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SHARP ASYMPTOTICS FOR POSITIVE SERIES

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Abstract. Suppose that $\sum\limits_{k=0}^{\infty} a_k$ is a series with $a_k \geq 0$. In [3] some asymptotic properties were obtained for series of the type $\sum\limits_{k=0}^{\infty} a_k\varphi(\sum\limits_{\nu=0}^{k} a_\nu)$ and $\sum\limits_{k=0}^{\infty} a_k\varphi(\sum\limits_{\nu=k}^{\infty} a_\nu)$ for a class of functions φ. It has been proved by L. Leindler [2] that these asymptotics are best possible for the case $\varphi(t) = t^{-\alpha}$. In this paper we show that the asymptotics obtained in [3] are best possible for more general $\varphi(t)$.

§1. INTRODUCTION AND STATEMENT OF RESULTS

Throughout this note we consider series $\sum\limits_{k=0}^{\infty} a_k$ with $a_k \geq 0$. As usual

$$S_n := \sum_{k=0}^{n} a_k; \qquad R_n := \sum_{k=n}^{\infty} a_k$$

are the partial sums and the remainders respectively, and suppose that $S_n > 0$ and $R_n > 0$ for all $n \in \mathbb{N}_0$.

In our previous paper [3] we investigated convergence – divergence properties of the series

$$\sum_{k=0}^{\infty} a_k\, \varphi(S_k) \quad \text{and} \quad \sum_{k=0}^{\infty} a_k \psi(R_k),$$

under certain assumptions on the functions φ and ψ. Among others we proved the following results.

Theorem A. *Let $\sum\limits_{k=0}^{\infty} a_k$ be divergent and φ be a positive and decreasing function on $[a_0, \infty)$. The following assertions hold :*

(1) If $\int\limits_{a_0}^{\infty} \varphi(t)\,dt < \infty$, *then* $\sum\limits_{k=0}^{\infty} a_k \varphi(S_k) < \infty$,

(2) If $\int\limits_{a_0}^{\infty} \varphi(t)\,dt = \infty$, *then* $\sum\limits_{k=1}^{\infty} a_k \varphi(S_{k-1}) = \infty$.

Theorem B. *Let* $\sum\limits_{k=0}^{\infty} a_k$ *be convergent and* ψ *be a positive and decreasing function on* $(0, R_0]$. *The following assertions hold :*

(1) If $\int\limits_{0}^{R_0} \psi(t)\,dt < \infty$, *then* $\sum\limits_{k=0}^{\infty} a_k \psi(R_k) < \infty$,

(2) If $\int\limits_{0}^{R_0} \psi(t)\,dt = \infty$, *then* $\sum\limits_{k=0}^{\infty} a_k \psi(R_{k+1}) = \infty$.

For $\varphi(t) = t^{-\alpha}$, $t \in [a_0, \infty)$, Theorem A reduces to the well-known Abel-Dini-theorem (see for instance [1], p. 299, or [5], Theorem 7.9). A classical result of Dini (see for instance [1], p. 301) is obtained for $\psi(t) = t^{-\alpha}$, $t \in (0, R_0]$, as a special case of Theorem B.

For the case $\varphi(t) = t^{-\alpha}$ and $\psi(t) = t^{-\alpha}$ L. Leindler [2] proved that Theorem A and Theorem B are in a certain sense best possible.

At the International Conference on Functions, Series, Operators (Alexits Memorial Conference, Budapest) the problem was set whether Theorem A and Theorem B are also best possible for more general functions φ and ψ. Using Leindler technic [2] we answer this question and prove the following results.

Theorem 1. *Let* φ *be a positive and decreasing function on* $[1, \infty)$.

(1) Suppose that $\int\limits_{1}^{\infty} \varphi(t)dt < \infty$ *and let* $\{\rho_k\}$ *be a sequence of positive factors with* $\lim\limits_{k\to\infty} \rho_k = \infty$. *Then there exists a sequence* $\{a_k\}$ *of positive numbers with* $a_0 = S_0 = 1$ *such that*

a) $\sum\limits_{k=0}^{\infty} a_k = \infty$,

b) $\sum\limits_{k=0}^{\infty} a_k \varphi(S_k) < \infty$,

c) $\sum\limits_{k=0}^{\infty} a_k \rho_k \varphi(S_k) = \infty$.

(2) Suppose that $\int\limits_{1}^{\infty} \varphi(t)dt = \infty$ *and let* $\{\varepsilon_k\}$ *be a sequence of positive factors with* $\lim\limits_{k\to\infty} \varepsilon_k = 0$. *Then there exists a sequence* $\{a_k\}$ *of positive numbers with* $a_0 = S_0 = 1$ *such that*

a) $\sum\limits_{k=0}^{\infty} a_k = \infty$,

b) $\sum\limits_{k=1}^{\infty} a_k \varphi(S_{k-1}) = \infty$, *c)* $\sum_{k=1}^{\infty} a_k \varepsilon_k \varphi(S_{k-1}) < \infty.$

