Известия НАН Армении. Математика, 39, № 2, 2004, 69–82

НЕРЕШЁННЫЕ ЗАДАЧИ

С 30 сентября по 7 октября 2003 года в Цахкадзоре (Армения) состоялась международная конференция "МАТЕМАТИКА в АРМЕНИИ : Достижения и перспективы". Конференция завершилась круглым столом, посвящённым обсуждением нерешённых проблем. Ниже представлены некоторые из нерешённых проблем, обсуждённых за круглым столом.

1. A QUESTION ON ERGODIC MEASURE PRESERVING TRANSFORMATIONS DEFINED ON THE INFINITE MEASURE SPACE

by A. Hajian (Boston, USA)

Since ergodic theory was not among the topics covered at this conference, allow me to present a few definitions needed for my question, references will not be mentioned. Let $(X, \mathcal{B}, m)$ be a σ-finite Lebesgue measure space. By a *measurable transformation* T on $(X, \mathcal{B}, m)$ I mean a 1-1 map of X onto itself, such that $A \in \mathcal{B} \iff TA \in \mathcal{B}$. A measurable transformation T on $(X, \mathcal{B}, m)$ is *ergodic* if $TA = A \implies m(A) = 0$ or $m(X - A) = 0$, and it is *measure preserving*, or *MP* for short, if $m(TA) = m(A)$ for all $A \in \mathcal{B}$. If one prefers, whenever I mention a finite or infinite measure space $(X, \mathcal{B}, m)$ one may think of the Lebesgue measure space of the unit interval with $m(X) = 1$ or the Lebesgue measure space of the whole real line with $m(X) = \infty$, respectively.

There are many examples of ergodic MP transformations defined on a finite measure space in the literature, and the classification of such transformations is well developed. Examples of ergodic MP transformations defined on an infinite measure space also exist. However, they are more complicated to construct, and the techniques used in their classification are quite different from the ones used in a finite measure space.

Birkhoff's Pointwise Ergodic Theorem : Let T be a MP transformation defined on the measure space $(X, \mathcal{B}, m)$ and let $f \in L^1(X, \mathcal{B}, m)$. Then,

$$f^*(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x)$$

exists a.e., $f^* \in L^1(X, \mathcal{B}, m)$, $f^*(Tx) = f^*(x)$ a.e., and in case $m(X) = 1$,

$$\int_X f^*(x)\, dm(x) = \int_X f(x)\, dm(x).$$

Consider the MP transformation T defined on the infinite measure space $(X, \mathcal{B}, m)$. Let us define a sequence of integers $\{n_i \mid i = 1, 2, \dots\}$ to be a *dissipative sequence* for the transformation T if $m(A) < \infty$ implies for almost all $x \in X$, $T^{n_i}x \in A$ for finitely many i only. The following property is true :

(D) *Every ergodic MP transformation defined on an infinite measure space admits dissipative sequences.*

The existence of dissipative sequences for ergodic MP transformations defined on an infinite measure space was not suspected until recently by mathematicians working in the field. I mention another property yet, which follows from the Pointwise Ergodic Theorem.

(B) *Let T be an ergodic MP transformation defined on the infinite measure space $(X, \mathcal{B}, m)$. Let $m(A) < \infty$, $m(B) < \infty$ and $\epsilon > 0$. Then there exists an integer $n > 0$ such that $m(T^n A \cap B) < \epsilon$.*

It is possible to prove (D) using the property (B). Both appear to be basic properties shared by all ergodic MP transformations defined on an infinite measure space, yet they eluded mathematicians working in the field for more than thirty years following the origins of ergodic theory in the early 1930's.

I would like to see a direct proof of either property (D) or property (B) from the definitions ; one that does not invoke the full force of the Pointwise Ergodic Theorem.

I feel that this may shed some light on our understanding the nature of ergodic MP transformations defined on an infinite measure space. It seems to me that the mathematicians who have been working in ergodic theory of finite measure spaces, when they consider ergodic MP transformations defined on an infinite measure space, somehow get influenced not to accept property (B) as natural. In this regard I mention the beautifully written book by Eberhard Hopf ; Ergodentheorie, Berlin, 1937.

2. DIVISIBILITY PROBLEM FOR ONE RELATOR MONOIDS

by S. I. Adian (Moscow, Russia)

We consider monoids presented by one defining relation in 2 generators :

$$M = < a, b;\ aU = bV > . \qquad (1)$$

Let $A_1 = aU$ and $A_2 = bV$. We say that a word X is *left side divisible* by Y in M if there exists a word Z such that $X = YZ$ in M.

The left side *divisibility problem* for M : to find an algorithm that tells for any two words X and Y, if X is left side divisible by Y in M or not ?

