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NONSTANDARD TRANSFINITE ELECTRICAL NETWORKS

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Abstract. The idea of a nonstandard, transfinite, linear electrical network is defined. To prove an existence and uniqueness theorem for a hyperreal current-voltage regime, the concept of a transfinite graph has to be lifted into a nonstandard setting. Nonstandard versions of Kirchhoff's laws are also examined. Finally, it is pointed out that certain nonlinear networks have unique hyperreal current-voltage regimes as well.

§1. INTRODUCTION

Our objective in this paper is to lift the ideas of transfinite graphs and electrical networks, as well as the fundamental theorem concerning the existence of a current-voltage regime in a transfinite network, into a nonstandard setting. For the sake of brevity, we shall restrict our attention to the first rank of transfiniteness. Such transfinite graphs are defined in [6], Sec. 3.2 and also in [7], Sec. 2.1, and the fundamental theorem for electrical networks having such graphs is stated by [6], Theorem 3.3-5 and also by [7], Theorem 5.2-8. A simpler version of these ideas for restricted transfinite connections can be found in [8].

Herein, we will state briefly the necessary definitions concerning transfinite graphs, alleviating thereby the need to refer to those prior works. In §2 we construct a nonstandard version of a conventional graph, and in §3 we do the same for the first rank transfinite graph, called a "1-graph". A nonstandard transfinite linear electrical network is defined in §4, wherein the existence of a hyperreal current-voltage regime is also established. Finally, Kirchhoff's laws in a nonstandard setting are examined in §5. In §6 we discuss the extensions to certain nonlinear networks.

We will employ a variety of concepts and results from nonstandard analysis whose definitions can be found in many books, as for example [2] – [5]. We adopt the ultrapower approach to nonstandard analysis and will mention the transfer principle only occasionally. Thus, it is understood that a nonprincipal ultrafilter $\mathcal{F}$ (also called a "free ultrafilter") has been chosen, and equivalence classes of sequences are defined with respect to $\mathcal{F}$. Terminology and symbolism vary somewhat in the literature on nonstandard analysis; we follow those used by [2]. For instance, $\mathbb{N}$ is the set of natural numbers, and $\mathbb{R}$ is the set of real numbers. An infinite sequence $a_0, a_1, a_2, \ldots$ indexed by the natural numbers is denoted by $\langle a_n : n \in \mathbb{N}\rangle$ or simply by $\langle a_n\rangle$. Then, a nonstandard entity is an equivalence class of such sequences whereby two sequences $\langle a_n\rangle$ and $\langle b_n\rangle$ are considered equivalent if $\{n : a_n = b_n\} \in \mathcal{F}$. Any such nonstandard entity is denoted by $[a_n]$, where a_n enters any sequence in the equivalence class.

§2. NONSTANDARD GRAPHS

A graph G is a pair $G = \{X, B\}$, where X is a set and B is a set of two-element subsets of X. Thus, a typical branch b is $b = \{x, y\}$, where $x, y \in X$ and $x \neq y$. X and B may be infinite sets. We will use electrical terminology by referring to the elements of X as **nodes** (instead of "vertices") and to the elements of B as **branches** (instead of "edges"). Given any branch $b = \{x, y\} \in B$, x and b are said to be **incident**, and similarly for y and b.

Next, let $\langle G_n : n \in \mathbb{N}\rangle$ be a given sequence of graphs. The nonstandard graph we shall construct will depend upon this choice of the sequence $\langle G_n : n \in \mathbb{N}\rangle$. For each n, we have $G_n = \{X_n, B_n\}$, where X_n is the set of nodes and B_n is the set of branches. We allow $X_n \cap X_m \neq \emptyset$ so that G_n and G_m may be subgraphs of a larger graph. In fact, we can view each G_n as being a subgraph of the union $G = \{\cup X_n, \cup B_n\}$ of all the G_n. As a special case, we may have $X_n = X_m$ and $B_n = B_m$ for all $n, m \in \mathbb{N}$ so that G_n may be the same graph for all $n \in \mathbb{N}$.

In the following, $\langle x_n\rangle = \langle x_n : n \in \mathbb{N}\rangle$ will denote a sequence of nodes with $x_n \in X_n$ for all $n \in \mathbb{N}$. A **nonstandard node** $\mathbf{x}$ is an equivalence class of such sequences of nodes, where two such sequences $\langle x_n\rangle$ and $\langle y_n\rangle$ are taken to be equivalent if $\{n : x_n = y_n\} \in \mathcal{F}$, in which case we write $\langle x_n\rangle = \langle y_n\rangle$ a.e. or say that $x_n = y_n$ "for almost all n." As was stated before, we also write $\mathbf{x} = [x_n]$, where it is understood that x_n enters any sequence in the equivalence class.

