Известия НАН Армении. Математика, 38, № 5, 2003, 39–46

ON THE HOMOGENEITY OF PRINCIPAL BUNDLES

R. Ch. Khatamy and M. Toomanian

Faculty of Mathematics, Tabriz University, Tabriz, IRAN
E-mail : *toomanian@tabrizu.ac.ir*, *chavosh@tabrizu.ac.ir*

Abstract. The paper considers the principal bundles and gives some results about the structure of geodesics in the base space of the principal bundles.

§1. INTRODUCTION

Let G be a Lie group. A (smooth) **principal bundle with structure group** G is a pair $(\wp, T)$ satisfying

(i) $\wp = (P, \pi, B, G)$ is a smooth fiber bundle.

(ii) $T : P \times G \longmapsto P$ is a right action of G on P.

(iii) $\wp$ admits a coordinate representation (U_α, ψ_α) such that

$$\psi_\alpha(x, ab) = \psi_\alpha(x, a)b, \quad x \in U_\alpha, \quad a, b \in G.$$

The action T is called **principal action** and the coordinate representation (U_α, ψ_α) is called **principal coordinate** (see [5], vol. I, p. 50).

Let G be a connected Lie group and K be a closed subgroup of G. The set G/K of left cosets of K in G possesses a unique differentiable structure, and is called homogeneous manifold.

Let $T : G \times M \longrightarrow M$ be a transitive action of G on a differentiable manifold M, and let K be the invariant subgroup of the point $x_0 \in M$. Then by the map $\phi : G/K \longrightarrow M$ with $\phi(gK) = gx_0$, taking $M = G/K$ reduces M to a homogeneous differentiable manifold. Given an affine connection ∇ on M, we are concerned with geodesics on (M, ∇) with respect to one parameter subgroups of G, called homogeneous geodesics (see [7], and also Definition 2.5).

By $\Im = (G, T, G/K, K)$ we denote the fiber bundle with group structure K (see Definition 2.4). In this paper we prove some results about the structure of homogeneous geodesics on the base space G/K of the fiber bundle $\Im$. First we consider the case where G is a semisimple Lie group, and then we remove this assumption.

§2. PRELIMINARIES

Let $\wp = (P, \pi, B, G)$ be a principal bundle with principal action T. A left action of G on a manifold F we write as

$$S : G \times F \to F.$$

Definition 2.1. A left action Q of G on the product manifold $P \times G$, given by

$$Q_a(z, y) = (z, y).a = (z.a, a^{-1}.y), \quad z \in P, \; y \in F, \; a \in G$$

is called **joint action of** G.
The set of orbits for a joint action we denote by $P \times_G F$ and define a map q by

$$q : P \times F \to P \times_G F.$$

Notice that q determines a map $\rho : P \times_G F \to B$ such that

$$\rho \circ q = \pi \circ \pi_p,$$

where $\pi_p : P \times F \to P$ is the projection and $\pi : P \to B$ is the bundle map.

Definition 2.2. A smooth fiber bundle $\Im = (P \times_G F, \rho, B, F)$ with a unique smooth structure on $P \times_G F$ is called **fiber bundle associated with** $\wp$.
Let P be a representation of a Lie group in a real vector space W. An Euclidean inner product $< \cdot, \cdot >$ in W is said to be invariant with respect to P, if

$$< p(x)u, p(x)v > = < u, v >, \quad x \in G, \quad u, v \in W.$$

Notice that for each $h \in T_eG$ the map $p'(h) : W \longrightarrow W$ is skew.
Let $M = G/K$ be a homogeneous manifold. G/K is **reductive**, if the Lie algebra $\mathcal{G}$ of G can be represented as a direct sum of the Lie algebra $\mathcal{K}$ of the subgroup K and a vector space $\mathcal{M}$ which is $ad_{\mathcal{K}}$-invariant, i.e.

1. $\mathcal{G} = \mathcal{M} + \mathcal{K}; \qquad \mathcal{M} \bigcap \mathcal{K} = \{0\}$,
2. $ad_{\mathcal{K}} \mathcal{M} \subset \mathcal{M}$. (1)

It follows from (1) that

$$[\mathcal{K}, \mathcal{M}] \subset \mathcal{M}. \tag{2}$$

Observe that if K is connected then (2) implies (1) (see [5], vol. I).