The following theorem was proved in [1] - [3].

Theorem 1. The word problem for any 1-relation monoid can be reduced to the left side divisibility problem for monoids M presented in 2 generators by 1 defining relation of the form $aU = bV$. For the solution of this problem it suffices to find an algorithm that for any word aW (or for any word bW) tells if the word is left side divisible in M by b (accordingly by a) or not.

The algorithm A was introduced in [2] for more general case of monoids presented by any cycle-free system of relations. Here we shall apply this algorithm to the case of monoids M. The algorithm A was used in several papers for solution of the left side divisibility problem for monoids M under some additional conditions.

To apply this algorithm one should find another algorithm B that decides for any word aW does the algorithm A terminate or not when we apply it to aW. For a given word aW the algorithm A finds a uniquely defined *prefix decomposition* which either is of the form

$$aW = P_1 P_2 \ldots P_k P_{k+1}, \tag{2}$$

where each P_i is the maximal nonempty, proper common prefix of the word $P_i P_{i+1} \ldots P_{k+1}$ and the appropriate relator aU or bV, or of the form

$$aW = P_1 P_2 \ldots P_k A_{j_k} W_{k+1}, \tag{3}$$

where the prefixes P_i are defined in a similar way, but the segment A_{j_k} is one of the relators of the monoid M. We call the segment A_{j_k} *the head* of the decomposition (3).

Let us describe in detail how our algorithm A works. Suppose we have an initial word aW. Consider the Maximal Common Prefix of two words aW and A_1 and denote it by

$$P_1 = MCP(aW, A_1). \tag{4}$$

We have $aW = P_1 W_1$ and $aU = P_1 U_1$ for some W_1 and U_1. Clearly P_1 is not empty. We consider the following two cases.

Case 1. If U_1 is empty, then $aW = aUW_1$. So we have a prefix decomposition of the form (3) for $k = 0$.

In this case the algorithm A replaces in aW the segment aU by bV. So we obtain $aW = bVW_1$ in M. Hence aW is left side divisible by b in M.

Case 2. Let U_1 be nonempty. Then P_1 is a proper prefix of aU.

If W_1 is empty then aW is a proper segment of the relator aU. It is easy to prove that the proper segment P_1 of aU is not divisible by b in M. Hence we can assume that U_1 and W_1 both are nonempty. It follows from (4) that in this case they have different initial letters a and b.

In this case to prolong the prefix P_1 of aU in P_1W_1 to the right side we should divide W_1 by b, if it starts by a or divide W_1 by a, if it starts by b. So the situation is similar to the initial one.

In a similar way we consider the nonempty word $P_2 = MCP(W_1, A_j)$, where A_{j_1} is the relator of M, which has an initial letter in common with W_1. Suppose $W_1 = P_2W_2$ and $A_j = P_2U_2$. Then again we consider two cases.

Case 2.1. If U_2 is empty, then $W_1 = A_jW_1$.

In this case we have the following prefix decomposition of the word aW :

$$aW = P_1A_{j_1}W_2, \tag{5}$$

where A_{j_1} is called *the head* of (5).

Case 2.2. Let U_2 be nonempty. In this case if W_2 is empty then $aW = P_1P_2$, where P_2 is a proper segment of the relator A_{j_1}. Hence for aW we obtain a prefix decomposition of the form (2). It is easy to prove that the word P_1P_2 is not divisible by b in M.

Hence we can assume that both U_2 and W_2 are nonempty. It follows from (4) that in this case they have different initial letters a and b. To prolong the prefix P_2 of A_{j_1} in $P_1P_2W_2$ in this case we should divide W_2 by b, if it starts by a, or divide W_2 by a, if it starts by b. So the situation again is similar to the initial one.

Hence we can consider the nonempty word $P_3 = MCP(W_2, A_{j_2})$, where A_{j_2} is one of the relators of M, which has a common initial letter with the word W_2, and so on. The length of the word W_i decreases, so after a finite number of steps we either find some prefix decomposition of the form (3) with the head A_{j_k}, or stop on some decomposition of the form (2).

It is easy to prove that if the decomposition of aW is of the form (2), then the word aW in M is not left side divisible by b.

If the decomposition is of the form (3) then the algorithm A replaces the head A_i in aW by another relator in (1) : aU should be replaced by bV or bV - by Ua. Hence we get one of the following elementary transformations in the monoid M :

$$aW = P_1P_2 \dots P_kaUW_{k+1} \longmapsto P_1P_2 \dots P_kbVW_{k+1} = W'$$

or

$$aW = P_1P_2 \dots P_kbVW_{k+1} \longmapsto P_1P_2 \dots P_kaUW_{k+1} = W'.$$

Clearly the result W' of this transformation is equal to aW in M. If the resulting word W' starts by the letter b (happens only if $k = 0$), then the algorithm A terminates by a positive answer. Otherwise the algorithm A repeats the same procedure with the word W'.

