(p,q(p,t),t) = 0$, hence $q(p,t)$ is minimum of $L(p,\cdot,t)$. Suppose now there exists a (p,q) such that $\|p - p_t\| < d$ where $L(p_t,q_t,t) > L(p,q,t)$. Then, because of $(p,q(p,t)) \in K_\delta(t)$:

$$L(p_t,q_t,t) \leq L(p,q(p,t),t) \leq L(p,q,t) < L(p_t,q_t,t),$$

which is a contradiction.

It appears that local uniform convexity of the Lagrangian in a C^1-neighborhood of an extremal is guaranteed through the Legendre–Riccati condition (see [13]). Namely, the following theorem is true.

Theorem 4.16 *(Uniform Strong Convexity of the Lagrangian) Let x^* be an extremal and let the Legendre–Riccati condition be satisfied, then there is a $\delta > 0$ and a $c > 0$ such that for all $(p,q),(u,v) \in K_t := K((x^*(t),\dot{x}^*(t)),\delta)$ and for all $t \in [a,b]$ we have*

$$\tilde{L}(\frac{p+u}{2},\frac{q+v}{2},t) \leq \frac{1}{2}\tilde{L}(p,q,t) + \frac{1}{2}\tilde{L}(u,v,t) - \frac{c}{8}(\|p-u\|^2 + \|q-v\|^2)$$

Theorem 4.17 *(Strong Local Minimum). Let x^* be an admissible, regular extremal and let the Legendre-Riccati condition be satisfied. Besides, let there exist a $\kappa > 0$ such that for all $t \in [a,b]$ and all p with $\|p - x^*(t)\| < \kappa$ the function $L(p,\cdot,t)$ is convex.*

Then x^* is a locally strong minimal solution of the variational problem, i.e. there is a positive d, such that for all $x \in K := \{x \in S|\ \|x - x^*\|_\infty < d\}$ we have :

$$\int_a^b L(x^*(t),\dot{x}^*(t),t)dt \leq \int_a^b L(x(t),\dot{x}(t),t)dt$$

§ 5. APPROXIMATION PROBLEMS

As an application we consider a class of **modular approximation problems** in the following sense : Let $\Phi : \mathbb{R}^2 \to \mathbb{R}$ be a twice differentiable convex function with positive definite Hessian, let $x \in C^2[a,b]$ be given and $S := \{v \in C^2[a,b] | v(a) = v(b) = 0\}$. We consider the approximation problem :

$$\text{minimize} \quad \int_a^b \Phi(x - v, \dot x - \dot v)dt \ , v \in S.$$

Then every admissible solution of the Euler–Lagrange equation

$$\frac{d}{dt}\Phi_r(x - v, \dot x - \dot v) = \Phi_x(x - v, \dot x - \dot v)$$

is a strong solution of the approximation problem, in particular uniquely determined.

Example 5.1. Let $a = 0$ and $b > 0$ and let $\Phi(p,q) := p^2 + q^2$ then for the Euler-Lagrange equation we obtain the linear second order ODE :

$$\ddot v - v = \ddot x - x =: f$$

with the solution

$$v(t) = \int_0^t \sinh(t - \tau) \cdot f(\tau)d\tau + v'(0)\sinh t,$$

where

$$v'(0) = -\frac{\int_0^b \sinh(b - \tau) \cdot f(\tau)d\tau}{\sinh b}$$

5.1. Parameter-free Approximation of Time-Series Data by Monotone Functions. In this subsection we treat a problem that occurs in the analysis of Time Series : we determine a parameter-free approximation of given data by a smooth monotone function in order to eliminate a monotone trend function from the data. In this way, investigations of cyclic behaviour of difference data ("Fourier-analysis") can be facilitated. In the discrete case, this type of approximation is known as monotone regression (see [3], p. 28 f).

In addition to the data themselves, our approximation also takes derivatives into account, employing the mechanism of variational calculus :

$$\text{minimize} \quad \int_a^b (v - x)^2 + (\dot v - \dot x)^2 dt \quad \text{on} \quad S.$$

where $S := \{v \in RCS^1[a,b] | \dot v \geq 0\}$.

Remark 5.2. The problem under consideration is related to Example 5.1. However, on the one hand, the class of functions is larger (piecewise smooth), but on the other hand we impose the additional restriction that the approximating function is increasing.
We also have the transversality conditions : $\eta(a) = 0 = \eta(b)$ (see the existence discussion in subsection 5.2).