Definition 2.3. Let G be a connected Lie group and K be a closed connected subgroup of G with Lie algebras $\mathcal{G}$ and $\mathcal{K}$, respectively. Then $\mathcal{G} \subseteq \mathcal{K}$ and hence we can write

$$\mathcal{G} = \mathcal{K}^{\perp} \oplus \mathcal{K}.$$

The algebra $\mathcal{K}^{\perp}$ is called the **orthogonal complement** of $\mathcal{K}$ in $\mathcal{G}$ with respect to Euclidean inner product $< \cdot, \cdot >$ in $\mathcal{G}$. We have

$$Ady = Ad^{\perp}y \oplus id_{\mathcal{K}}, \quad y \in \mathcal{G},$$

where $Ad^{\perp}y$ stands for the restriction of Ady to $\mathcal{K}^{\perp}$.

Definition 2.4. Let K be a closed subgroup G. The fiber bundle $\Im = (G, \pi, G/K, K)$ with right multiplication action of K on G is a principal bundle with structure group K, and is called **principal homogeneous bundle** (see [2], p. 45).

Definition 2.5. Let ∇ be an affine connection on $M = G/K$, which is invariant under the action $T : G \times M \longrightarrow M$. Then a geodesic $\gamma : I \longrightarrow M$ is called **homogeneous**, if for some $X \in \mathcal{G} = T_1\mathcal{G}$ there exists an one-parameter subgroup $t \longrightarrow \exp tX$, $t \in R$, such that

$$\gamma(t) = T(\exp tX, x), \quad \gamma(0) = x, \quad t \in I.$$

A connected Riemannian manifold M is homogeneous, if either the isometry group $I(M)$ or a connected subgroup G of $I(M)$ acts transitively on M. In this case, if $x_0 \in M$ and K is the stabilizer of x_0, then $G/K = M$. Moreover, G will act effectively on G/K from the left. The point $x_0 = \{K\}$ is called the origin of the homogeneous Riemannian manifold M.

Now let $M = G/K$ be a reductive homogeneous Riemannian manifold and let $\mathcal{G} = \mathcal{M} + \mathcal{K}$ be its Lie algebra decomposition. The natural map $\phi : G \longrightarrow G/K = M$ will induce a linear epimorphism $(d\phi)_e : T_eG \longrightarrow T_{x_0}M$ and the vector space $\mathcal{M}$ will be identified with $T_{x_0}M$.

For a Riemannian M, the inner product on $T_{x_0}M$ induces an inner product C on $\mathcal{M}$, which is $ad_{\mathcal{K}}$-invariant. According to Definition 2.1, the geodesic γ on M passing through x_0 is homogeneous if and only if for all $t \in R$

$$\gamma(t) = (\exp tX)(x_0), \quad \text{for some} \quad X \in g, \ X \neq 0.$$

Definition 2.6. Let G be a Lie group and $\mathcal{G}$ be its Lie algebra. A vector $X \in \mathcal{G}$ $(X \neq 0)$ is called a **geodesic vector**, if the curve $\gamma(t) = (\exp tX)(x_0)$ is a geodesic on M (see [7]).

In view of Definition 2.6 there is a correspondence between the geodesic vectors and the homogeneous geodesics passing through $x_0 \in M$. Let C be an inner product on $\mathcal{M}$, induced by the inner product on $T_{x_0}M$. The following lemma holds.

Lemma 2.7 ([7]). *Let $X \in \mathcal{G}$ and let $[X,Y]_{\mathcal{M}}$ be the component of $[X,Y]$ in $\mathcal{M}$ with respect to reductive decomposition. Then X is geodesic if and only if*

$$C(X,[X,Y]_{\mathcal{M}}) = 0 \quad \text{for all} \quad Y \in \underline{\mathcal{G}}.$$

Definition 2.8. Let C be a bilinear symmetric form on a finite dimensional vector space V. The **radical** of C is a vector subspace of V, such that

$$rad\,C = \{v \in V; C(v,u) = 0 \quad \text{for all} \quad u \in V\}.$$

Definition 2.9. The **Killing form** B on $\mathcal{G}$ is defined to be a bilinear symmetric form given by $B(X,Y) = Tr(ad_X ad_Y)$.
The **Cartan's criterion for solvability** of $\mathcal{G}$ asserts that $\mathcal{G}$ is solvable if and only if $B(X,Y) = 0$ for every $Y \in [\mathcal{G},\mathcal{G}]$ and $X \in \mathcal{G}$ (see [4], p. 669).
The next result was proved in ([8], p. 55).

Theorem 2.10. *Every Lie algebra $\mathcal{G}$ has largest solvable ideal, which is denoted by $rad\ \mathcal{G}$. Every Lie group G has largest connected normal solvable Lie subgroup such that its Lie algebra is $rad\,\mathcal{G}$.*
The subgroup of the Lie group G in Theorem 2.10 is called **radical of** G and is denoted by $Rad\,G$.