Theorem 2. ([2]) If the word aW is left side divisible by b in M, then the algorithm $A(aW)$ terminates with positive result, and in this case we obtain the shortest proof of the left side divisibility of the word aW by b in M.

Conjecture 1. There exists an algorithm B that decides for any word aW, if the algorithm $A(aW)$ terminates or not.

REFERENCES

1. S. I. Adian, "Defining relations and algorithmic problems for groups and semigroups" (in Russian), Proc. Steklov Inst. Math., vol. 85, 1966 [English version : American Mathematical Society, 1967].
2. S. I. Adian, "Word transformations in a semigroup that is given by a system of defining relations" (in Russian), Algebra i Logika, vol. 15, pp. 611 – 621, 1976 [English transl. : Algebra and Logic vol. 15, 1976].
3. S. I. Adian, G. U. Oganesian, "On the word and divisibility problems in semigroups with one defining relation" (in Russian), Mat. Zametki, vol. 41, pp. 412 – 421, 1987 [English transl. : Math. Notes, vol. 41, 1987].
4. S. I. Adian, V. G. Durnev, "Decision problems for groups and semigroups" (in Russian), Uspechi Mat. Nauk, vol. 55, no. 2, pp. 207 – 296, 2000.

3. SPHERICAL SYMMETRY FOR EXTREMAL SOLUTIONS OF FREE BOUNDARY PROBLEMS

by H. Shahgholian (Stockholm, Sweden)

Two well-known free boundary problems we present from an "extremal" point of view. Let $D \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain of unit volume say, and $\lambda > 0$ to be a fixed constant.

The first problem we consider is the so-called Bernoulli free boundary problem, which prescribes a flow inside the region D,

$$\begin{cases} \Delta u = 0, & \text{on} \quad \{u > 0\} \cap D, \\ u = 1 & \text{on} \quad \partial D, \\ |\nabla u| = \lambda & \text{on} \quad \Gamma, \end{cases}$$

where $\Gamma = \partial\{u > 0\} \cap D$ (see [1], [2].)

The second problem is the obstacle problem, describing the equilibrium position of a stretched membrane subject to a constant force and the zero obstacle $u \geq 0$,

$$\begin{cases} \Delta u = \lambda \chi_{\{u>0\}} & \text{in} \quad D, \\ u = 1 & \text{on} \quad \partial D, \end{cases}$$

where χ stands for the characteristic function (see [4]).

For both problems one can show the existence of solution for larger values of λ. In the first problem the solution is not necessarily unique. For the second problem the solution is always unique.

Now for each domain D exists the smallest value for $\lambda > 0$ such that the free boundary $\Gamma_\lambda := \partial\{u_\lambda > 0\} \cap D$ is non-void. Here u_λ refers to solution for the parameter λ, in both problems.

Conjecture : Let $\lambda^* = \min\{\lambda : \Gamma_\lambda \neq \emptyset, |D| = 1\}$. *We conjecture that the extremal domain* D^* *must be a ball of appropriate radius.*

This conjecture for the first problem appeared for the first time in [3]. The conjecture for the second problem has not been considered earlier.

REFERENCES

1. A. Acker, "Uniqueness and monotonicity of solutions for the interior Bernoulli free boundary problem in the convex, n-dimensional case", Nonlinear Anal., vol. 13, no. 12, pp. 1409 – 1425, 1989.
2. H. W. Alt, L. A. Caffarelli, "Existence and regularity for a minimum problem with free boundary", J. Reine Angew. Math., vol. 325, pp. 105 – 144, 1981.
3. M. Flucher, M. Rumpf, "Bernoulli's free-boundary problem, qualitative theory and numerical approximation", J. Reine Angew. Math., vol. 486, pp. 165 – 204, 1997.
4. A. Friedman, Variational Principles and Free Boundary Problems, Wiley, New York, 1982.

4. A LIST OF PROBLEMS RELATED TO CONVERGENCE OF FOURIER SERIES

by M. Lacey (Atlanta, USA)

4.1. Almost everywhere convergence of Fourier series

The famous result of Carleson concerning the convergence of Fourier series has received a recent proof by Lacey and Thiele [8].

Theorem 1. For all $f \in L^2(\mathbb{T})$ *one has*

$$f(x) = \lim_{N\to\infty} \sum_{|n|<N} \widehat{f}(n) e^{2\pi i n x} \quad a.e.$$

I would hesitate to say that the proof is easier, but many would agree that it is more conceptual. A recent survey [4] of the proof is far more leisurely than the condensed original reference [8], and comes with exercises.

