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SEVERAL UNCERTAINTY PRINCIPLES FOR TWO-STEP NILPOTENT LIE GROUPS

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Abstract. Some Hardy's uncertainty principles are proved for two-step nilpotent Lie groups. The group Fourier transforms are considered.

§1. INTRODUCTION

There is a classical theorem due to Hardy (see [5]).

Theorem A. Let a measurable function $f : \mathbb{R} \longrightarrow \mathbb{C}$ satisfy

(i) $$|f(x)| \leq Ce^{-a\pi x^2}, \quad x \in \mathbb{R},$$

and

(ii) $$|\hat{f}(y)| \leq Ce^{-b\pi y^2}, \quad y \in \mathbb{R},$$

where $C, a, b > 0$ and $\hat{f}$ is the Fourier transform of f. If $ab > 1$, then $f = 0$ almost everywhere. If $ab = 1$, then $f(x) = Ce^{-a\pi x^2}$. If $ab < 1$, then there exist infinitely many linearly independent functions satisfying (i) and (ii).

Cowling and Price [3] and Hörmander [6] established some general versions of Theorem A. Bagchi and Ray [2] extended the result of [3], [6] to $\mathbb{R}^n$, see the following Theorem B and Theorem C. In the sequel, for $x \in \mathbb{R}^n$, we denote $(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n) \in \mathbb{R}^{n-1}$ by x_j'.

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Theorem B. Let $f:\mathbb{R}^n \longmapsto \mathbb{C}$ be measurable and for some j $(1\le j\le n)$

(i)
$$\left\|e^{a\pi x_j^2}u(x_j{}')f(x)\right\|_{L^p(\mathbb{R}^n)}<+\infty,$$

(ii)
$$\left\|e^{b\pi y_j^2}v(y_j{}')\hat f(y)\right\|_{L^q(\mathbb{R}^n)}<+\infty,$$

where $a,b>0$, $u,v:\mathbb{R}^{n-1}\longmapsto \mathbb{C}$ are measurable with $u\ge\delta>0$, $v\ge\sigma>0$, where δ,σ are constants, $1/u\in L^{p'}(\mathbb{R}^{n-1})$, $p^{-1}+p'^{-1}=1$, $1/v\in L^{q'}(\mathbb{R}^{n-1})$, $q^{-1}+q'^{-1}=1$. If $ab\ge 1$, then $f=0$ almost everywhere.

Theorem C. Let $f\in L^1(\mathbb{R}^n)$ and for some j $(1\le j\le n)$, f and $\hat f$ satisfy

(i)
$$|f(x)|\le Cu(x_j{}')e^{-a\pi|x_j|^p},$$

(ii)
$$|\hat f(y)|\le Cv(y_j{}')e^{-b\pi|y_j|^q},$$

where $p^{-1}+p'^{-1}=1$, $u,v(\ge 0)\in L^1(\mathbb{R}^{n-1})$. If $(ap)^{1/p}(bq)^{1/q}>2$, then $f=0$ almost everywhere.

Recently considerable attention was devoted to extending forms of the uncertainty principle to some non-commutative groups (see [9] – [12]). There was a Hardy's uncertainty principle for two-step nilpotent Lie groups in [1] :

Theorem D. Suppose that a and b are constants such that $ab>1/4$, and that k and l are positive integers. Let f be a locally integrable function on N satisfying

(i)
$$\int_{\mathfrak{a}}|f(V,Z)|(1+|V|)^{-k}dV\le Ce^{-a|Z|^2},\quad Z\in\beta,$$

(ii)
$$\int_{r_\lambda^*}\|\pi_{\lambda,\mu}(f)\|_{op}(1+|\mu|)^{-k}d\mu\le C'r(\lambda)^l e^{-b\pi|\lambda|^2},\quad \lambda\in\Lambda,$$

where $\|\cdot\|_{op}$ denotes the operator norm and the meaning of the other notations are explained in the next section. Then $f=0$ almost everywhere.

The purpose of this paper is to establish several uncertainty principles on two-step nilpotent Lie groups different from Theorem D. Now we describe the main results and give the necessary facts about the two-step nilpotent Lie groups in Section 2.

Theorem 1. Let N be the two-step nilpotent Lie group, $S(N)$ be the Schwartz space on N and $f \in S(N)$. Suppose that $a, b > 0$, $1 \leq q < \infty$ and for some $j (1 \leq j \leq k)$, f satisfies

(i) $$|f(V,Z)| \leq C e^{-a\pi\|(V,Z)\|^4}, \quad (V,Z) \in N.$$

(ii) $$\int_{\beta^*} \int_{\Gamma_\lambda^*} e^{qb\pi(|\lambda|^2+|\mu|^2)} |v(\lambda_j{}')|^q \|\pi_{\lambda,\mu}(f)\|_{HS}^q \, Sym_{2m} B(\lambda)^{1/2} d\mu \, d\lambda < \infty.$$

where $v : \mathbb{R}^{k-1} \longrightarrow \mathbb{C}$ is measurable, $v \geq \sigma > 0$ for some constant σ and $1/v \in L^{q'}(\mathbb{R}^{k-1})$, $q^{-1} + q'^{-1} = 1$.

(1) If $q = 2$ and $ab \geq 1$, then $f = 0$ almost everywhere;

(2) If $q > 2$ and $ab > 1$, then $f = 0$ almost everywhere;

(3) If $1 \leq q < 2$ and $ab > 2$, then $f = 0$ almost everywhere.

