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# ZERO-$\sqrt{3}$ AND ZERO-2 LAWS FOR REPRESENTATIONS OF LOCALLY COMPACT ABELIAN GROUPS

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**Abstract.** A well-known "zero-2 law" states that if $(T(t))_{t\in\mathbb{R}}$ is a strongly continuous one-parameter group of bounded operators on a Banach space $X$, and if $\limsup_{t\to0^+}\|I-T(t)\|<2$, then $\lim_{t\to0^+}\|I-T(t)\|=0$. We discuss here analogous problems for unitary representations $\theta$ of a general topological group $\mathcal{U}$ on a unitary Banach algebra $A$. Let 1 be the unit element of $\mathcal{U}$, and let $I$ be the unit element of $A$. Elementary geometric considerations show that the situation with the spectral radius $\rho(I-\theta(u))$ as $u\to1$ is quite simple, since there are only four possibilities : $\lim_{u\to1}\rho(I-\theta(u))=0$, $\limsup_{u\to1}\rho(I-\theta(u))=\sin(\frac{n\pi}{2n+1})\geq\sqrt{3}$ for some $n\geq1$, $\limsup_{u\to1}\rho(I-\theta(u))=2$ and $\limsup_{u\to1}\rho(I-\theta(u))=+\infty$. If the group $\mathcal{U}$ admits "continuous division by 2," the second case is impossible and a "zero-2 law" holds for $\limsup_{u\to1}\rho(I-\theta(u))$. Another phenomenon holds for unitary representations of an Abelian locally compact group $(H,+)$ on a Banach algebra $A$. Using a classical result of Gelfand, which is equivalent to the fact that points are sets of synthesis for the algebra of absolutely convergent Fourier series, it is possible to show that if $\lim_{h\to0}\rho(I-\theta(h))=0$, then either $\lim_{h\to0}\|I-\theta(h)\|=0$, i.e. the representation is continuous with respect to the norm of $A$, or $\limsup_{h\to0}\|I-\theta(h)\|=+\infty$. So if we consider any unitary representation of $(H,+)$, then either $\lim_{h\to0}\|I-\theta(h)\|=0$, or $\limsup_{h\to0}\|I-\theta(h)\|\geq\sqrt{3}$. If a locally compact Abelian group admits continuous division by 2, then either $\lim_{h\to0}\|I-\theta(h)\|=0$, or $\limsup_{h\to0}\|I-\theta(h)\|\geq2$. If we restrict attention to representations which are bounded on some neighborhood of 0, we obtain a more precise result : either the representation is continuous with respect to the norm of $A$, or

$$\limsup_{h\to0}\|p(\theta(u))\|\geq\limsup_{h\to0}\rho(p(\theta(u))=\max_{|z|=1}|p(z)|$$

for every polynomial $p$.

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## §1. INTRODUCTION

A well-known "zero-2 law", see [14], p. 60, shows that if $(T(t))_{t\in\mathbb{R}}$ is a strongly continuous one-parameter group of bounded operators on a Banach space $X$, and if $\limsup_{t\to 0^+}\|I-T(t)\|<2$, then $\lim_{t\to 0^+}\|I-T(t)\|=0$, which means that the infinitesimal generator $A$ of the semigroup is bounded, so that the map $z\longmapsto e^{zA}$ defines a holomorphic extension of the semigroup on the complex plane. This is a consequence of a theorem of Kato [9] which shows that if a $C_0$-semigroup $(T(t))_{t>0}$ of bounded operators satisfies $\limsup_{t\to 0^+}\|I-T(t)\|<2$, then the semigroup admits a holomorphic semigroup extension to some angular sector

$$\Omega_\theta=\{z\in\mathbf{C}\setminus\{0\}\mid |arg(z)|<\theta\}.$$

Analogous results for exponentially bounded weakly measurable semigroups have been obtained by Beurling [1].