The questions in the area of the pointwise convergence of Fourier series are rather challenging. In two and higher dimensions, one of the most outstanding problem in the convergence of Fourier series is :

Question 1. Is it the case that the maximal operator

$$\sup_{r>0}\left|\int_{|\xi|<r}\widehat{f}(\xi)e^{i\xi\cdot x}\,d\xi\right|$$

maps $L^2(\mathbb{R}^2)$ into weak $L^2(\mathbb{R}^2)$?

In the known proofs of Carleson's theorem, the truncations of singular integrals play a distinguished role.

Question 2. Is it the case that the maximal operator

$$\sup_{k\in\mathbb{N}}\left|\int_{|\xi|<1+2^{-k}}\widehat{f}(\xi)e^{i\xi\cdot x}\,d\xi\right|$$

maps $L^2(\mathbb{R}^2)$ into weak $L^2(\mathbb{R}^2)$?

It is conceivable that a positive answer here could lead to a proof of spherical convergence of Fourier series.

It is of course natural to consider the summation by a polygon, namely one replaces the unit ball above by the interior of a polygon. If the polygon has finitely many sides, a brief argument of C. Fefferman shows that the situation can be reduced to the one dimensional situation, and so there is a positive answer for all L^p, $1<p<\infty$.

It is of interest to find a positive result for an infinite-sided polygon. The natural first choice is the polygon, which has slope -2^j in the angle from $\pi 2^{-j-1}$ to $\pi 2^{-j}$. It is a fact due to Cordoba and R. Fefferman that the lacunary-sided polygon is a bounded L^p multiplier, for all $1<p<\infty$. That is the operator T_{lac} maps L^p into itself for $1<p<\infty$:

$$T_{\text{lac}}f(x)=\int_{P_{\text{lac}}}\widehat{f}(\xi)e^{i\xi\cdot x}\,d\xi.$$

This fact is in turn linked to the boundedness of the maximal function in a lacunary set of directions :

$$M_{\text{lac}}f(x)=\sup_{k\in\mathbb{N}}\sup_{t>0}(2t)^{-1}\int_{-t}^{t}|f(x-u(1,2^{-k}))|\,du.$$

Note that this is an one-dimensional maximal function computed in a set of directions in the plane that, in a strong sense, is zero dimensional.

Conjecture 1. For $2\le p<\infty$, the maximal function

$$\sup_{t>0}\left|\int_{tP_{\text{lac}}}\widehat{f}(\xi)e^{i\xi\cdot x}\,d\xi\right|$$

maps $L^p(\mathbb{R}^2)$ into itself. Even a restricted version of this conjecture remains quite challenging.

Conjecture 2. For $2 \leq p < \infty$, the maximal function

$$\sup_{k \in \mathbb{N}} \left| \int_{(1+2^{-k})P_{sec}} \widehat{f}(\xi) e^{i\xi \cdot x} \, d\xi \right|$$

maps $L^p(\mathbb{R}^2)$ into itself.

Another question considers regular polygons with N sides : to find norm bounds on these two maximal operators, on L^2 say, that grow logarithmically in N.

Conjecture 3. For integers N, let P_N be the centrally symmetric polygon with N sides of equal length. Then for $4/3 \leq p < 4$ there is a constant $k(p)$ such that

$$\sup_{t>0} \left| \int_{tP_N} \widehat{f}(\xi) e^{i\xi \cdot x} \, d\xi \right|$$

maps $L^p(\mathbb{R}^2)$ into itself, with norm bounded by $\sim (\log N)^{k(p)}$.

The method of Fefferman applied to this polygon would give the norm bound of N for all $1 < p < \infty$. One would expect that at L^2, the bound would be much smaller of course. The logarithmic growth would be expected for the range of p's indicated above. The operator of Fourier restriction to the regular N polygon has a L^p norm that is dominated by logarithmic factors only for $4/3 < p < 4$.

Consider the partial summation operator given by

$$S_N f(x) = \int \int_{(-\infty, N) \times (-\infty, N^2)} \widehat{f}(\xi) e^{2\pi i x \cdot \xi} \, d\xi.$$

One could replace N^2 by some other reasonably nice increasing function φ on $\mathbb{R}_+$. The argument by C. Fefferman proving pointwise convergence for finite-sided polygons also implies that $S_N f$ converges pointwise to f for $f \in L^2$, but not L^p, for $p \neq 2$. To carry out Fefferman's argument one must preform the Fourier projection on the region $\{(\xi_1, \xi_2) : |\xi_1| < \xi_2^2\}$, and by C. Fefferman's disc multiplier example, this is a bounded operator on L^p only if $p = 2$.