Theorem 2. Let f be in $S(N)$. Suppose that $a, b > 0$, $1 \leq p < \infty$ and for some j $(1 \leq j \leq k)$, f satisfies

(i) $$\int_{\beta} \int_{\alpha} e^{pa\pi\|(V,Z)\|^4} |u(Z_j{}')|^p |f(V,Z)|^p \, dV \, dZ < \infty, \quad (V,Z) \in N.$$

(ii) $$\int_{\beta^*} \int_{\Gamma_\lambda^*} e^{qb\pi(|\lambda|^2+|\mu|^2)} |v(\lambda_j{}')|^q \|\pi_{\lambda,\mu}(f)\|_{HS}^q \, Sym_{2m} B(\lambda)^{1/2} d\mu \, d\lambda < \infty.$$

where $\lambda \neq 0$, $u, v : \mathbb{R}^{k-1} \longrightarrow \mathbb{C}$ are measurable with $u \geq \delta > 0$, $v \geq \sigma > 0$, where δ, σ are constants, $1/u \in L^{p'}(\mathbb{R}^{k-1})$, $p^{-1} + p'^{-1} = 1$, $1/v \in L^{q'}(\mathbb{R}^{k-1})$, $q^{-1} + q'^{-1} = 1$.

(1) If $q \geq 2$ and $ab > 1$, then $f = 0$ almost everywhere;

(2) If $1 \leq q < 2$ and $ab > 2$, then $f = 0$ almost everywhere.

Theorem 3. Let $a, b > 0$ and for some $j (1 \leq j \leq k)$, $f \in S(N)$ satisfies

(i) $$|f(V,Z)| \leq C e^{-a\pi\|(V,Z)\|^{2p}}, \quad (V,Z) \in N,$$

(ii) $$\|\pi_{\lambda,\mu}(f)\|_{HS} \leq C v(\lambda_j{}') e^{-b\pi(|\lambda|+|\mu|)^q}, \quad \lambda \neq 0.$$

where $v \in L^2(\mathbb{R}^{k-1})$, $p \geq 2$, $p^{-1} + q^{-1} = 1$. If $(ap)^{1/p}(bq)^{1/q} > 2$, then $f = 0$ almost everywhere.

The paper is organized as follows: In Section 2 we recall the Lie algebras and Lie groups under consideration, including the group Fourier transforms. The importance of the group Fourier transforms to the study of partial operators on N is well known (see [8] and references therein). We will prove Theorem 1, Theorem 2 and Theorem 3 in Section 3, Section 4 and Section 5, respectively.

In what follows, we often use partial Fourier transforms. If A is a subspace of B and f is a function in B, then we denote by F_A the Euclidean Fourier transform of f with respect to the variables in A. Also we employ the "variable constant convention", i.e. all constants are positive, but not necessarily equal.

§2. TWO-STEP NILPOTENT LIE GROUPS

We collect some facts about two-step nilpotent Lie algebras and Lie groups. Let η be a real two-step nilpotent Lie algebra, that is, $[\eta, \eta] \neq \{0\}$ and $[\eta, [\eta, \eta]] = \{0\}$. We write η as the sum of subspaces

$$\eta = \alpha \oplus \beta,$$

where β is the center of η and α is any subspace of η complementary to β, $\dim \beta = k$. We take an inner product on η that renders α and β are orthogonal.

Let N be the connected, simply connected Lie group with Lie algebra η. By the exponential map, we parameterize N by $\alpha \oplus \beta$ and write (V, Z) for $exp(V + Z)$, where $V \in \alpha$, $Z \in \beta$. The Baker-Campbell-Hausdorff formula gives the product law in N

$$(V, Z) \cdot (\tilde{V}, \tilde{Z}) = \left(V + \tilde{V}, Z + \tilde{Z} + \frac{1}{2}\left[V, \tilde{V}\right]\right), \quad V, \tilde{V} \in \alpha, \quad Z, \tilde{Z} \in \beta. \tag{2.1}$$

»From (2.1)

$$(V, Z)^{-1} = (-V, -Z), \quad (V, Z) \in \alpha \oplus \beta. \tag{2.2}$$

The norm on N is defined by

$$\|(V, Z)\| = (|V|^4 + |Z|^2)^{1/4}, \quad (V, Z) \in \alpha \oplus \beta. \tag{2.3}$$

We denote by dV and dZ the Lebesgue measures on α and β, respectively. Then $dVdZ$ is a Haar measure on N. We write $S(N)$ for the Schwartz space on N.

We now recall the unitary representations of the group N. Fixing $\lambda \in \beta^*$ (β^* is the dual of β), we define the skew-symmetric linear mapping $B(\lambda)$ on α by

$$\langle B(\lambda)U, V\rangle = \lambda([U, V]), \quad U, V \in \alpha. \tag{2.4}$$

Denote the kernel of $B(\lambda)$ by r_λ and the orthogonal complement of r_λ in α by m_λ. Since $B(\lambda)$ is skew-symmetric, m_λ is $B(\lambda)$-invariant and its dimension $\dim m_\lambda$ is even. Let Λ be the Zariski-open subspace subset of β^* of the vector λ for which $\dim m_\lambda$

is maximum, $\dim m_\lambda = 2m$ for all $\lambda \in \Lambda$. $Sym_{2m} B(\lambda)$ stands for the symmetric function of degree $2m$ in the roots of $B(\lambda)$.