Theorem 2. *Let ψ be a positive and decreasing function on $(0, 1]$.*

(1) Suppose that $\int_0^1 \psi(t)dt < \infty$ and let $\{\rho_k\}$ be a sequence of positive factors with $\lim_{k\to\infty} \rho_k = \infty$. Then there exists a sequence $\{a_k\}$ of positive numbers such that

a) $\sum_{k=0}^{\infty} a_k = R_0 = 1,$

b) $\sum_{k=0}^{\infty} a_k \psi(R_k) < \infty,$

c) $\sum_{k=0}^{\infty} a_k \rho_k \psi(R_k) = \infty.$

(2) Suppose that $\int_0^1 \psi(t)\,dt = \infty$ and let $\{\varepsilon_k\}$ be a sequence of positive factors with $\lim_{k\to\infty} \varepsilon_k = 0$. Then there exists a sequence $\{a_k\}$ of positive numbers such that

a) $\sum_{k=0}^{\infty} a_k = R_0 = 1,$

b) $\sum_{k=0}^{\infty} a_k \psi(R_{k+1}) = \infty,$

c) $\sum_{k=0}^{\infty} a_k \varepsilon_k \psi(R_{k+1}) < \infty.$

§2. PROOF OF THEOREM 1

a) We first prove (1). Since $\lim_{k\to\infty} \rho_k = \infty$, there exists a strictly increasing sequence $\{\mu_m\}$ of natural numbers with $\mu_1 := 1$ and

$$\rho_k \geq \frac{1}{\varphi(m+1)} \qquad \text{for all} \quad k \geq \mu_m;\ m = 2, 3, \dots .$$

Let the sequence $\{a_k\}$ be defined by $a_0 := 1$ and

$$a_k := \frac{1}{\mu_{m+1} - \mu_m} \qquad \text{for} \quad \mu_m \leq k < \mu_{m+1};\ m = 1, 2, \dots .$$

We obviously obtain $\sum_{k=0}^{\infty} a_k = \infty$ and it follows from [3] that $\sum_{k=0}^{\infty} a_k \varphi(S_k) < \infty$. For $\mu_m \leq k < \mu_{m+1}$ we have $S_k \leq S_{\mu_{m+1}-1} = m + 1$ and therefore $\varphi(S_k) \geq \varphi(m+1)$. For $N \geq 2$ this implies

$$\sum_{k=0}^{\mu_{N+1}-1} a_k\, \rho_k \varphi(S_k) \geq \sum_{m=2}^{N} \sum_{k=\mu_m}^{\mu_{m+1}-1} a_k \rho_k \varphi(S_k) \geq$$

$$\geq \sum_{m=2}^{N} \frac{1}{\varphi(m+1)} \sum_{k=\mu_m}^{\mu_{m+1}-1} a_k \varphi(S_k) \geq \sum_{m=2}^{N} \sum_{k=\mu_m}^{\mu_{m+1}-1} a_k = N - 2$$

Therefore $\sum\limits_{k=0}^{\infty} a_k \rho_k \varphi(S_k) = \infty$.

b) To prove (2) we choose a strictly increasing sequence $\{\mu_m\}$ of natural numbers with $\mu_1 = 1$ and

$$0 < \varepsilon_k < \frac{1}{m^2 \varphi(m)} \qquad \text{for all} \quad k \geq \mu_m;\ m = 2, 3, \dots .$$

We define the sequence $\{a_k\}$ putting $a_0 := 1$ and

$$a_k := \frac{1}{\mu_{m+1} - \mu_m} \qquad \text{for all} \quad k \geq \mu_m;\ m = 1, 2, \dots .$$

We again obtain $\sum\limits_{k=0}^{\infty} a_k = \infty$ and it follows now from [3] that $\sum\limits_{k=1}^{\infty} a_k \varphi(S_{k-1}) = \infty$. For $\mu_m \leq k < \mu_{m+1}$ we have $S_{k-1} \geq S_{\mu_m - 1} = m$ and therefore $\varphi(S_{k-1}) \leq \varphi(m)$, implying

$$\sum_{k=\mu_2}^{\infty} a_k \varepsilon_k \varphi(S_{k-1}) = \sum_{m=2}^{\infty} \sum_{k=\mu_m}^{\mu_{m+1}-1} a_k \varepsilon_k \varphi(S_{k-1}) \leq$$

$$\leq \sum_{m=2}^{\infty} \frac{1}{m^2 \varphi(m)} \sum_{k=\mu_m}^{\mu_{m+1}-1} a_k \varphi(S_{k-1}) \leq \sum_{m=2}^{\infty} \frac{1}{m^2} \sum_{k=\mu_m}^{\mu_{m+1}-1} a_k = \sum_{m=2}^{\infty} \frac{1}{m^2}.$$

Therefore $\sum\limits_{k=1}^{\infty} a_k \varepsilon_k \varphi(S_{k-1}) < \infty$. Theorem 1 is proved.