But this observation does not prevent the pointwise convergence result from being true for $p \neq 2$. Thus,

Conjecture 4 (see [8]). For $1 < p \neq 2 < \infty$ the maximal function $\sup_N |S_N f|$ maps L^p into itself.

4.2. Bilinear Hilbert transform

The canonical instance of the bilinear Hilbert transform is

$$H(f, g)(x) = \text{p.v.} \int f(x + y) g(x - y) \, \frac{dy}{y}$$

The analysis of this transform requires the techniques of Carleson theorem. See [9] – [12] for a proof of this theorem.

Theorem 2 (see [13]). For $1 < p, q \leq \infty$, *if* $0 < 1/r = 1/p + 1/q < 3/2$, *then*

$$\|H(f,g)\|_r \sim \|f\|_p \|g\|_q.$$

There are some puzzling aspects of the theory that have not as of yet been resolved. For instance, what happens for $0 < r < 2/3$ in Theorem 2?

Conjecture 5. The bilinear Hilbert transform is unbounded on $L^p \times L^p$ *for* $1 < p < 4/3$.

The method developed with Thiele fails in this range. The point is that in the positive direction, we expand the bilinear Hilbert transform as an unconditionally convergent series. If the target space is L^r with $0 < r < 2/3$, we can show that the series is no longer unconditional. The relevant example is published in [5].

The case of the trilinear Hilbert transform is equally puzzling. Consider now

$$H_\alpha(f,g,h)(x) = \text{p.v.} \int f(x+y)g(x-y)h(x-\alpha y)\,\frac{dy}{y}.$$

We have inserted a shift parameter α into this formula. We are mostly interested in $\alpha \neq \{\pm 1\}$. We can show that the series that is suggested by the bilinear Hilbert transform is *not* unconditionally convergent.

Conjecture 6. For some irrational α, $H_\alpha(f,g,h)$ *is unbounded on* $L^3 \times L^3 \times L^3$.

This would still be interesting if one replaced the Hilbert transform by some other nice singular integral of one's choosing.

Theorem 3 (C. Thiele). Consider the elementary integral operator

$$I_\alpha(f,g,h)(x) = \int_{-1}^{1} f(x+y)g(x-y)h(x-\alpha y)\,dy.$$

There is an irrational α *so that this operator is unbounded on* $L^p \times L^p \times L^p$ *for* $1 < p < 3/2$.

The bilinear version of this integral operator is very well behaved, so that I_α will have the correct behavior as soon as the target space is L^r for $r > 1$, where one can use Holder's inequality. This kind of counterexample cannot arise if α is rational (C. Thiele).

REFERENCES

1. S-Y. A. Chang, "Carleson measure on the bi-disc", Ann. of Math. (2), vol. 109, no. 3, pp. 613 – 620, 1979.

2. S-Y. A. Chang, R. Fefferman, "Some recent developments in Fourier analysis and H^p-theory on product domains", Bull. Amer. Math. Soc. (N.S.), vol. 12, no. 1, pp. 1 – 43, 1985.
3. S-Y. A. Chang, R. Fefferman, "A continuous version of duality of H^1 with BMO on the bidisc", Ann. of Math. (2), vol. 112, no. 1, pp. 179 – 201, 1980.
4. M. T. Lacey, "Carleson's Theorem : Proof, Complements, Variations", 2003.
5. M. T. Lacey, "The bilinear maximal functions map into L^p for $2/3 < p \leq 1$", Annals of Mathematics, Series 2, vol. 151, no. 1, pp. 35 – 57, 2000.
6. M. T. Lacey, X. Li, "Maximal theorems for directional Hilbert transform on the plane", Preprint.
7. M. T. Lacey, X. Li, "Hilbert transform on $C^{1+\epsilon}$ families of lines", Preprint.
8. M. T. Lacey, Ch. Thiele, "A proof of boundedness of the Carleson operator", Math. Res. Lett., vol. 7, no. 4, pp. 361 – 370, 2000.
9. M. T. Lacey, Ch. Thiele, "L^p estimates for the bilinear Hilbert transform", Proc. Nat. Acad. Sci. U.S.A., vol. 94, no. 1, pp. 33 – 35, 1997.
10. M. T. Lacey, Ch. Thiele, "L^p estimates on the bilinear Hilbert transform for $2 < p < \infty$", Ann. of Math. (2), vol. 146, no. 3, pp. 693 – 724, 1997.
11. M. T. Lacey, Ch. Thiele, "On Calderón's conjecture for the bilinear Hilbert transform", Proc. Natl. Acad. Sci. USA, vol. 95, no. 9, pp. 4828 – 4830, 1998.
12. M. T. Lacey, Ch. Thiele, "On Calderón's conjecture", Ann. of Math. (2), vol. 149, no. 2, pp. 475 – 496, 1999.
13. Ch. Thiele, "Singular integrals meet modulation invariance", Proceedings of the International Congress of Mathematicians, vol. II (Beijing, 2002), pp. 721 – 732, Higher Ed. Press, Beijing, 2002.
14. Ch. Thiele, "The quartile operator and pointwise convergence of Walsh series", Trans. Amer. Math. Soc., vol. 352, no. 12, pp. 5745 – 5766, 2000.