Reflexivity and symmetry of this relation being obvious, consider transitivity : Given that $\langle x_n\rangle = \langle y_n\rangle$ a.e. and that $\langle y_n\rangle = \langle z_n\rangle$ a.e., we have $N_{xy} = \{n : x_n = y_n\} \in \mathcal{F}$ and $N_{yz} = \{n : y_n = z_n\} \in \mathcal{F}$. By the properties of the ultrafilter, $N_{xy} \cap N_{yz} \in \mathcal{F}$. Moreover, $N_{xz} = \{n : x_n = z_n\} \supseteq (N_{xy} \cap N_{yz})$. Therefore, $N_{xz} \in \mathcal{F}$. Hence, $\langle x_n\rangle = \langle z_n\rangle$ a.e. ; transitivity holds. We let ${}^*X$ denote the set of nonstandard nodes.

Next, we define the nonstandard branches : Let $\mathbf{x} = [x_n]$ and $\mathbf{y} = [y_n]$ be two nonstandard nodes. This time, let $N_{xy} = \{n : \{x_n, y_n\} \in B_n\}$ and $N_{xy}^c = \{n : \{x_n, y_n\} \notin B_n\}$. Since $\mathcal{F}$ is an ultrafilter, exactly one of N_{xy} and N_{xy}^c is a member of $\mathcal{F}$. If it is N_{xy}, then $\mathbf{b} = [\{x_n, y_n\}]$ is defined to be a **nonstandard branch** ; that is, $\mathbf{b}$ is an equivalence class of sequences $\langle b_n \rangle$ where $b_n = \{x_n, y_n\}$, $n = 0, 1, 2, \ldots$. In this case, we also write $\mathbf{x}, \mathbf{y} \in \mathbf{b}$ and $\mathbf{b} = \{\mathbf{x}, \mathbf{y}\}$. We let $^*B$ denote the set of nonstandard branches. On the other hand, if $N_{xy}^c \in \mathcal{F}$, then $[\{x_n, y_n\}]$ is not a nonstandard branch. We shall now show that this definition is independent of the representatives chosen for the nodes. Let $[\{x_n, y_n\}]$ and $[\{v_n, w_n\}]$ represent the same nonstandard branch. We want to show that, if $\langle x_n \rangle = \langle v_n \rangle$ a.e., then $\langle y_n \rangle = \langle w_n \rangle$ a.e. Suppose $\langle y_n \rangle \neq \langle w_n \rangle$ a.e. Then

$$\{n : x_n = v_n\} \cap \{n : y_n \neq w_n\} \in \mathcal{F}.$$

Thus, there is at least one n for which the three nodes $x_n = v_n$, y_n, and w_n are all incident to the same standard branch – in violation of the definition of a branch. Similarly, if all of $\langle x_n \rangle$, $\langle y_n \rangle$, $\langle v_n \rangle$, $\langle w_n \rangle$ are different a.e., then there would be a standard branch having four incident nodes – again a violation.

Next, we show that we have an equivalence relation for the set of all sequences of standard branches. Reflexivity and symmetry being obvious again, consider transitivity : Let $\mathbf{b} = [\{x_n, y_n\}]$, $\bar{\mathbf{b}} = [\{\bar{x}_n, \bar{y}_n\}]$, $\acute{\mathbf{b}} = [\{\acute{x}_n, \acute{y}_n\}]$, and assume that $\mathbf{b} = \bar{\mathbf{b}}$ and $\bar{\mathbf{b}} = \acute{\mathbf{b}}$. We want to show that $\mathbf{b} = \acute{\mathbf{b}}$. We have

$$N_{b\bar{b}} = \{n : \{x_n, y_n\} = \{\bar{x}_n, \bar{y}_n\}\} \in \mathcal{F}, \quad N_{\bar{b}\acute{b}} = \{n : \{\bar{x}_n, \bar{y}_n\} = \{\acute{x}_n, \acute{y}_n\}\} \in \mathcal{F},$$

Moreover,

$$N_{b\acute{b}} = \{n : \{x_n, y_n\} = \{\acute{x}_n, \acute{y}_n\}\} \supseteq (N_{b\bar{b}} \cap N_{\bar{b}\acute{b}}) \in \mathcal{F}.$$

Therefore, $N_{b\acute{b}} \in \mathcal{F}$. Thus, $\mathbf{b} = \acute{\mathbf{b}}$, as desired.