Given $\lambda \in \Lambda$, there are orthogonal vectors $E_1(\lambda), \cdots, E_m(\lambda), \bar{E}_1(\lambda), \cdots, \bar{E}_m(\lambda)$ in m_λ and values $b_1(\lambda), \cdots, b_m(\lambda)$ in $\mathbb{R}^+$ such that

$$B(\lambda)E_i(\lambda) = b_i(\lambda)\bar{E}_i(\lambda), \quad B(\lambda)\bar{E}_i(\lambda) = -b_i(\lambda)E_i(\lambda), \quad Sym_{2m} B(\lambda) = \prod_{i=1}^{m} b_i(\lambda)^2.$$

Denote by γ_λ and $\bar{\gamma}_\lambda$ the subspace $span\{E_1(\lambda), \cdots, E_m(\lambda)\}$ and $span\{\bar{E}_1(\lambda), \cdots, \bar{E}_m(\lambda)\}$, respectively, and write $V \in \alpha$ as $W + R$ or $X + Y + R$, where $W \in m_\lambda$, $X \in \gamma_\lambda$, $Y \in \bar{\gamma}_\lambda$, $R \in r_\lambda$. The elements $\exp(W + R + Z)$ and $\exp(X + Y + R + Z)$ of N we denote by (W, R, Z) and (X, Y, R, Z), respectively.
For $\lambda \in \Lambda$ and $\mu \in r_\lambda^*$ (the dual of r_λ), the irreducible unitary representation is defined to be

$$[\pi_{\lambda,\mu}(X, Y, R, Z)\varphi](\bar{X}) = e^{-2\pi i\lambda(Z+[\bar{X}+\frac{1}{2}X, Y])}e^{-2\pi i\mu(R)}\varphi(X + \bar{X}), \quad \varphi \in L^2(\gamma_\lambda). \tag{2.5}$$

Define the Fourier transform of a function $f \in L^1(N)$ by

$$\pi_{\lambda,\mu}(f) = \int_N f(n)\pi_{\lambda,\mu}(n)\, dn, \quad \lambda \in \Lambda, \quad \mu \in r_\lambda^*. \tag{2.6}$$

If $f \in S(N)$, then $\pi_{\lambda,\mu}(f)$ is a Hilbert-Schmidt operator. In fact, (2.5) infers

$$\pi_{\lambda,\mu}(f)\varphi(\bar{X}) = \int_{\gamma_\lambda}\int_{\bar{\gamma}_\lambda}\int_{r_\lambda}\int_{\beta} f(X, Y, R, Z)e^{-2\pi i\lambda(Z+[\bar{X}+\frac{1}{2}X, Y])-2\pi i\mu(R)} \times$$

$$\times\varphi(X + \bar{X})\, dZ\, dR\, dY\, dX = \int_{\gamma_\lambda} K_{f,\lambda,\mu}(\bar{X}, X)\varphi(X)\, dX, \tag{2.7}$$

where

$$K_{f,\lambda,\mu}(\bar{X}, X) =$$

$$= \int_{\bar{\gamma}_\lambda}\int_{r_\lambda}\int_{\beta} f(X - \bar{X}, Y, R, Z)e^{-2\pi i\lambda(Z+\frac{1}{2}[\bar{X}+X, Y])-2\pi i\mu(R)}\, dZ\, dR\, dY =$$

$$= F_{\bar{\gamma}_\lambda\oplus r_\lambda\oplus\beta} f(X - \bar{X}, \frac{1}{2}B(\lambda)(X + \bar{X}), \mu, \lambda) \tag{2.8}$$

is a kernel operator. The Plancherel formula in Schwartz space gives

$$\|\pi_{\lambda,\mu}(f)\|_{HS}^2 = \int_{\gamma_\lambda}\int_{\gamma_\lambda} |K_{f,\lambda,\mu}(\bar{X}, X)|^2 dX\, d\bar{X} =$$

$$= Sym_{2m} B(\lambda)^{-1/2} \int_{m_\lambda} |F_{\gamma_\lambda\oplus\beta}(W, \mu, \lambda)|^2 dW.$$

where $\|\pi_{\lambda,\mu}(f)\|_{HS}$ is the Hilbert-Schmidt norm, so that

$$\int_{r_\lambda^*} \|\pi_{\lambda,\mu}(f)\|_{HS}^2 \, Sym_{2m} B(\lambda)^{1/2} d\mu = \int_\alpha |F_\beta f(V,\lambda)|^2 dV. \tag{2.9}$$

Also, (2.9) implies the Plancherel formula on N

$$\int_{\beta^*} \int_{r_\lambda^*} \|\pi_{\lambda,\mu}(f)\|_{HS}^2 \, Sym_{2m} B(\lambda)^{1/2} d\mu \, d\lambda = \int_N |f(n)|^2 dn. \tag{2.10}$$

Polarizing (2.9) yields

$$\int_{r_\lambda^*} tr(\pi_{\lambda,\mu}(f)\pi_{\lambda,\mu}(g)^*) Sym_{2m} B(\lambda)^{1/2} d\mu = \int_\alpha F_\beta f(V,\lambda)\overline{F_\beta g(V,\lambda)} dV. \tag{2.11}$$

For a function $g \in S(N)$ and any positive integer k, there exists a constant C and a positive integer L such that (see [1])

$$(1+|\mu|)^k tr|\pi_{\lambda,\mu}(g)| \le C r(\lambda)^L \|g\|_*, \quad \lambda \in \Lambda, \quad \mu \in r_\lambda^*, \tag{2.12}$$

where

$$r(\lambda) = \sum_{i=1}^m (b_i(\lambda)^{-2} + b_i(\lambda)^2),$$

and $\|\cdot\|_*$ denotes a sum of Schwartz seminorms. Since $r(\lambda)$ is a symmetric rational function in the eigenvalues of $B(\lambda)$, it is a rational function of λ and satisfies

$$Sym_{2m} B(\lambda) = \prod b_i(\lambda)^2 \le m^{-m} \left(\sum_{i=1}^m b_i(\lambda)^2 \right)^m \le$$

$$\le m^{-m} \left\{ \sum_{i=1}^m \left[b_i(\lambda)^{-2} + b_i(\lambda)^2 \right] \right\}^m = m^{-m} r(\lambda)^m. \tag{2.13}$$