5. НЕКОТОРЫЕ ВОПРОСЫ ПО РЯДАМ ФРАНКЛИНА

Г. Г. Геворкян (Ереван, Армения)

Классическая система Франклина, введённая им в 1928 году [1], как первый пример ортонормированного базиса в пространстве $C([0,1])$, является полной ортонормированной системой непрерывных, кусочно-линейных функций с двоичными узлами. Её свойства изучались многими авторами с различной точки зрения. В частности, известно, что система Франклина образует базис в пространстве $C[0,1]$. В 1975 году, С. В. Бочкарёв [2] доказал, что система Франклина является безусловным базисом в пространстве $L^p(0,1)$ при $1 < p < \infty$ а П. Войтащик [3] доказал, что система Франклина является безусловным базисом в пространстве $H^1[0,1]$. Дополнительную информацию о рядах Франклина можно найти в [4]. Пусть $\sigma_N = \{s_i : 0 \leq i \leq N\}$ – разбиение отрезка $[0,1]$ точками s_i кратность которых не превосходит 2, т.е.

$$0 < s_0 < s_1 \leq s_2 \leq \ldots \leq s_{N-1} < s_N = 1, \tag{1}$$

$$s_i < s_{i+2}, \quad 0 \leq i \leq N-2. \tag{2}$$

Обозначим через $S(\sigma)$ пространство кусочно-линейных функций на $[0,1]$, ассоциированное с последовательностью узлов σ, т.е. $S(\sigma)$ есть пространство функций, которые линейны на каждом интервале (s_i, s_{i+1}), непрерывны слева в точках s_i (непрерывны справа в точке s_0) и непрерывны во всех точках s_i, удовлетворяющих как $s_{i-1} < s_i$ так и $s_i < s_{i+1}$. Отметим, что $S(\sigma)$ является линейным пространством размерности $N+1$. Теперь пусть $\sigma_N = \{s_i : 0 \le i \le N\}$ и $\sigma^*_{N+1} = \{s^*_i : 0 \le i \le N+1\}$ суть два разбиения интервала $[0,1]$, удовлетворяющие (1), (2) и такие, что σ^*_{N+1} получается из σ_N добавлением единственного узла s^*. Отметим, что s^* может быть отличной от всех точек из σ_N (в этом случае, для некоторого i имеем $s^* = s^*_i$ и $s^*_{i-1} < s^*_i < s^*_{i+1}$), или для некоторого i, $s^* = s_i$ (в этом случае, $s^*_{i-1} < s^*_i = s^* = s^*_{i+1} < s^*_{i+2}$). Известно, что существует единственная функция $\varphi \in S(\sigma^*_{N+1})$, ортогональная к $S(\sigma_N)$ в пространстве $L^2[0,1]$, $||\varphi||_2 = 1$ и $\varphi(s^*) > 0$. Эта функция φ называется **общей функцией Франклина, соответствующей паре разбиений** $(\sigma_N, \sigma^*_{N+1})$.

Определение 1. Пусть $T = \{t_n : n \ge 0\}$ – последовательность точек из интервала $[0.1]$. Последовательность T называется **допустимой**, если $t_0 = 0$, $t_1 = 1$, $t_n \in (0,1)$ для всех $n > 1$, и для каждого $t \in (0,1)$ существуют не более двух различных индексов $n_1 > n_2 > 1$ таких, что $t = t_{n_1} = t_{n_2}$, и T плотно в $[0,1]$.

Для допустимой последовательности точек $T = \{t_n : n \ge 0\}$ и $n > 1$, пусть $\sigma_n = \{t_{n,i} : 0 \le i \le n\}$ – разбиение интервала $[0,1]$, полученное из точек последовательности T перенумерованных в неубывающем порядке, считая кратности точек. Ясно, что каждое σ_n удовлетворяет условиям (1), (2), и σ_n образуется из σ_{n-1} добавлением одного узла t_n.