Definition 2.11. A Lie group G is called **semisimple**, if $Rad\,G = \{e\}$. A Lie algebra $\mathcal{G}$ is called **semisimple**, if $rad\,\mathcal{G} = \{0\}$.
If a Lie algebra $\mathcal{G}$ is semisimple, then $[\mathcal{G},\mathcal{G}] = \mathcal{G}$. It is well known, that $\mathcal{G}$ is semisimple iff $B(X,Y)$ is non degenerate for all X and Y from $\mathcal{G}$. In other words $\mathcal{G}$ is semisimple if and only if the radical of the corresponding Killing form is identical zero (see [4], [5]).

§3. MAIN RESULTS

Let G be a connected Lie group, $T : G \times M \longrightarrow M$ be a transitive action of G on a differentiable manifold M and K be the invariant subgroup of the point $x_0 \in M$. The Lie algebras of K and G are denoted by $\mathcal{K}$ and $\mathcal{G}$ respectively. The adjoint representation of G leads to a representation Ad_K of K in $\mathcal{G}$. Since the Lie algebra $\mathcal{K}$ is stable under the map $Ad_K(a)$, $a \in K$, we get a representation $Ad^{\perp}$ of K in $\mathcal{G}/\mathcal{K}$. The sequence

$$0 \to \mathcal{K} \to \mathcal{G} \to \mathcal{G}/\mathcal{K}$$

is a short exact and K-equivariant with respect to the representations Ad, Ad_K and $Ad^{\perp}$ of K.
The next result is known (see, e.g. [2], vol. I, pp. 45, 94).

Proposition 3.1. Under the above hypotheses the vector bundle

$$\xi = (G \times_K \mathcal{G}/\mathcal{K}, \rho_\xi, \mathbf{G}/\mathbf{K}, \mathcal{G}/\mathcal{K})$$

is a fiber bundle associated with

$$\Im = (G, \pi, G/K, K),$$

and is strongly isomorphic to the tangent bundle $\tau_{G/K} = (T_{G/K}, \pi_{G/K}, G/K, \mathbb{R}^m)$, where $m = \dim(G/K)$.

By Proposition 3.1, instead of the tangent bundle $\tau_{G/K} = (T_{G/K}, \pi_{G/K}, G/K, \mathbb{R}^m)$ we can consider the base space G/K of $\Im = (G, \pi, G/K, K)$ and fiber space $\mathcal{G}/\mathcal{K}$ of $\xi = (G \times_K \mathcal{K}, \rho_\xi, \mathbf{G}/\mathbf{K}, \mathcal{G}/\mathcal{K})$.

Every Lie algebra $\mathcal{G}$ possesses largest nilpotent ideal (see 1], p. 58). Moreover, every Lie group G has largest connected normal and nilpotent subgroup, such that its Lie algebra is the largest nilpotent ideal in $\mathcal{G}$ (see [1], p. 59).

Definition 3.2. The largest nilpotent ideal in the Lie algebra $\mathcal{G}$ is called **weak radical of** $\mathcal{G}$ and is denoted by $W_{rad}\mathcal{G}$. The largest Lie subgroup which is normal nilpotent and its Lie algebra is $W_{rad}\mathcal{G}$, is called **weak radical of** G and is denoted by $W_{Rad}G$. According to Definitions 2.8 and 2.9 :

$$W_{rad}\mathcal{G} \subseteq \mathrm{rad}\mathcal{G}, \qquad \mathcal{W}_{\mathcal{R}\dashv\lceil}\mathcal{G} \subseteq \mathrm{Rad}\mathcal{G}.$$

Definition 3.3. A Lie group G (Lie algebra $\mathcal{G}$, respectively) is called weakly semisimple if $W_{Rad}G = \{e\}$ ($W_{rad}\mathcal{G} = \{0\}$, respectively).

Observe that every semisimple Lie group is weakly semisimple. The converse is not always true.