Finally, we define a **nonstandard graph** $^*G$ to be the pair $^*G = \{^*X, {}^*B\}$; we also write $^*G = [G_n]$. As an example, let all the G_n be the same one-way infinite path P. That path is an alternating sequence of nodes x_k and branches b_k :

$$P = \langle x_0, b_0, x_1, b_1, x_2, b_2, \ldots \rangle$$

where all the x_k and b_k are distinct and b_k is incident to x_k and x_{k+1} for every k. We can identify x_k with k. Then, the nonstandard graph $^*G = \{^*X, {}^*B\}$ has the hypernatural numbers as its nodes, and there is a nonstandard branch connected between each pair of consecutive hypernatural numbers. There are no other nonstandard branches. Note that the nonstandard nodes can be partitioned into galaxies, just as are the hypernatural numbers. Thus, there is no "next galaxy" after the first one consisting of the standard nodes : $x_0, x_1, x_2, \ldots$. In fact, between any two galaxies there is another galaxy.

In general, however, the graphs G_n may be arbitrary and may be different from each other so that the resulting nonstandard graph may have a far more complicated structure than does our simple example. Nevertheless, the nonstandard nodes can still be partitioned into galaxies whereby two such nodes are in the same galaxy if there is a finite nonstandard path connecting them.

A special case arises when almost all the G_n are (possibly different) finite graphs. In conformity with the terminology used for hyperfinite internal subsets of $^*R$, we will refer to the resulting nonstandard graph $^*G$ as a **hyperfinite graph**. As a result, we can lift many theorems concerning finite graphs to hyperfinite graphs. It is just a matter of writing the standard theorem in an appropriate form using symbolic logic and then applying the transfer principle ([9] has several such results). We let $^*G_f$ denote the set of hyperfinite graphs.

§3. NONSTANDARD TRANSFINITE GRAPHS

A "1-graph" is a transfinite graph of the first rank of transfiniteness. Let us briefly define it before turning to its nonstandard generalization. Let $G^0 = \{X^0, B\}$ be conventional graph containing at least one one-way infinite path. The nodes of x^0 will be called "0-nodes". We partition the set of all such paths into equivalence classes by taking two as equivalent if they are identical except for finitely many nodes and branches. Each such equivalence class is a "0-tip" for G^0.

Next, we partition the set of 0-tips for G^0 in an arbitrary fashion. To each set of that partition we may (or may not) assign a 0-node of X^0 with the proviso that, if a 0-node is assigned to a set of the partition, it is not assigned to any other set of the partition. Each set of the partition augmented by thus assigned 0-node (if such exists) is called a "1-node". It can be viewed as a connection among the infinite extremities of G^0 and possibly with a particular 0-node as well.

Thus, if G^0 has many infinite components, the 1-nodes serve as connections among the infinite extremities of those components, yielding thereby a transfinite graph. The resulting 1-graph is denoted by $G^1 = \{X^0, B, X^1\}$, where X^0 and B are the sets of 0-nodes and branches of G^0 and X^1 is the set of 1-nodes. We refer to the 0-tips and the assigned 0-nodes (those occurring in 1-nodes) as the **extremities** of the 0-graph $\{X^0, B\}$. If e and f are extremities in the same 1-node x^1 of G^1, we will say that e and f are **shorted together** by x^1 and will write $e \asymp f$.

We turn now to the definition of a nonstandard 1-graph. We start with a given sequence $\langle G_n^1 : n \in \mathbb{N} \rangle$ of 1-graphs $G_n^1 = \{X_n^0, B_n, X_n^1\}$. $G_n^0 = \{X_n^0, B\}$ is the 0-graph from which G_n^1 was constructed. Our next step is to make an ultrapower construction to get the nonstandard 1-nodes. We consider sequences of extremities of G_n^0, $\langle e_n \rangle$ being one such sequence and e_n being an extremity of G_n^0. Two such sequences $\langle e_n \rangle$ and $\langle f_n \rangle$ are taken to be equivalent if $e_n = f_n$ for almost all n. This partitions the set of all such sequences into equivalence classes. Indeed, reflexivity and symmetry are obvious, and transitivity follows as usual (i.e., if $e_n = f_n$ a.e. and if $f_n = g_n$ a.e., then $e_n = g_n$ a.e.). Each equivalence class is taken to be a nonstandard extremity $\mathbf{e} = [e_n]$ where $\langle e_n \rangle$ is any sequence in that equivalence class.