Определение 2. Пусть T – допустимая последовательность точек. **Общей системой Франклина**, соответствующей последовательности узлов T, называется последовательность функций $\{f_n : n \ge 0\}$, задаваемых по формулам $f_0(t) = 1$, $f_1(t) = \sqrt{3}(2t-1)$, а для $n \ge 2$, f_n – общая функция Франклина, соответствующая паре разбиений (σ_{n-1}, σ_n).

Определение 3. Последовательность $T = \{t_n : n \ge 0\}$ точек из $[0,1]$, удовлетворяющая условиям

1) $t_0 = 0$, $t_1 = 1$, $t_2 \in (0,1)$,

2) $t_{2^k+1} < t_{2^k+2} < \ldots < t_{2^{k+1}}$, $k = 1, 2, \ldots$,

3) между двумя соседними точками из $\{t_i : 0 \le i \le 2^k\}$, существует одна точка из $\{t_i : 2^k + 1 \le i \le 2^{k+1}\}$,

называется **квази–двоичным разбиением** $[0.1]$, а соответствующая общая система Франклина называется **квази-двоичной системой Франклина**. Если разбиение $T = \{t_n : n \geq 0\}$ – двоичное, т.е. $t_0 = 0$, $t_1 = 1$, и

$$t_n = \frac{2m-1}{2^{k+1}}, \quad n = 2^k + m, \quad 1 \leq m \leq 2^k, \quad k = 0, 1, ...,$$

то мы получаем классическую систему Франклина.

Проблема 1. Пусть $\{f_n : n \geq 0\}$ – классическая система Франклина. Из условия $\sum_{n=0}^{\infty} a_n f_n(t) = 0$ при любом $t \in [0,1]$ вытекает ли $a_n = 0$ для $n = 0, 1, ...$?

Известно (см. [5], [6]), что

$$\left|\sum_{n=0}^{N} f_n(0) f_n(t)\right| < CNq^{nt}, \quad 0 < q < 1,$$

где C и q – некоторые абсолютные постоянные. Следовательно, существует ряд $\sum_{n=0}^{\infty} a_n f_n(t)$, сходящийся к нулю для любого $t \in (0,1]$ и $a_0 \neq 0$. Отметим, что (см. [5], [6])

$$|f_n(0)| < C\sqrt{n}, \quad \varlimsup_{n\to\infty} \frac{f_n(0)}{\sqrt{n}} > 0.$$

Проблема 2. Пусть $\{f_n : n \geq 0\}$ – классическая система Франклина. Из условий $\sum_{n=0}^{\infty} a_n f_n(t) = 0$ для любого $t \in (0,1]$ и $a_n = o(\sqrt{n})$ вытекает ли $a_n = 0$ для $n = 0, 1, ...$?

Проблема 3. Пусть $\{f_n : n \geq 0\}$ – классическая система Франклина и E некоторе конечное или счётное множество. Из условий $\sum_{n=0}^{\infty} a_n f_n(t) = 0$ для любого $t \in [0,1] \setminus E$ и $a_n = o(\sqrt{n})$ вытекает ли $a_n = 0$ для $n = 0, 1, ...$?

Используя классическую систему Франклина $\{f_n : n \geq 0\}$, С. В. Бочкарёв [7] построил базис в пространстве A комплекснозначных функций, аналитических внутри единичного круга и непрерывных в замкнутом единичном круге. Для формулировки этого результата рассмотрим функции (см. [7])

$$F_n(x) = \begin{cases} f_n(x/\pi) & \text{для} \quad x \in [0,\pi], \\ f_n(-x/\pi) & \text{для} \quad x \in [-\pi, 0], \end{cases} \tag{3}$$

$$G_0 = \frac{1+i}{2\sqrt{\pi}}, \quad G_n(x) = \frac{1}{\sqrt{2\pi}}\left[F_n(x) + i\widetilde{F}_n(x)\right], \tag{4}$$

где

$$\widetilde{F}_n(x) = -\lim_{\varepsilon\to+0} \frac{1}{\pi}\int_{\varepsilon}^{\pi} \frac{F_n(x+t) - F_n(x-t)}{2\tan t/2}\, dt. \tag{5}$$

Теорема (С. В. Бочкарёв). Пусть $P(r,x)$ – ядро Пуассона. Система функций

$$G_n(z) = G_n(re^{ix}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} G_n(t)P(r,x-t);dt. \tag{6}$$

образует базис в пространстве A

Проблема 4. Пусть $T = \{t_n : n \geq 0\}$ – последовательность простых узлов, т.е. $t_i \neq t_j$ для $i \neq j$ и пусть $\{f_n : n \geq 0\}$ – общая система Франклина, соответствующая T. Образует ли система функций $G_n(z)$, определённых по (3) – (6) базис в пространстве A ?