§3. PROOF OF THEOREM 1

Denote

$$f_V(Z) = f(V,Z), \quad f_V^*(Z) = \overline{f(V,-Z)},$$

$$h(Z) = \int_\alpha (f_V * f_V^*)(Z) \, dV = \int_\alpha \int_\beta f_V(\tilde{Z}) \overline{f_V(\tilde{Z}-Z)} \, d\tilde{Z} \, dV.$$

By the assumption (i),

$$|h(Z)| \le \int_\alpha \int_\beta |f_V(\tilde{Z})| \cdot |f_V(\tilde{Z}-Z)| \, d\tilde{Z} \, dV \le$$

$$\le C\int_\alpha\int_\beta e^{-a\pi(2|V|^4+|\tilde{Z}|^2+|\tilde{Z}-Z|^2)}\, d\tilde{Z}\, dV \le$$

$$\le C\int_\alpha\int_\beta e^{-a\pi[2|V|^4+\frac{1}{2}(2|\tilde{Z}|-|Z|)^2+\frac{1}{2}|Z|^2]}\, d\tilde{Z}\, dV =$$

$$= Ce^{-\frac{a}{2}\pi|Z|^2}\int_\alpha\int_\beta e^{-a\pi[2|V|^4+\frac{1}{2}(2|\tilde{Z}|-|Z|)^2]}\, d\tilde{Z}\, dV \le Ce^{-\frac{a}{2}\pi|Z|^2} \le Ce^{-\frac{a}{2}\pi Z_j^2}, \quad (3.1)$$

where we used that

$$|\tilde{Z}|^2+|\tilde{Z}-Z|^2 \ge |\tilde{Z}|^2+(|\tilde{Z}|-|Z|)^2 = \frac{1}{2}(2|\tilde{Z}|-|Z|)^2+\frac{1}{2}|Z|^2$$

and $e^{-a\pi[2|V|^4+\frac{1}{2}(2|\tilde{Z}|-|Z|)^2]}$ is integrable on N.

On the other hand, it follows from (2.9) that

$$\left|\tilde{h}(\lambda)\right| = \left|\int_\beta e^{-2\pi i\lambda(Z)}\int_\alpha\int_\beta f(V,\tilde{Z})\cdot\overline{f(V,\tilde{Z}-Z)}\, d\tilde{Z}\, dV\, dZ\right| =$$

$$= \left|\int_\beta\int_\alpha\int_\beta e^{-2\pi i\lambda(\tilde{Z})} f(V,\tilde{Z})\cdot e^{-2\pi i\lambda(Z-\tilde{Z})}\overline{f(V,\tilde{Z}-Z)}\, d\tilde{Z}\, dV\, dZ\right| =$$

$$= \left|\int_\alpha F_\beta f(V,\lambda)\cdot\overline{F_\beta f(V,\lambda)}\, dV\right| = \int_\alpha |F_\beta f(V,\lambda)|^2\, dV =$$

$$= \int_{r_\lambda^*} ||\pi_{\lambda,\mu}(f)||_{HS}^2\, Sym_{2m} B(\lambda)^{1/2}\, d\mu. \quad (3.2)$$

We consider the three cases separately.

Case 1 : $q=2$, $ab\ge 1$. By $v(\lambda_j{}')\ge\sigma>0$, the assumption (ii) and (3.2), we have

$$\int_{\beta^*} e^{2b\pi\lambda_j^2}|v(\lambda_j{}')|\cdot|\tilde{h}(\lambda)|\, d\lambda \le \frac{1}{\sigma}\int_{\beta^*} e^{2b\pi(|\lambda|^2+|\mu|^2)}|v(\lambda_j{}')|^2|\tilde{h}(\lambda)|\, d\lambda =$$

$$= \frac{1}{\sigma}\int_{\beta^*}\int_{r_\lambda^*} e^{2b\pi(|\lambda|^2+|\mu|^2)}|v(\lambda_j{}')|^2\, ||\pi_{\lambda,\mu}(f)||_{HS}^2\, Sym_{2m} B(\lambda)^{1/2} d\mu\, d\lambda < \infty. \quad (3.3)$$

»From (3.1) and (3.3), we see that h satisfies the conditions of Theorem B for $p=\infty$ and $q=2$. We note that the assumption (i) in Theorem B becomes $|f(x)|\le Ce^{-a\pi x_j^2}$ if $p=\infty$. Hence $h=0$ almost everywhere and then $\tilde{h}=0$. From (3.2), we have $||\pi_{\lambda,\mu}(f)||_{HS}=0$ for almost all $\lambda\in\Lambda$, $\mu\in r_\lambda^*$, and then by the Plancherel formula (2.10), $f=0$ almost everywhere.