Given any sequence $\langle e_n \rangle$, let $N_{t^0} = \{n : e_n \text{ is a 0-tip}\}$ and $N_{x^0} = \{n : e_n \text{ is a 0-node}\}$. Thus, $N_{t^0} \cap N_{x^0} = \emptyset$ and $N_{t^0} \cup N_{x^0} = \mathbb{N}$. So, exactly one of N_{t^0} and N_{x^0} is a member of $\mathcal{F}$. If it is N_{t^0} (resp. N_{x^0}), $\langle e_n \rangle$ is a representative of a **nonstandard 0-tip** (resp. a **nonstandard 0-node**).

Now, let $\mathbf{e} = [e_n]$ and $\mathbf{f} = [f_n]$ be two nonstandard extremities, and let $\mathbb{N}_{ef} = \{n : e_n \asymp f_n\}$ and $\mathbb{N}_{ef}^c = \{n : e_n \not\asymp f_n\}$. Exactly one of $\mathbb{N}_{ef}$ and $\mathbb{N}_{ef}^c$ is a member of $\mathcal{F}$. If it is $\mathbb{N}_{ef}$ (resp. $\mathbb{N}_{ef}^c$), we say that $\mathbf{e}$ is **shorted** to $\mathbf{f}$ (resp. $\mathbf{e}$ is **not shorted** to $\mathbf{f}$), and we write $\mathbf{e} \asymp \mathbf{f}$ (resp. $\mathbf{e} \not\asymp \mathbf{f}$). Furthermore, we take it that every $\mathbf{e}$ is shorted to itself : $\mathbf{e} \asymp \mathbf{e}$. This shorting is an equivalence relation for the set of all nonstandard extremities, as can be shown much as before ; indeed, for transitivity, assume $\mathbf{e} \asymp \mathbf{f}$ and $\mathbf{f} \asymp \mathbf{g}$. Since

$$\{n : e_n \asymp f_n\} \cap \{n : f_n \asymp g_n\} \subseteq \{n : e_n \asymp g_n\},$$

we have $\mathbf{e} \asymp \mathbf{g}$. The resulting equivalence classes are the **nonstandard 1-nodes**.

This definition can be shown to be independent of the representative sequences chosen for the nonstandard extremities. To be specific, let $\mathbf{e} = [e_n] = [\bar{e}_n]$ and $\mathbf{f} = [f_n] = [\bar{f}_n]$. Set $\mathbb{N}_e = \{n : e_n = \bar{e}_n\} \in \mathcal{F}$ and $\mathbb{N}_f = \{n : f_n = \bar{f}_n\} \in \mathcal{F}$. Assume $[e_n] \asymp [f_n]$. Thus, $\mathbb{N}_{ef} = \{n : e_n \asymp f_n\} \in \mathcal{F}$. We want to show that $\mathbb{N}_{\bar{e}\bar{f}} = \{n : \bar{e} = \bar{f}\} \in \mathcal{F}$ and thus $[\bar{e}_n] \asymp [\bar{f}_n]$. We have $(\mathbb{N}_e \cap \mathbb{N}_f \cap \mathbb{N}_{ef}) \subseteq \mathbb{N}_{\bar{e}\bar{f}}$, whence our conclusion.

Altogether we have defined a nonstandard 1-node $\mathbf{x}^1$ to be any set in the partition of the set of nonstandard extremities induced by the shorting $\asymp$, with every nonstandard 1-node having at least one nonstandard 0-tip. With ${}^*X^1$ standing for the set of nonstandard 1-nodes, we define the **nonstandard 1-graph** ${}^*\mathcal{G}^1$ to be the triplet ${}^*\mathcal{G}^1 = \{{}^*X^0, {}^*B, {}^*X^1\}$.

Let us observe the each nonstandard 1-node $\mathbf{x}^1$ contains no more than one nonstandard 0-node, and, if it does contain such a 0-node, it does not share that nonstandard 0-node with any other nonstandard 1-node. Indeed, if $\mathbf{x}^1$ had two nonstandard 0-nodes, then, for at least one n, two 0-nodes in G_n^1 would have to be shorted together within a 1-node of G_n^1, a violation of the definition of standard 1-nodes. Our second observation follows in the same way because for no n will a 1-node in G_n^1 share a 0-node with another 1-node in G_n^1.

§4. NONSTANDARD 1-NETWORKS AND THEIR HYPERREAL CURRENT–VOLTAGE REGIMES

A 1-network $\mathbf{N}^1$ is a 1-graph where every branch b is assigned an orientation (that is, a direction from one of the nodes to another node). A resistor r_b is required to be a positive real number, whereas a voltage source e_b can be any real number, possibly 0. Also, b has a current i_b and a voltage v_b measured with respect to b's orientation in such a fashion that Ohm's law holds : $v_b + e_b = r_b i_b$.