§3. PROOF OF THEOREM 2

a) We first prove (1). Since $\lim\limits_{k\to\infty} \rho_k = \infty$, there exists a strictly increasing sequence $\{\mu_m\}$ of natural numbers with $\mu_1 := 1$ and

$$\rho_k \geq \frac{2^{m+1}}{\psi(2^{-m})} \qquad \text{for all} \quad k \geq \mu_m; \quad m = 2, 3 \dots .$$

Let the sequence $\{a_k\}$ be defined by $a_0 := \frac{1}{2}$ and

$$a_k := \frac{1}{2^{m+1} \cdot (\mu_{m+1} - \mu_m)} \qquad \text{for } \mu_m \leq k < \mu_{m+1};\ m = 1, 2, \dots .$$

A short calculation gives $\sum\limits_{k=0}^{\infty} a_k = 1$ and from [3] we get $\sum\limits_{k=0}^{\infty} a_k \psi(R_k) < \infty$. For $k \geq \mu_m$ we obtain

$$R_k = \sum_{\nu=k}^{\infty} a_\nu \leq \sum_{\nu=\mu_m}^{\infty} a_\nu = \sum_{j=m}^{\infty} \sum_{\nu=\mu_j}^{\mu_{j+1}-1} a_\nu = \sum_{j=m}^{\infty} \frac{1}{2^{j+1}} = \frac{1}{2^m}$$

and therefore $\psi(R_k) \geq \psi(2^{-m})$. For $N \geq 2$ this implies

$$\sum_{k=0}^{\infty} a_k \rho_k \psi(R_k) \geq \sum_{m=2}^{N} \sum_{k=\mu_m}^{\mu_{m+1}-1} a_k \rho_k \psi(R_k) \geq \sum_{m=2}^{N} \frac{2^{m+1}}{\psi(2^{-m})} \sum_{k=\mu_m}^{\mu_{m+1}-1} a_k \psi(R_k) \geq$$

$$\geq \sum_{m=2}^{N} \frac{2^{m+1}}{\psi(2^{-m})} \psi(2^{-m}) \sum_{k=\mu_m}^{\mu_{m+1}-1} a_k = N-2.$$

Therefore $\sum\limits_{k=0}^{\infty} a_k \rho_k \psi(R_k) = \infty$.

b) To prove (2) we choose a strictly increasing sequence $\{\mu_m\}$ of natural numbers with $\mu_1 = 1$ and

$$0 < \varepsilon_k \leq \frac{1}{\psi(2^{-m-1})} \qquad \text{for all} \quad k \geq \mu_m;\ m = 2, 3, \ldots.$$

We define the sequence $\{a_k\}$ putting $a_0 := \frac{1}{2}$ and

$$a_k := \frac{1}{2^{m+1} \cdot (\mu_{m+1} - \mu_m)} \qquad \text{for} \quad \mu_m \leq k < \mu_{m+1};\ m = 1, 2, \ldots.$$

Again we have $\sum\limits_{k=0}^{\infty} a_k = 1$ and from [3] it follows $\sum\limits_{k=0}^{\infty} a_k \psi(R_{k+1}) = \infty$. For $k < \mu_{m+1}$ we get

$$R_{k+1} = \sum_{\nu=k+1}^{\infty} a_\nu \geq \sum_{\nu=\mu_{m+1}}^{\infty} a_\nu = \sum_{j=m+1}^{\infty} \sum_{\nu=\mu_j}^{\mu_{j+1}-1} a_\nu = \sum_{j=m+1}^{\infty} \frac{1}{2^{j+1}} = \frac{1}{2^{m+1}}$$

and therefore $\psi(R_{k+1}) \leq \psi(2^{-m-1})$, implying

$$\sum_{k=\mu_2}^{\infty} a_k \varepsilon_k \psi(R_{k+1}) = \sum_{m=2}^{\infty} \sum_{k=\mu_m}^{\mu_{m+1}-1} a_k \varepsilon_k \psi(R_{k+1}) \leq$$

$$\leq \sum_{m=2}^{\infty} \frac{1}{\psi(2^{-m-1})} \sum_{k=\mu_m}^{\mu_{m+1}-1} a_k \psi(R_{k+1}) \leq \sum_{m=2}^{\infty} \frac{\psi(2^{-m-1})}{\psi(2^{-m-1})} \sum_{k=\mu_m}^{\mu_{m+1}-1} a_k = \sum_{m=2}^{\infty} \frac{1}{2^{m+1}}.$$

Therefore $\sum\limits_{k=0}^{\infty} a_k \varepsilon_k \psi(R_{k+1}) < \infty$. Theorem 2 is proved.

Резюме. Предположим, что $\sum\limits_{k=0}^{\infty} a_k$ есть ряд с $a_k \geq 0$. В [3] получены некоторые асимптотические свойства для рядов типа $\sum\limits_{k=0}^{\infty} a_k \varphi(\sum\limits_{\nu=0}^{k} a_\nu)$ и $\sum\limits_{k=0}^{\infty} a_k \varphi(\sum\limits_{\nu=k}^{\infty} a_\nu)$ для некоторого класса функций φ. Л. Лейндлером [2] было доказано, что эти асимптотики являются наилучшими возможными для случая $\varphi(t) = t^{-\alpha}$. В настоящей статье доказано, что асимптотики, полученные в [3] являются наилучшими возможными для более общего $\varphi(t)$.

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