Case 2 : $q>2$, $ab>1$. Let $\varepsilon>0$ be such that $b'=b-\varepsilon>0$ and $ab'>1$. By (3.2),

$$\int_{\beta^*} e^{qb'\pi|\lambda|^2}|v(\lambda_j{}')|^q|\tilde{h}(\lambda)|^{q/2}d\lambda =$$

$$= \int_{\beta^*} e^{qb'\pi|\lambda|^2}|v(\lambda_j')|^q \left(\int_{r_\lambda^*} ||\pi_{\lambda,\mu}(f)||_{HS}^2 Sym_{2m} B(\lambda)^{1/2} d\mu\right)^{q/2} d\lambda \le$$

$$\le \int_{\beta^*} e^{qb'\pi|\lambda|^2}|v(\lambda_j')|^q Sym_{2m} B(\lambda)^{q/4} \left(\int_{r_\lambda^*} e^{2b'\pi|\mu|^2} ||\pi_{\lambda,\mu}(f)||_{HS}^2 d\mu\right)^{q/2} d\lambda =$$

$$= \int_{\beta^*} e^{qb'\pi|\lambda|^2}|v(\lambda_j')|^q Sym_{2m} B(\lambda)^{q/4} \left(\int_{r_\lambda^*} e^{-2\epsilon\pi|\mu|^2} e^{2b\pi|\mu|^2} ||\pi_{\lambda,\mu}(f)||_{HS}^2 d\mu\right)^{q/2} d\lambda.$$

Using Hölder's inequality and the assumption (ii)

$$\int_{\beta^*} e^{qb'\pi\lambda_j^2}|v(\lambda_j')|^{q/2}|\hat{h}(\lambda)|^{q/2} d\lambda \le \int_{\beta^*} e^{qb'\pi|\lambda|^2}\sigma^{-q/2}|v(\lambda_j')|^q|\hat{h}(\lambda)|^{q/2} d\lambda \le$$

$$\le \sigma^{-q/2} \int_{\beta^*} e^{qb'\pi|\lambda|^2}|v(\lambda_j')|^q Sym_{2m} B(\lambda)^{q/4} \times$$

$$\times \left(\int_{r_\lambda^*} e^{-2\epsilon q/(q-2)\pi|\mu|^2} d\mu\right)^{q/2-1} \left(\int_{r_\lambda^*} e^{qb\pi|\mu|^2} ||\pi_{\lambda,\mu}(f)||_{HS}^q d\mu\right) d\lambda \le$$

$$\le C \int_{\beta^*} \int_{r_\lambda^*} e^{qb\pi(|\lambda|^2+|\mu|^2)}|v(\lambda_j')|^q ||\pi_{\lambda,\mu}(f)||_{HS}^q Sym_{2m} B(\lambda)^{1/2} d\mu d\lambda < \infty, \quad (3.4)$$

where we used that $e^{-2\epsilon q/(q-2)\pi|\mu|^2}$ is integrable on r_λ^* and

$$e^{-\epsilon q\pi|\lambda|^2} Sym_{2m} B(\lambda)^{1/2(q/2-1)}$$

is bounded on Λ. By (3.1) and (3.4), h satisfies the conditions of Theorem B for $p = \infty$ and $q/2 (q > 2)$. So $h = 0$ almost everywhere and then $f = 0$ almost everywhere.
Case 3 : $1 \le q < 2$, $ab > 2$. By (3.2), (2.11), (2.12) and

$$tr(\pi_{\lambda,\mu}(f)\pi_{\lambda,\mu}(f)^*) \le ||\pi_{\lambda,\mu}(f)||_{HS} tr|\pi_{\lambda,\mu}(f)|$$

(see [1]), we obtain

$$\left|\hat{h}(\lambda)\right| = \left|\int_a F_\beta f(V,\lambda)\overline{F_\beta f(V,\lambda)}\, dV\right| = \left|\int_{r_\lambda^*} tr(\pi_{\lambda,\mu}(f)\pi_{\lambda,\mu}(f)^*) Sym_{2m} B(\lambda)^{1/2} d\mu\right| \le$$

$$\le \int_{r_\lambda^*} ||\pi_{\lambda,\mu}(f)||_{HS} tr|\pi_{\lambda,\mu}(f)| Sym_{2m} B(\lambda)^{1/2} d\mu \le$$

$$\le C\, r(\lambda)^L ||f||_* \int_{r_\lambda^*} ||\pi_{\lambda,\mu}(f)||_{HS} (1+|\mu|)^{-k} Sym_{2m} B(\lambda)^{1/2} d\mu \le$$

$$\leq C\,r(\lambda)^L \|f\|_\circ \left(\int_{\Gamma_\lambda^\circ} \|\pi_{\lambda,\mu}(f)\|_{HS}^q \, Sym_{2m}\, B(\lambda)^{1/2} d\mu\right)^{1/q} \times$$

$$\times \left(\int_{\Gamma_\lambda^\circ} (1+|\mu|)^{-q'k} d\mu\right)^{1/q'} (Sym_{2m}\, B(\lambda)^{1/2(1-1/q)},$$

where q and q' satisfy $1/q + 1/q' = 1$ and $q' > 2$. By (2.13), there exist a positive integer l, such that

$$(Sym_{2m}\, B(\lambda)^{1/2(1-1/q)} \leq C\,r(\lambda)^l.$$

Then

$$|\hat{h}(\lambda)| \leq C\,r(\lambda)^{L+l} \|f\|_\circ \left(\int_{\Gamma_\lambda^\circ} \|\pi_{\lambda,\mu}(f)\|_{HS}^q \, Sym_{2m}\, B(\lambda)^{1/2} d\mu\right)^{1/q} \tag{3.5}$$