To get a **nonstandard 1-network**, we first start with a sequence $\langle \mathbf{N}_n^1 : n \in \mathbb{N} \rangle$ of 1-networks, with each 1-network $\mathbf{N}_n^1$ having $G_n^1 = \{X_n^0, B_n, X_n^1\}$ for its 1-graph and with every branch b_n of G_n^1 having the parameters r_{b_n} and e_{b_n} as well as a current i_{b_n} and voltage v_{b_n}, as stated above. The currents and voltages of b_n are again measured with respect to a given orientation so that

$$v_{b_n} + e_{b_n} = r_{b_n} i_{b_n}. \tag{1}$$

We will now state a previously established theorem concerning the existence and uniqueness of the current i_{b_n} in every branch b_n of $\mathbf{N}_n^1$. To do this, we need to construct for each $\mathbf{N}_n^1$ a solution space $\mathcal{L}_n$ that will be searched for a unique branch-current vector satisfying a form of Tellegen's equation (see (3)). Below we let $\sum_{b_n \in B_n}$ denote the summation over all the branches b_n in B_n. $\mathcal{I}_n$ will denote the linear space over the field $\mathbb{R}$ of all finite-powered branch-current vectors in $\mathbf{N}_n^1$, so $i_n = \{i_{b_n} : b_n \in B_n\} \in \mathcal{I}_n$ whenever

$$\|i_n\|^2 = \sum_{b_n \in B_n} i_{b_n}^2 r_{b_n} < \infty.$$

We assign the norm $\|i_n\|$ to the members of $\mathcal{I}_n$ and make $\mathcal{I}_n$ a real Hilbert space with the inner product $(i_n, s_n) = \sum_{b_n \in B_n} i_{b_n} s_{b_n} r_{b_n}$.

The (unit) **loop current** for a given oriented loop L in $\mathbf{N}_n^1$ is a vector $i_n = \{i_{b_n} : b_n \in B_n\}$ of branch currents i_{b_n} such that $i_{b_n} = 1$ (resp. $i_{b_n} = -1$, resp. $i_{b_n} = 0$) if the branch b_n is in L with the same orientation as L (resp. b_n is in L with the opposite orientation, resp. b_n is not in L). If L is a 0-loop, the loop current for L is a member of $\mathcal{L}_n$, but, if L is a 1-loop, the loop current for L will be in $\mathcal{I}_n$ if and only if the sum of branch resistances r_{b_n} for the branches in L is finite.

Let $\mathcal{L}_n^o$ be the span of all the loop currents in $\mathcal{I}_n$. Finally, let $\mathcal{L}_n$ be the closure of $\mathcal{L}_n^o$ in $\mathcal{I}_n$: $\mathcal{L}_n$ is a subspace of $\mathcal{I}_n$ and is a Hilbert space by itself with the same norm and inner product as those of $\mathcal{I}_n$.

Next, let $e_n = \{e_{b_n} : b_n \in B_n\}$ be the vector of branch voltage sources in $\mathbf{N}_n^1$. We say that e_n is of **finite total isolated power** if

$$\sum_{b_n \in B_n} e_{b_n}^2 g_{b_n} < \infty, \tag{2}$$

where $g_{b_n} = 1/r_{b_n}$. We let $\mathcal{E}_{f,n}$ denote the set of all e_n satisfying (2).

We have the following fundamental theorem for $\mathbf{N}_n^1$. (See [6], Theorem 3.3-5 or [7], Theorem 5.2-8.)

Theorem 1. *If $e_n \in \mathcal{E}_{f,n}$, then there exists a unique branch-current vector $i_n \in \mathcal{L}_n$, such that*

$$\sum_{b_n \in B_n} r_{b_n} i_{b_n} s_{b_n} = \sum_{b_n \in B_n} e_{b_n} s_{b_n} \tag{3}$$

for every $s_n = \{s_{b_n} : b_n \in B_n\} \in \mathcal{L}_n$.