Linear supplement : $F(t, p) = \eta(t) \cdot p$, then

$$\frac{d}{dt} F(\cdot, v(\cdot)) = F_t + F_p \cdot \dot{v} = \dot{\eta} v + \eta \dot{v}.$$

Using $\eta(a) = \eta(b) = 0$ we obtain that the supplement is constant on S :

$$\int_a^b \dot{\eta} v + \eta \dot{v} dt = [\eta(t) v(t)]_a^b = 0.$$

Let $\tilde{L}(p, q) := L(p, q) - \dot{\eta} p - \eta q = \frac{1}{2}(x-p)^2 - \dot{\eta} p + \frac{1}{2}(\dot{x}-q)^2 - \eta q$. Pointwise minimization of $\tilde{L}$ with respect to p and q is broken down into two separate parts :

1. $\min\{\frac{1}{2}(x-p)^2 - \dot{\eta} p | p \in \mathbb{R}\}$ with the result : $\dot{\eta} = p - x$,
2. $\min\{\frac{1}{2}(\dot{x}-q)^2 - \eta q | q \in \mathbb{R}_{\geq 0}\}$.

In order to perform the minimization in 2, we consider the function $\psi(q) := \frac{1}{2}(\dot{x} - q)^2 - \eta q$. Setting the derivative equal to zero, we obtain for $c := \dot{x} + \eta$:

$$\psi_q = q - \dot{x} - \eta = q - c = 0.$$

For $c \geq 0$ we obtain c as the minimal solution of the parabola ψ. For $c < 0$ we have $q = 0$ as the minimum of ψ on $\mathbb{R}_{\geq 0}$ because of the strict monotonicity of the parabola to the right of the global minimum. For $c \geq 0$ we obtain the linear inhomogeneous system of linear differential equations

$$\begin{pmatrix} \dot{\eta} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \eta \\ v \end{pmatrix} + \begin{pmatrix} -x \\ \dot{x} \end{pmatrix}.$$

For $c(t) = \dot{x}(t) + \eta(t) < 0$, this inequality holds, because of the continuity of c, on an interval I, i.e. $\dot{v}(t) = 0$ on I, i.e. $v(t) = \gamma$ there. Hence $\dot{\eta}(t) = \gamma - x(t)$ on I, i.e.

$$\eta(t) - \eta(t_1) = \int_{t_1}^{t} (\gamma - x(\tau)) d\tau = \gamma(t - t_1) - \int_{t_1}^{t} x(\tau) d\tau$$

Algorithm (shooting procedure) : Choose $\gamma = v(a)$, notation : $c(t) = \dot{x}(t) + \eta(t)$, note : $\eta(a) = 0$.

Start : $c(a) = \dot{x}(a)$

if $\dot{x}(a) \geq 0$ then set $t_1 = a$. **goto 1.**

if $\dot{x}(a) < 0$ then set $t_0 = a$. **goto 2.**

1. solve initial value problem

$$\begin{pmatrix} \dot{\eta} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \eta \\ v \end{pmatrix} + \begin{pmatrix} -x \\ \dot{x} \end{pmatrix}.$$

with the initial values $\eta(t_1), v(t_1)$, for which we have the following explicit solutions (see below)

$$\eta(t) = = (v(t_1) - x(t_1)) \sinh(t - t_1) + \eta(t_1) \cosh(t - t_1)$$
$$v(t) = = x(t) + (v(t_1) - x(t_1)) \cosh(t - t_1) + \eta(t_1) \sinh(t - t_1)$$

Let t_0 be the first root with change of sign of $c(t)$ such that $t_0 < b$. **goto 2**

2. $\eta(t) = v(t_0)(t - t_0) - \int_{t_0}^{t} x(\tau)d\tau + \eta(t_0)$. Let t_1 be the first root with change of sign of $c(t)$ such that $t_1 < b$. **goto 1.**

For given $\gamma = v(a)$, this algorithm yields a pair of functions (η_γ, v_γ). Our aim is, to determine a γ such that $\eta_\gamma(b) = 0$. In other words : let $\gamma \mapsto f(\gamma) := \eta_\gamma(b)$, then we have to solve the (1-D)- equation $f(\gamma) = 0$.

Solution of Differential equation in 2 : Let $\Phi(t)$ be a fundamental system then the solution of the (inhomogeneous) system is given by

$$\Psi(t) = \Phi(t) \int_{t_1}^{t} \Phi^{-1}(\tau) b(\tau) d\tau + \Phi(t) D,$$

where $D = \Phi^{-1}(t_1)\Psi(t_1)$. In our case we have the fundamental system

$$\Phi(t) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix},$$

and hence

$$\Phi^{-1}(t) = \begin{pmatrix} \cosh t & \sinh t \\ -\sinh t & \cosh t \end{pmatrix}$$

which yields (using corresponding addition theorems) :

$$\Phi(t)\Phi^{-1}(\tau) = \begin{pmatrix} \cosh(t-\tau) & \sinh(t-\tau) \\ \sinh(t-\tau) & \cosh(t-\tau) \end{pmatrix}.$$