Let $q(D)$ be a differential operator on β with constant coefficients. Its Fourier transform $q(\lambda)$ is the denominator of the rational function $r(\lambda)$ (see [1]). We choose a function $H \in C_0^\infty(\beta)$ with

$$supp H \subset \left\{\bar{Z} : |\bar{Z}| < \varepsilon, \quad \varepsilon > 0 \text{ is arbitrarily small}\right\},$$

and denote $H_1 = q(D)^{L+l} H$. By (3.1) and (3.5)

$$|H_1 * h(Z)| = \left|\int_{|\bar{Z}|<\varepsilon} H_1(\bar{Z}) h(Z-\bar{Z}) d\bar{Z}\right| \leq C \left|\int_{|\bar{Z}|<\varepsilon} H_1(\bar{Z}) e^{-\frac{1}{2}\pi|Z-\bar{Z}|^2} d\bar{Z}\right| \leq$$

$$\leq C e^{-\frac{1}{2}\pi|Z|^2} \left|\int_{|\bar{Z}|<\varepsilon} H_1(\bar{Z}) e^{\frac{1}{2}\pi|\bar{Z}|^2} d\bar{Z}\right| \leq C e^{-\frac{1}{2}\pi|Z|^2} \leq C e^{-\frac{1}{2}\pi Z_i^2}; \tag{3.6}$$

$$\left|\widehat{H_1 * h}(\lambda)\right| = |\hat{H}_1(\lambda)| \cdot |\hat{h}(\lambda)| \leq C|P(\lambda)| \left(\int_{\Gamma_\lambda^\circ} \|\pi_{\lambda,\mu}(f)\|_{HS}^q \, Sym_{2m}\, B(\lambda)^{1/2} d\mu\right)^{1/q}, \tag{3.7}$$

where $P(\lambda)$ is a polynomial. As before, let $\varepsilon > 0$ be such that $b' = b - \varepsilon > 0$ and $ab' > 2$. By (3.7), the assumption (ii) and the boundedness of $e^{-\varepsilon q\pi|\lambda|^2}|P(\lambda)|^q$,

$$\int_{\beta^*} e^{qb'\pi\lambda_j^2} |v(\lambda_j{}')|^q \left|\widehat{H_1 * h}(\lambda)\right|^q d\lambda \leq$$

$$\leq C \int_{\beta^*}\int_{\Gamma_\lambda^\circ} e^{qb\pi|\lambda|^2} e^{-\varepsilon q\pi|\lambda|^2} |v(\lambda_j{}')|^q |P(\lambda)|^q \|\pi_{\lambda,\mu}(f)\|_{HS}^q \, Sym_{2m}\, B(\lambda)^{1/2} d\mu\, d\lambda \leq$$

$$\leq C \int_{\beta^*}\int_{\Gamma_\lambda^\circ} e^{qb\pi(|\lambda|^2+|\mu|^2)} |v(\lambda_j{}')|^q \|\pi_{\lambda,\mu}(f)\|_{HS}^q \, Sym_{2m}\, B(\lambda)^{1/2} d\mu\, d\lambda < \infty. \tag{3.8}$$

By (3.6) and (3.8), $H_1 * h$ satisfies the conditions of Theorem B for $p = \infty$ and q $(1 \leq q < 2)$, so $H_1 * h = 0$ almost everywhere, $\widehat{H_1 * h} = \hat{H}_1 \cdot \hat{h} = 0$. Since $\hat{H}_1 = 0$ on a set of zero measure, we conclude that $\hat{h} = 0$ almost everywhere and then $f = 0$ almost everywhere.

§4. PROOF OF THEOREM 2

Let $(V, Z), (\tilde{V}, \tilde{Z}) \in N$. By (2.4).

$$\left|\left[\tilde{V}, V\right]\right| = \frac{1}{|\lambda|}\left|< B(\lambda)\tilde{V}, V >\right| \leq$$

$$\leq \frac{\|B(\lambda)\|}{|\lambda|}\|(\tilde{V}, \tilde{Z})\| \cdot \|(V, Z)\| = 2c\left\|\left(\tilde{V}, \tilde{Z}\right)\right\| \cdot \|(V, Z)\|,$$

where $\|B(\lambda)\|$ denotes the norm of the matrix $B(\lambda)$, $c = \frac{\|B(\lambda)\|}{2|\lambda|}$. Then

$$\left\|(V, Z)\left(-\tilde{V}, -\tilde{Z}\right)\right\| = \left\|\left(V - \tilde{V}, Z - \tilde{Z} + \frac{1}{2}\left[\tilde{V}, V\right]\right)\right| \geq$$

$$\geq \|(V, Z)\| - \left\|\left(\tilde{V}, \tilde{Z}\right)\right\| - c^{1/2}\left\|\left(\tilde{V}, \tilde{Z}\right)\right\|^{1/2}\|(V, Z)\|^{1/2}. \tag{4.1}$$

We choose $g \in C_0(N)$ to satisfy

$$supp\, g \subset \left\{\left(\tilde{V}, \tilde{Z}\right) : \left\|\left(\tilde{V}, \tilde{Z}\right)\right\| \leq \epsilon^2\right\},$$

and let $\|(V, Z)\| > 1$. Then (4.1) implies

$$\left\|(V, Z)\left(\tilde{V}, \tilde{Z}\right)^{-1}\right\| \geq \|(V, Z)\|(1 - (1 + c^{1/2})\epsilon), \quad (\tilde{V}, \tilde{Z}) \subset supp\, g.$$