Note that the branch voltages $v_{b_n} = r_{b_n} i_{b_n} - e_{b_n}$ are also determined by this theorem. We wish to obtain a nonstandard version of this theorem that is applicable to a nonstandard 1-network ${}^*\mathbf{N}^1$ obtained from the given sequence $\langle \mathbf{N}_n^1 \rangle$ of 1-networks through an ultrapower construction. Upon constructing the nonstandard 1-graph ${}^*G^1 = \{{}^*X^0, {}^*B, {}^*X^1\}$ from $\langle G_n^1 \rangle$ as in the preceding section, the branch parameters too undergo an ultrapower construction to become hyperreal parameters. Thus, each nonstandard branch $\mathbf{b}$ of ${}^*G^1$ has a hyperreal positive resistor $\mathbf{r_b}$ and possibly a nonzero hyperreal branch voltage source $\mathbf{e_b}$. In particular, for $\mathbf{b} = [\{x_n^0, y_n^0\}] \in {}^*B$, we have $\mathbf{r_b} = [r_{b_n}]$, where $r_{b_n} > 0$ is the resistance of the branch $b_n = \{x_n^0, y_n^0\}$ for almost all n, and similarly $\mathbf{e_b} = [e_{b_n}]$, where $e_{b_n} \in \mathbb{R}$ is the branch voltage source for b_n, again for almost all n. All this yields the **nonstandard 1-network** ${}^*\mathbf{N}^1 = [\mathbf{N}_n^1]$, whose 1-graph is ${}^*G^1$ and whose branch parameters are the hyperreals $\mathbf{r_b}$ and $\mathbf{e_b}$. Furthermore, i_{b_n} and v_{b_n} denote the branch current and the branch voltage for the branch $b_n \in B_n$, and these yield the hyperreal branch current $\mathbf{i}_b = [i_{b_n}]$ and the hyperreal branch voltage $\mathbf{v}_b = [v_{b_n}]$ for each $\mathbf{b} = [b_n] \in {}^*B$.

Every member $i_n = \{i_{b_n} : b_n \in B_n\}$ of any space $\mathcal{L}_n$ ($n \in \mathbb{N}$) determines a function mapping the set B_n of branches in $\mathbf{N}_n^1$ into $\mathbb{R}$, and thus by means of an ultrapower construction of $[i_n] = \mathbf{i} = \{\mathbf{i}_b : b \in {}^*B\}$ determines an internal function mapping ${}^*B$ into ${}^*\mathbb{R}$ with regard to the nonstandard network ${}^*\mathbf{N}^1_n$. In particular, $\mathbf{i}_b = [i_{b_n}]$, where $\{i_{b_n} : b_n \in B_n\}$ is a member of $\mathcal{L}_n$ for almost all n. All this yields a solution space ${}^*\mathcal{L} = [\mathcal{L}_n]$ consisting of the nonstandard current vectors $\mathbf{i} = \{\mathbf{i}_b : \mathbf{b} \in {}^*B\}$.

In order to invoke Theorem 1, we also assume that, for almost all n, the branch voltage sources e_{b_n} together have finite total isolated power (i.e., (2) is satisfied for almost all n). We let ${}^*\mathcal{E}_f$ denote the set of such nonstandard branch-voltage-source vectors; that is, each member of ${}^*\mathcal{E}_f$ is a vector $\mathbf{e} = \{\mathbf{e_b} : \mathbf{b} \in {}^*B\}$, where $\mathbf{e_b} = [e_{b_n}]$ and the e_{b_n} satisfy (2) for almost all n. Then, Theorem 1 holds again for almost all n. For the nonstandard 1-network ${}^*\mathbf{N}^1$ this can be restated as follows :

Theorem 2. *If* $\mathbf{e} \in {}^*\mathcal{E}_f$, *then there exists a unique branch-current vector* $\mathbf{i} = \{\mathbf{i_b} : \mathbf{b} \in {}^*B\} \in {}^*\mathcal{L}$ *such that*

$$\sum_{\mathbf{b} \in {}^*B} \mathbf{r_b i_b s_b} = \sum_{\mathbf{b} \in {}^*B} \mathbf{e_b s_b} \tag{4}$$

for every $\mathbf{s} = \{\mathbf{s_b} : \mathbf{b} \in {}^*B\} \in {}^*\mathcal{L}$.

Each side of (4) is well-defined as the hyperreal having the sequence of real numbers given by (3) for a representative sequence. Note that the uniquely determined branch-current vector $\mathbf{i}$ determines a unique branch-voltage vector $\mathbf{v} = \{\mathbf{v_b} : \mathbf{b} \in {}^*B\}$ by means of Ohm's law :

$$\mathbf{v} = \mathbf{r_b i_b} - \mathbf{e_b}.$$

Theorem 2 could also have been obtained from Theorem 1 by appending asterisks in accordance with the transfer principle.