Definition 3.4. Let B be a Killing form on $\mathcal{G}$. The **weak radical of** B is defined to be

$$W_{rad}B = \{X \in [\mathcal{G}, \mathcal{G}], B(X, Y) = 0 \quad \text{for all} \quad Y \in \mathcal{G}\}.$$

Theorem 3.5. *Let G be a connected transitive Lie subgroup of the isometry group $I(M)$ of a Riemannian manifold $M = G/K$. Let $\Im = (G, \pi, G/K, K)$ be a principal homogeneous bundle and $\xi = (G \times_K \mathcal{G}/\mathcal{K}, \rho_\xi, \mathbf{G}/\mathbf{K}, \mathcal{G}/\mathcal{K})$ be a bundle associated with $\Im$. If G is semisimple and $m = \dim G/K$, then there are m orthogonal homogeneous geodesics passing through $x_0 = \{K\}$.*

Proof : Let $\Im = (G, \pi, G/K, K)$ be a principal homogeneous bundle and G be a connected Lie subgroup of $I(M)$ acting transitively on M. Let $\xi = (G \times_K \mathcal{G}/\mathcal{K}, \rho_\xi, \mathrm{G/K}, \mathcal{G}/\mathcal{K})$ be a bundle associated with $\Im$ and B be the Killing form on $\mathcal{G}$.

It is known (see [6]) that if $M = G/K$ is a homogeneous Riemannian manifold such that $\subseteq I(M)$ is solvable, then there exists a homogeneous geodesic passing through any point $x_0 \in M$.

Since $\Im = (G, \pi, G/K, K)$ is principal homogeneous bundle, there exists a homogeneous geodesic passing through every point in the base space.

Let $\mathcal{G}/\mathcal{K}$ be the fiber space of ξ and B be the Killing form on $\mathcal{G}$. Since $radB$ is solvable we have $radB \subseteq rad\mathcal{G}$ (see [3], p. 22). On the other hand, since B is non degenerate on $\mathcal{K}$, taking $\mathcal{M}$ the orthogonal complement of $\mathcal{K}$ with respect to B, we conclude that $\mathcal{G} = \mathcal{M} + \mathcal{K}$ is a reductive decomposition. (Here $\mathcal{K}$ is the Lie algebra of K, while $\mathcal{M}$ is a vector subspace of T_eG (see Proposition 3.1)). By Proposition 2 from [7] the Killing form B is non degenerate on $\mathcal{M}$. Hence taking into account that $\mathcal{M}$ is the orthogonal complement of $\mathcal{K}$ with respect to B we get $radB \subseteq \mathcal{M}$.

First we consider the case where $radB \subset \mathcal{M}$. By means of the inner product C on $\mathcal{M}$ we define an endomorphisms $\phi : \mathcal{M} \longrightarrow \mathcal{M}$ by (see [4], p. 669) :

$$B(X,Y) = C(\phi(X), Y), \quad X, Y \in \mathcal{M}.$$

Since the matrices of ϕ and B in the basis orthogonal with respect to C coincide, the matrix of ϕ is symmetric. Hence the eigenvalues $\lambda_1, \cdots, \lambda_m$ are real, the corresponding eigenvectors $v_1, \cdots, v_m$ form an orthogonal basis with respect to C and

$$B(v_i, v_j) = C(\lambda_i v_i, v_j) = \lambda_i C(v_i, v_j) = 0 \quad \text{for} \quad i \neq j.$$

If for some index l we have $B(v_l, v_l) = 0$, then $v_l \in radB$. Let $\lambda_l \in (\mathcal{M} - radB)$, implying $\lambda_l \neq 0$, so for any $Z \in \mathcal{G}$ we have

$$C(v_l, [v_l, Z]) = \frac{1}{\lambda_l} C(\phi(v_l), [v_l, Z]_{\mathcal{M}}) = \frac{1}{\lambda_l} B(v_l, [v_l, Z]_{\mathcal{M}}) =$$

$$= \frac{1}{\lambda_l} B(v_l, [v_l, Z]) = \frac{1}{\lambda_l} B([v_l, v_l], Z) = 0,$$

i.e. v_l is a geodesic vector.

Next, since $\mathcal{G}$ is semisimple we have $[\mathcal{G}, \mathcal{G}] = \mathcal{G}$ and $rad\,B = 0$ (see Definition 2.10). But $B(X,Y) = C(\phi(X), Y)$, hence $\mathrm{Ker}\phi = rad\,B$ and ϕ is isomorphism. Thus, all the eigenvalues $\lambda_i \neq 0$ $1 \leq i \leq m$ and the eigenvectors $v_1, \cdots, v_m$ are geodesic vectors, i.e. there are m orthogonal homogeneous geodesics passing through $x_0 = \{K\}$.

Let now $rad\,B = \mathcal{M}$. There is a solvable Lie group of isometries acting transitively on M. Since $\Im = (G, \pi, G/K, K)$ is a principal homogeneous bundle, there exists a homogeneous geodesic passing through every point of the base space. Taking into account that $\mathcal{G}$ is semisimple, we get $m = dimG/K$ orthogonal homogeneous geodesics passing through $x_0 = \{K\}$. Theorem 3.5 is proved.