We obtain

$$\Psi(t) = \int_{t_1}^{t} \begin{pmatrix} \cosh(t-\tau) & \sinh(t-\tau) \\ \sinh(t-\tau) & \cosh(t-\tau) \end{pmatrix} \begin{pmatrix} -x(\tau) \\ \dot{x}(\tau) \end{pmatrix} d\tau +$$

$$+\begin{pmatrix} \cosh(t-t_1) & \sinh(t-t_1) \\ \sinh(t-t_1) & \cosh(t-t_1) \end{pmatrix} \begin{pmatrix} \eta(t_1) \\ v(t_1) \end{pmatrix}.$$

i.e.

$$\eta(t) = \int_{t_1}^{t} \dot{x}(\tau)\sinh(t-\tau) - x(\tau)\cosh(t-\tau)d\tau +$$

$$+\eta(t_1)\cosh(t-t_1) + v(t_1)\sinh(t-t_1),$$

$$v(t) = \int_{t_1}^{t} \dot{x}(\tau)\cosh(t-\tau) - x(\tau)\sinh(t-\tau)d\tau +$$

$$+\eta(t_1)\sinh(t-t_1) + v(t_1)\cosh(t-t_1)$$

The integrals can be readily solved using the product rule :

$$\eta(t) = [x(\tau)\sinh(t-\tau)]_{t_1}^{t} + \eta(t_1)\cosh(t-t_1) + v(t_1)\sinh(t-t_1),$$

$$v(t) = [x(\tau)\cosh(t-\tau)]_{t_1}^{t} + \eta(t_1)\sinh(t-t_1) + v(t_1)\cosh(t-t_1).$$

We finally obtain the following explicit solutions

$$\eta(t) == (v(t_1) - x(t_1))\sinh(t-t_1) + \eta(t_1)\cosh(t-t_1),$$

$$v(t) == x(t) + (v(t_1) - x(t_1))\cosh(t-t_1) + \eta(t_1)\sinh(t-t_1).$$

5.2. Existence. If we minimize the strongly convex functional $f(v) := \int_a^b (v-x)^2 + (\dot{v}-\dot{x})^2 dt$ on the closed and convex subset S of the Sobolev Space $W_{2,1}^1[a,b]$, where $S := \{v \in W_{2,1}^1[a,b] | \dot{v} \geq 0\}$, then, according to [8], p. 289, Satz 2, f has a unique minimal solution, which is in particular absolutely continuous (see [7], p. 35). In order to establish the existence of the function η we make use of Pontryagin's Maximum Principle (see [7], p. 126 ff, compare also p. 208, Satz 1). Let

$$\min J(v,u) := \int_a^b L(v,u)dt, \quad \dot{v} = u,$$

such that $u(t) \in U := \mathbb{R}_{\geq 0}$, $h_0 = h_1 = 0$. We consider Pontryagin's function

$$H(v,u,\eta,\lambda_0) = \eta \cdot u - \lambda_0 \cdot L(v,u).$$

Let (v_*, u_*) be a minimal solution of J, then there is a number $\lambda_0 \geq 0$ and an absolutely continuous function η (not both identical to zero), such that the so-called adjoint equation :

$$\dot{\eta} = -H_v = \lambda_0 L_v(v_*, u_*) = \lambda_0(v_* - x)$$

is satisfied, together with the transversality conditions

$$\eta(a) = \eta(b) = 0.$$

It turns out that in our situation $\lambda_0 > 0$, for suppose $\lambda_0 = 0$, then $\eta = 0$ and $\eta(a) = 0$, i.e. $\eta = 0$, a contradiction to the above cited theorem.

5.3. Smoothness Considerations. The adjoint equation now assumes the form :

$$\dot{\eta} = v_* - x$$

which implies that $\eta \in C^1[a, b]$ and $c \in C[a, b]$. If $x \in C^1[a, b]$ then $c \in C^1[a, b]$. According to Pontryagin's Principle, $\dot{v}_*$ is pointwise maximum almost everywhere of the function H. As we have performed the pointwise minimization for all $t \in [a, b]$, according to the stability principle of convex optimization (see [8]), the resulting solution function v_* is in $C^1[a, b]$ ($\tilde{L}$ is continuous with respect to t).•

Резюме. Задача поточечной минимизации лагранжиана решается одновременной оптимизацией относительно обеих переменных — состояния и контроля. Условие Лежандра-Рикатти гарантирует существование эквивалентной выпуклой вариационной задачи, делающей возможным применение соответствующих принципов стабильности. Этот подход также допускает элементарное применение фундаментальных теорем вариационного исчисления, не используя теорию экстремальных полей. Этот подход применяется в задачах модулярного и непараметрического приближения временных рядов монотонными функциями. В статье предлагается метод, опирающийся на вариационное исчисление, для определения гладкой монотонной функции, которая аппроксимирует заданный временной ряд в смысле наименьших квадратов.

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