§5. KIRCHHOFF'S LAWS

Kirchhoff's laws can also be lifted in a nonstandard way for the current-voltage regime dictated by Theorem 2. First, consider Kirchhoff's current law. The nonstandard 0-node $\mathbf{x}^0 = [x^0_n]$ is called **maximal** if x^0_n is maximal in $\mathbf{N}^1_n$ for almost all n (that is, if x^0_n is not contained in any 1-node of $\mathbf{N}^1_n$). Below $\sum_{b_n \ni x^0_n}$ will denote a summation over all branches b_n that are incident at x^0_n. Also, x^0_n is called **restraining** if the sum of the conductances $g_b = 1/r_b$ for the branches incident at x^0_n is finite (in symbols, if $\sum_{b_n \ni x^0_n} g_{b_n} < \infty$). We say that $\mathbf{x}^0$ is **restraining** if x^0_n is restraining for almost all n.

Under the assumptions on $\mathbf{N}^1_n$ required for Theorem 1, Kirchhoff's current law is satisfied at every restraining maximal 0-node x^0_n as follows :

$$\sum_{b_n \ni x^0_n} \pm i_{b_n} = 0, \tag{5}$$

where the plus (resp. minus) sign is used if b_n is incident away from (resp. toward) x_n^0. Furthermore, (5) converges absolutely, as established in [6], Theorem 3.4-1 or [7], Theorem 5.3-1.

Turning to the nonstandard case, we first observe again that every branch b_n in $\mathbf{N}_n^1$ has an orientation. So, for $\mathbf{x}^0 = [x_n^0]$ and $\mathbf{b} = [b_n]$, every branch b_n incident at x_n^0 is either oriented away from x_n^0 a.e. or is oriented toward x_n^0 a.e. Thus, $\mathbf{b}$ acquires an orientation either away from $\mathbf{x}^0$ or toward $\mathbf{x}^0$. Also (5) holds for almost all n. We set

$$\sum_{\mathbf{b} \ni \mathbf{x}^0} \pm \mathbf{i_b} = \left[\sum_{b_n \ni x_n^0} \pm i_{b_n} \right].$$

In this way, we get Kirchhofff's current law for ${}^*\mathbf{N}^1$:

Theorem 3. *If $\mathbf{x}^0$ is a restraining maximal 0-node in ${}^*\mathbf{N}^1$, then under the regime dictated by Theorem 2*

$$\sum_{\mathbf{b} \ni \mathbf{x}^0} \pm \mathbf{i_b} = 0, \tag{6}$$

where the summation converges absolutely (i.e., $\sum_{\mathbf{b} \ni \mathbf{x}^0} |\mathbf{i_b}| < \infty$).

Next, we discuss a nonstandard version of Kirchhoff's voltage law for ${}^*\mathbf{N}^1 = [\mathbf{N}_n^1]$. For this purpose we need to define nonstandard loops. Let ${}^*\mathbf{N}^1 = [\mathbf{N}_n^1]$ be a nonstandard 1-network, and let $G_{s,n}$ be a branch-induced subnetwork of $\mathbf{N}_n^1$. Then, the **relative degree** $d_x(G_{s,n})$ of a node x (0-node or 1-node) in $G_{s,n}$ is the cardinality of the set of branches and 0-tips in $G_{s,n}$ that are incident to x. Finally, a loop L (0-loop or 1-loop) in $\mathbf{N}_n^1$ is a connected subgraph $G_{s,n}$ having at least three branches and whose every node x has a relative degree equal to 2 (i.e., $d_x(G_{s,n}) = 2$ for every node x in $G_{s,n}$).

Any sequence $\langle G_{s,n} \rangle$ of subgraphs $G_{s,n}$ in the $\mathbf{N}_n^1$ determines a nonstandard subgraph ${}^*G_s$ of the nonstandard graph ${}^*G^1$ of ${}^*\mathbf{N}^1$ in the same way as $\langle G_n^1 \rangle$ determines the nonstandard 1-graph ${}^*G^1$ of ${}^*\mathbf{N}^1$. (A nonstandard branch $\mathbf{b} = [b_n]$ is in ${}^*G_s$ if and only if b_n is in $G_{s,n}$ for almost all n, and similarly for 0-nodes, 0-tips, and 1-nodes.) Then, ${}^*G_s$ is a nonstandard loop (0-loop or 1-loop) if, for almost all n, $G_{s,n}$ is connected, has at least three branches, and the relative degrees of all its relatively maximal nodes equal 2. In this case, we write $\mathbf{L} = {}^*G_s = [L_n]$, where $L_n = G_{s,n}$ is a loop in $\mathbf{N}_n^1$ for almost all n.