Denoting

$$\phi(\epsilon) = (1 + c^{1/2})\epsilon, \quad e_a(V, Z) = e^{a\pi\|(V,Z)\|^4},$$

$$u(V, Z) = u(Z_1, \cdots, \hat{Z}_j, \cdots, Z_k) = u(Z_j'),$$

we obtain

$$(|g| * (e_a|u| \cdot |f|))(V, Z) =$$

$$= \int_\alpha \int_\beta \left|g\left(\tilde{V}, \tilde{Z}\right)\right| e^{a\pi\|(V,Z)(\tilde{V},\tilde{Z})^{-1}\|^4} \left|u\left(Y_j'\right)\right| \left|f\left((V, Z)\left(\tilde{V}, \tilde{Z}\right)^{-1}\right)\right| d\tilde{Z}d\tilde{V} \geq$$

$$\geq \delta e^{a\pi(1-\phi(\epsilon))^4\|(V,Z)\|^4}(|g| * |f|)(V, Z), \tag{4.2}$$

where $Y = Z - \tilde{Z} - \frac{1}{2}[V, \tilde{V}]$. Noting that $e_a|u| \cdot |f|$ is an L^p function, by the assumption (i) in Theorem 2, g is an $L^{p'}$ function $(1/p + 1/p' = 1)$. Then it follows that $|g| * (e_a|u| \cdot |f|)$ is an L^∞ function.

From (4.2), we have

$$|(g * f)(V, Z)| \leq (|g| * |f|)(V, Z) \leq \frac{1}{\delta}\, C e^{-a\pi(1-\phi(\epsilon))^4\|(V,Z)\|^4},$$

where $C = \|(|g| * (e_a|u| \cdot |f|))\|_{L^\infty}$, $\|(V, Z)\| > 1$. By continuity of $g * f$ ($f \in L^1(N)$, $g \in L^\infty(N)$),

$$|(g * f)(V, Z)| \leq C e^{-a\pi(1-\phi(\epsilon))^4 \|(V,Z)\|^4}, \quad (V, Z) \in N. \tag{4.3}$$

Since $\pi_{\lambda,\mu}(g * f) = \pi_{\lambda,\mu}(g) \cdot \pi_{\lambda,\mu}(f)$ and $\pi_{\lambda,\mu}(g)$ is a bounded linear operator on $L^2(\gamma_\lambda)$, we get

$$\|\pi_{\lambda,\mu}(g * f)\|_{HS} \leq \|\pi_{\lambda,\mu}(g)\|_{op} \|\pi_{\lambda,\mu}(f)\|_{HS} \leq \|g\|_{L^1(N)} \|\pi_{\lambda,\mu}(f)\|_{HS}. \tag{4.4}$$

where we used

$$\|\pi_{\lambda,\mu}(g)\|_{op} \leq \int_N |g(n)| \, \|\pi_{\lambda,\mu}(n)\| \; dn \leq$$

$$\leq \int_N |g(n)| \left| e^{-2\pi i \lambda(Z + [\bar{X} + \frac{1}{2}X, Y])} e^{-2\pi i \mu(R)} \right| \; dn =$$

$$= \int_N |g(n)| dn = \|g\|_{L^1(N)}, \quad n = (X, Y, R, Z) \in N, \quad \bar{X} \in \gamma_\lambda.$$

By (4.4) and the assumption (ii) in Theorem 2,

$$\int_{\beta^*} \int_{r_\lambda^*} e^{qb\pi(|\lambda|^2 + |\mu|^2)} |v(\lambda_j')|^q \|\pi_{\lambda,\mu}(g * f)\|_{HS}^q \, Sym_{2m} B(\lambda)^{1/2} d\mu \, d\lambda \leq$$

$$\leq \|g\|_{L^1(N)}^q \int_{\beta^*} \int_{r_\lambda^*} e^{qb\pi(|\lambda|^2 + |\mu|^2)} |v(\lambda_j')|^q \|\pi_{\lambda,\mu}(f)\|_{HS}^q \, Sym_{2m} B(\lambda)^{1/2} d\mu \, d\lambda < \infty. \tag{4.5}$$

For $ab > 1$ (or $ab > 2$), we can take ε such that $ab(1-\phi(\varepsilon))^4 > 1$ (or $ab(1-\phi(\varepsilon))^4 > 2$). By (4.3) and (4.5), the proof of Theorem 2 is reduced to the proof of Theorem 1. So, $g * f = 0$ almost everywhere. Since g is an approximate identity, $f = 0$ almost everywhere.

§5. PROOF OF THEOREM 3

Using the function h in the proof of Theorem 1 and the assumption (i) in Theorem 3, we infer

$$|h(Z)| \leq \int_\alpha \int_\beta \left| f\left(V, \tilde{Z}\right) \right| \cdot \left| f\left(V, \tilde{Z} - Z\right) \right| \; d\tilde{Z} \, dV \leq$$

$$\leq C \int_\alpha \int_\beta e^{-a\pi(|V|^4 + |\tilde{Z}|^2)^{p/2}} e^{-a\pi(|V|^4 + |\tilde{Z} - Z|^2)^{p/2}} d\tilde{Z} \, dV.$$

Two inequalities

$$a^p + b^p \geq 2^{1-p/2}(a^2 + b^2)^{p/2} \quad \text{(see [1])} \quad \text{and} \quad (a+b)^{p/2} \geq a^{p/2} + b^{p/2},$$

for $a, b > 0$, $p \geq 2$ imply

$$\left|\widetilde{Z}\right|^{p} + \left|\widetilde{Z} - Z\right|^{p} \geq 2^{1-p/2}\left\{\left|\widetilde{Z}\right|^{2} + \left|\widetilde{Z} - Z\right|^{2}\right\}^{p/2}$$

and

$$\left\{\frac{1}{2}\left(2\left|\widetilde{Z}\right| - |Z|\right)^{2} + \frac{1}{2}|Z|^{2}\right\}^{p/2} \geq \left\{\frac{1}{2}\left(2\left|\widetilde{Z}\right| - |Z|\right)^{2}\right\}^{p/2} + \left\{\frac{1}{2}|Z|^{2}\right\}^{p/2}.$$