Проблема 5. Если ответ на Проблему 4 отрицателен, то найти необходимые и достаточные условия в терминах T, при которых система функций $G_n(z)$ образует базис в пространстве A.

Теперь рассмотрим периодические, общие системы Франклина с простыми узлами. Пусть $T = \{t_n : n \geq 0\}$ – последовательность точек из $(0,1]$ такая, что $t_i \neq t_j$ при $i \neq j$. Обозначим через $S_n(t)$ пространство кусочно–линейных и непрерывных 1-периодических функций на $[0,1]$ с узлами $t_1, ..., t_n$.

Определение 4. Общая периодическая система Франклина, соответствующая последовательности узлов T есть последовательность функций $\{f_n : n \geq 0\}$, определённых по формулам $f_0(t) = 1$ а при $n \geq 2$

$$f_n \in S_n(T), \quad f_n \perp S_{n-1}(T), \quad ||f_n||_2 = 1, \quad f_n(t_n) > 0.$$

В [8] было доказано, что всякая общая система Франклина является безусловным базисом в пространстве $L^p[0,1]$, $1 < p < \infty$.

Проблема 6. Является ли периодическая общая система Франклина безусловным базисом в пространстве $L^p[0,1]$, $1 < p < \infty$?

В [9] были найдены необходимые и достаточные условия на допустимую последовательность точек T, при которых соответствующая общая система Франклина является базисом или безусловным базисом в пространстве $H^1[0,1]$.

Проблема 7. Пусть $\{f_n : n \geq 1\}$ – периодическая общая система Франклина, соответствующая узлам $T = \{t_n : n \geq 1\}$. Найти необходимые и достаточные условия на последовательность точек T, при которых система функций $F_n(x) = f_n\left(\frac{x}{2\pi}\right)$ образует базис или безусловный базис в $H^1(D)$, где D – единичная окружность.

В 1985 С. В. Бочкарёв [10] доказал, что если $\{f_n : n \geq 1\}$ является периодической классической системой Франклина на $[0, 2\pi]$, то система функций $\frac{1}{2\pi}$, $\widetilde{f_2}, \widetilde{f_3}, ...$

образует базис в пространстве непрерывных и 2π-периодических функций, где $\widetilde{f}$ – гармонически сопряжённая к f.

Проблема 8. Пусть $\{f_n : n \geq 1\}$ – периодическая общая система Франклина на $[0, 2\pi]$. Образует ли система функций $\frac{1}{2\pi}$, $\widetilde{f}_2, \widetilde{f}_3, \dots$ базис в пространстве непрерывных и 2π-периодических функций ?

Проблема 9. Если ответ на Проблему 8 отрицателен, то найти необходимое и достаточное условие на последовательность $T = \{t_n : n \geq 1\}$, при котором система функций $\frac{1}{2\pi}$, $\widetilde{f}_2, \widetilde{f}_3, \dots$ образует базис в пространстве непрерывных и 2π-периодических функций.

REFERENCES

1. Ph. Franklin, "A set of continuous orthogonal functions", Math. Ann., vol. 100, pp. 522 – 528, 1928.
2. S. V. Bochkarev, "Some inequalities for the Franklin series", Anal. Math., vol. 1, pp. 249 – 257, 1975.
3. P. Wojtaszczyk, "The Franklin system is unconditional basis in H^1", Ark. Mat., vol. 10, pp. 293 – 300, 1982.
4. Z. Ciesielski, A. Kamont, "Survey on the orthonormal Franklin system", In : Approximation Theory, pp. 84 – 132, Darba, Sofia, 2002.
5. Z. Ciesielski, "Properties of the orthonormal Franklin system", Studia Math., vol. 23, pp. 141 – 157, 1963.
6. Z. Ciesielski, "Properties of the orthonormal Franklin system II", Studia Math., vol. 27, pp. 289 – 323, 1966.
7. С. В. Бочкарёв, "Существование базиса в пространстве аналитических в круге функций, и некоторые свойства системы Франклина", Мат. Сборник, том 95 (137), 1974.
8. G. G. Gevorkian, A. Kamont, "Unconditionality of general Franklin systems in $L^p[0, 1]$, $1 < p < \infty$", Studia Math., to appear.
9. G. G. Gevorkian, A. Kamont, "General Franklin systems as bases in H^1", Studia Math., to appear.
10. С. В. Бочкарёв, "Сопряжённая система Франклина являющаяся базисом в пространстве непрерывных функций", ДАН СССР, том 285, стр. 521 – 526, 1985.