In the following, $\sum_{b_n \dashv L_n}$ denotes a sum over all the branches in the standard loop L_n. L_n is called **permissive** if $\sum_{b_n \dashv L_n} r_{b_n} < \infty$. Furthermore, we assign an orientation to each loop L_n. Under the regime dictated by Theorem 1, Kirchhoff's voltage law is satisfied around every permissive loop L_n in $\mathbf{N}_n^1$. In symbols,

$$\sum_{b_n \dashv L_n} \pm v_{b_n} = 0, \tag{7}$$

where the plus (resp. minus) sign is used if the orientations of b and L_n agree (resp. disagree). This too is a known result; see [6], Theorem 3.4-3 or [7], Theorem 5.3-4. With regard to the nonstandard case, the nonstandard loop $\mathbf{L} = [L_n]$ is called **permissive** if L_n is permissive for almost all n. Also, $\mathbf{L}$ acquires an orientation with regard to its nonstandard branches $\mathbf{b} = [b_n]$ in the following way. For almost all n, b_n is in L_n, and the orientation of b_n either agrees a.e or disagrees a.e with the orientation of L_n. So, if $\mathbf{v_b}$ is the hyperreal voltage of the nonstandard oriented branch $\mathbf{b}$ in $\mathbf{L}$, we have unambiguously the voltage $+\mathbf{v_b}$ or $-\mathbf{v_b}$ measured with respect to this implicitly defined orientation of $\mathbf{L}$. Upon setting

$$\sum_{\mathbf{b} \dashv \mathbf{L}} \pm \mathbf{v_b} = \left[\sum_{b_n \dashv L_n} \pm v_{b_n} \right],$$

we obtain the following nonstandard version of Kirchhoff's voltage law.

Theorem 4. *If $\mathbf{L}$ is an oriented permissive loop in ${}^*\mathbf{N}^1$, then under the regime dictated by Theorem 2*

$$\sum_{\mathbf{b} \dashv \mathbf{L}} \pm \mathbf{v_b} = 0, \tag{8}$$

where the summation converges absolutely (i.e., $\sum_{\mathbf{b} \dashv \mathbf{L}} |\mathbf{v_b}| < \infty$)).

Finally, let us note an immediate corollary. If x_n^0 is a 0-node of finite degree for almost all n, then $\mathbf{x}^0 = [x_n^0]$ is restraining. Also, if L_n is a finite 0-loop for almost all n, then $\mathbf{L} = [L_n]$ is permissive. It follows that Kirchhoff's laws will always hold for nonstandard 0-networks having hyperfinite graphs.

Corollary 5. If the nonstandard 0-network ${}^*\mathbf{N}^0$ has a hyperfinite graph, then Kirchhoff's laws are satisfied at all its nodes and around all its loops.

§6. A FINAL COMMENT

Finally, let us simply take note of the following nonlinear result obtained by transferring Duffin's theorem (see [1] or [8], Sec. 6.4). Let ${}^*\mathbf{N}^0 = [\mathbf{N}_n^0]$ be any nonstandard nonlinear 0-network such that its nonstandard graph ${}^*G^0 = [G_n^0]$ is hyperfinite and, for almost all n, the resistance characteristic $R_{b_n} : i_{b_n} \mapsto v_{b_n}$ for each branch $b_n \in B_n$ in G_n^0 is a continuous, strictly monotonically increasing bijection of $\mathbb{R}$ onto $\mathbb{R}$. Then, the hyperreal current-voltage regime for ${}^*\mathbf{N}^0$ is determined by Kirchhoff's current law at each nonstandard 0-node $\mathbf{x}^0 = [x_n^0]$, by Kirchhoff's voltage law around each nonstandard 0-loop, and by the replacement of Ohm's law by the expression $\mathbf{v_b} = \mathbf{R_b}(\mathbf{i_b})$ for each nonstandard branch $\mathbf{b} = [b_n]$, where now $\mathbf{R_b} = [R_{b_n}]$ is the internal nonlinear resistance characteristic, that is, $v_{b_n} = R_{b_n}(i_{b_n})$ for almost all n.

Резюме. Реализована идея нестандартной, трансконечной, линейной электрической сети. Чтобы доказать теорему существования и единственности для гипервещественного режима напряжения тока, понятие трансконечного графа должно быть распространено на нестандартные положения. Проверяются также нестандартные версии законов Хирчова. Наконец, указывается, что некоторые нелинейные сети также имеют единственные гипервещественные режимы напряжения тока.

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