Since $e^{-2a\pi|V|^{2p}}$ is integrable on α and $e^{-a\pi 2^{(1-p)}(2|\widetilde{Z}|-|Z|)^{p}}$ is integrable on β, we conclude

$$|h(Z)| \leq C\int_{\beta}\int_{\alpha} e^{-a\pi(|V|^{2p}+|\widetilde{Z}|^{p})}e^{-a\pi(|V|^{2p}+|\widetilde{Z}-Z|^{p})}dV\, d\widetilde{Z} \leq$$

$$= C\int_{\beta}\int_{\alpha} e^{-2a\pi|V|^{2p}}e^{-a\pi(|\widetilde{Z}|^{p}+|\widetilde{Z}-Z|^{p})}dV d\widetilde{Z} \leq$$

$$\leq C\int_{\beta} e^{-a\pi(|\widetilde{Z}|^{p}+|\widetilde{Z}-Z|^{p})}d\widetilde{Z} \leq C\int_{\beta} e^{-a\pi 2^{(1-p/2)}\{|\widetilde{Z}|^{2}+|\widetilde{Z}-Z|^{2}\}^{p/2}}\, d\widetilde{Z} \leq$$

$$\leq C\int_{\beta} e^{-a\pi 2^{(1-p/2)}\{\frac{1}{2}(2|\widetilde{Z}|-|Z|)^{2}+\frac{1}{2}|Z|^{2}\}^{p/2}}d\widetilde{Z} \leq$$

$$\leq C\, e^{-a\pi 2^{(1-p)}|Z|^{p}}\int_{\beta} e^{-a\pi 2^{(1-p)}|(2|\widetilde{Z}|-|Z|)|^{p}}d\widetilde{Z} \leq C\, e^{-a\pi 2^{(1-p)}|Z|^{p}} \leq$$

$$\leq C\, e^{-a\pi 2^{(1-p)}(Z_{1}^{2}+\cdots+Z_{j-1}^{2}+Z_{j+1}^{2}+\cdots+Z_{n}^{2})^{p/2}}e^{-a\pi 2^{(1-p)}|Z_{j}|^{p}}.$$

On the other hand, $(|\lambda|+|\mu|)^{q} \geq |\lambda|^{q}+|\mu|^{q}$ $(1 \leq q \leq 2)$, so (3.2) and the assumption (ii) in Theorem 3 yield

$$\left|\hat{h}(\lambda)\right| = \int_{r_{\lambda}^{*}} \|\pi_{\lambda,\mu}(f)\|_{HS}^{2} Sym_{2m}B(\lambda)^{1/2}d\mu \leq$$

$$\leq C\, v^{2}(\lambda_{j}{}')Sym_{2m}B(\lambda)^{1/2}\int_{r_{\lambda}^{*}} e^{-2b\pi(|\lambda|+|\mu|)^{q}}\, d\mu \leq$$

$$\leq C\, v^{2}(\lambda_{j}{}')Sym_{2m}B(\lambda)^{1/2}e^{-2b\pi|\lambda|^{q}}\int_{r_{\lambda}^{*}} e^{-2b\pi|\mu|^{q}}d\mu \leq$$

$$\leq C\, v^{2}(\lambda_{j}{}')Sym_{2m}B(\lambda)^{1/2}e^{-2b\pi|\lambda|^{q}},$$

where we used that $e^{-2b\pi|\mu|^{q}}$ is integrable on r_{λ}^{*}. For $(ap)^{1/p}(bq)^{1/q} > 2$, we take $\varepsilon > 0$ to have $b = b' + \varepsilon$ and $(ap)^{1/p}(b'q)^{1/q} > 2$. The boundedness of $Sym_{2m}B(\lambda)^{1/2}e^{-2\varepsilon\pi|\lambda|^{q}}$ implies

$$\left|\hat{h}(\lambda)\right| \leq C\, v^{2}(\lambda_{j}{}')Sym_{2m}B(\lambda)^{1/2}e^{-2(b'+\varepsilon)\pi|\lambda|^{q}} =$$

$$= C\, v^2(\lambda_j{}')e^{-2b'\pi|\lambda|^q} Sym_{2m} B(\lambda)^{1/2} e^{-2c\pi|\lambda|^q} \leq C\, v^2(\lambda_j{}')e^{-2b'\pi|\lambda_j|^q}.$$

Since

$$(a2^{(1-p)}p)^{1/p}(2b'q)^{1/q} = (ap)^{1/p}(b'q)^{1/q}2^{(1-p)/p+1/q} > 2, \quad v^2 \in L^1(\mathbb{R}^{k-1})$$

and

$$e^{-a\pi 2^{(1-p)}(z_1^2+\cdots+z_{j-1}^2+z_{j+1}^2+\cdots+z_k^2)^{p/2}} \in L^1(\mathbb{R}^{k-1}),$$

we obtain from Theorem C that $h = 0$ almost everywhere, and then $\|\pi_{\lambda,\mu}(f)\|_{HS} = 0$ almost everywhere. By the Plancherel formula (2.10), $f = 0$ almost everywhere.

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