Theorem 3.6. *With the hypotheses of Theorem 3.5 let*

$$\Im = (G, \pi, G/K, K)$$

be a principal homogeneous bundle and

$$\xi = (G \times_K \mathcal{G}/\mathcal{K}, \rho_\xi, \mathbf{G}/\mathbf{K}, \mathcal{G}/\mathcal{K})$$

be a bundle associated with $\Im$. *Let* $\mathcal{G}' = [\mathcal{G}, \mathcal{G}]$ *and let* $x_0 = \{K\}$ *be the origin of* M, *where* $\mathcal{G}' = \mathcal{S} + \mathcal{P}$ *is a reductive decomposition of* $\mathcal{G}'$. *If* G *is weakly semisimple, then there are* $r = \dim S$ *orthogonal homogeneous geodesics passing through* x_0.

Proof : Consider the base space G/K of $\Im = (G, \pi, G/K, K)$ and fiber space $\mathcal{G}/\mathcal{K}$ of $\xi = (G \times_K \mathcal{K}, \rho_\xi, \mathcal{G}/\mathcal{K}, \mathcal{G}/\mathcal{K})$. Since $[\mathcal{G}, \mathcal{G}] \subseteq \mathcal{G}$ and $\mathcal{G}$ is weak by semisimple, then $W_{rad}\mathcal{G} = 0$. By Definition 3.4

$$W_{rad}B = \{X \in [\mathcal{G}, \mathcal{G}], B(X, Y) = 0, \quad \text{for all} \quad Y \in \mathcal{G}\}.$$

This implies $W_{rad}B \subseteq rad\,B$, and the equality holds when $\mathcal{G}$ is semisimple ($rad\,B = 0$). If $\mathcal{G}$ is not semisimple, there are elements from $rad\,B$ that belong to neither $W_{rad}B$ nor $[\mathcal{G}, \mathcal{G}]$.

Since G is connected, G' is a normal Lie subgroup of G and its Lie algebra is $\mathcal{G}' = [\mathcal{G}, \mathcal{G}]$. In the reductive decomposition of $\mathcal{G}'$ with respect to the restriction of Killing form on $\mathcal{G}'$ we set $\mathcal{G}' = \mathcal{S} + \mathcal{P}$, where S is a subspace of $\mathcal{M}$ with dimension r, while $\mathcal{P}$ is the Lie algebra of the closed subgroup P of G' such that $P = G' \cap K$.

Now if $v_l \in S - (W_{rad}B)$ then $B(v_l, v_l) \neq 0$ and by Theorem 3.5 v_l is a geodesic vector. Since $\mathcal{G}$ is weak semisimple $W_{rad}\mathcal{G} = 0$ and so $W_{rad}B = 0$. Therefore $v_1, v_2, \cdots, v_r$ are independent geodesic vectors passing through the origin x_0.

Using Gram-Schmidt method we can get r orthogonal geodesics passing through x_0. This completes the proof of Theorem 3.6.

Acknowledgement. The authors would like to express their appreciation to professor O. Kowalski for his invaluable suggestions.

Резюме. В статье рассматриваются главные пучки и приводятся некоторые результаты о структуре геодезических в основном пространстве главных пучков.

REFERENCES

1. V. V. Gorbatsevich, A. L. Onishchik and E. B. Vinberg, Foundation of Lie Theory and Lie Transformation Groups, Springer-Verlag, 1997.
2. W. Greub, S. Halperin, and R. Vanstone, Connections, Curvature and Cohomology, Academic Press, vol. I, 1973.
3. J. E. Humphreys, Introduction to Lie Algebras and Representation Theory, Springer-Verlag, 1972.
4. A. Knapp, Representation Theory of Semisimple Groups, Princeton Univ. Press, 1986.
5. S. Koboyashi and K.Nomizu, Foundation of Differential Geometry I, II, New York, 1963, 1969.
6. O. Kowalski and J. Szenthe, "On the existence of homogeneous geodesics in homogeneous Riemannian manifolds", Geometriae Dedicata, vol. 84, pp. 331 – 332, 2001.
7. O. Kowalski and L. Vanhecke, "Riemannian manifolds with homogeneous geodesics", Bull. Un. Math. Ital., vol. 5, pp. 184 – 246, 1991.
8. A. L. Onishchik and E. B. Vinberger, Lie Groups and Algebraic Groups, Springer-Verlag, 1990.

Поступила 22 октября 2003