Another approach, due to Neuberger [12] and Pazy [13], consists in showing that if a $C_0$- semigroup $(T(t))_{t>0}$ of bounded operators satisfies $\limsup_{t\to 0^+}|I-T(t)|<2$, or the weaker condition $|I-T(t)|\le 2-t\log(\frac{1}{t})u(t)$ for $t$ sufficiently small, where $\lim_{t\to 0^+}u(t)=+\infty$, then $AT(t)$ is bounded for every $t>0$, which implies that $A$ is bounded if $T(t)$ is invertible for some, or equivalently all, $t>0$.
In a recent paper [4], the author observed that if $(T(t))_{t>0}$ is any one-parameter group in a unitary Banach algebra $A$, and if $\rho(x)$ denotes the spectral radius of $x\in A$, then there are only three possibilities : either $\lim_{t\to 0^+}\rho(I-T(t))=0$, or $\limsup_{t\to 0^+}\rho(I-T(t))=2$, or $\limsup_{t\to 0^+}\rho(I-T(t))=+\infty$, which leads to an elementary proof of the "zero-2 law" for strongly continuous groups of bounded operators. For general one-parameter groups bounded near 0 a standard renorming argument which goes back do Feller [7] gives a smaller equivalent norm on the closed subalgebra generated by the semigroup for which $\lim_{t\to 0}\|T(t)\|=1$ (in which case $\limsup_{t\to 0}\|I-T(t)\|=\limsup_{t\to 0^+}\|I-T(t)\|$), and a very elementary computation, based on the identity

$$(I+T(h))^2-(I-T(h))^2=4T(h)$$

shows that either $\lim_{t\to 0^+}\|I-T(t)\|=0$, or $\limsup_{t\to 0^+}\|I-T(t)\|\ge\sqrt{3}$. (A similar "zero-$\sqrt{3}$ law holds for $\liminf_{t\to 0^+}\Delta(t)$, where $\Delta(t)=\limsup_{h\to 0^+}\|T(t+h)-T(t)\|$, and the elementary tricks used in [4] were suggested to the author by Baxter's proof [2] of the inequality $\Delta(s+t)\le\Delta(s)\Delta(t)$, stated in [2] for $C_0$-semigroups but in fact valid for arbitrary semigroups in a Banach algebra). So it is a natural question to ask whether a stronger "zero-2 law" holds for arbitrary one-parameter groups. For groups of the form $(T(s))_{s\in S}$, where $S$ is a dense additive subgroup of $\mathbb{R}$, the "zero-$\sqrt{3}$" law holds, and a nice observation of Borichev [3], see Example 2.4 below, shows that $\sqrt{3}$ turns out to be optimal. However we will see, that the "zero-2 law" holds for all semigroups $(T(t))_{t\in\mathbb{R}}$. In fact, we completely clarify here the situation for representations of locally compact Abelian groups. There are two distinct phenomena. The first phenomenon concerns unitary representations $\theta = \mathcal{U} \longmapsto A$ of a topological group $\mathcal{U}$ on a Banach algebra $A$, i.e. maps $\theta = \mathcal{U} \longmapsto A$ such that $\theta(1) = I$, where 1 denotes the unit element of $\mathcal{U}$ and $I$ the unit element of $A$, and such that $\theta(uv) = \theta(u)\theta(v)$ for $u, v \in \mathcal{U}$. We assume that they are locally spectrally bounded, in the sense that there exists a neighborhood $U$ of 1 and $M > 0$ satisfying $\rho(\theta(u)) \leq M$ for every $u \in U$. We introduce in Section 2 the set $\Gamma(\theta)$ is equal to $\{\lambda \in \mathbf{C} \mid \liminf_{u\to 1} \text{dist}(\lambda, \text{spec}(\theta(u)) = 0\}$. Elementary observations show that either $\Gamma(\theta) =$ the unit circle $\mathbf{T}$, or there exists a finite family $p_1, ..., p_k$ of positive integers such that $\Gamma(\theta) = \cup_{1\leq j\leq k}\Gamma_{p_j}$, where $\Gamma_{p_j} := \{z \in \mathbf{C} \mid z^{p_j} = 1\}$. The case $p_1 = 1, k = 1$ gives $\Gamma(\theta) = 1$. In the case $\Gamma(\theta) \neq \mathbf{T}$ we obtain a finite union of vertices of regular polygons containing 1 and contained in the unit circle $\mathbf{T}$. This shows that we have the following possibilities :

1) $\lim_{u\to 1}\rho(I - \theta(u)) = 0$,
2) $\limsup_{u\to 1}\rho(I - \theta(u)) = 2\sin(\frac{n\pi}{2n+1}) \geq \sqrt{3}$ for some $n \geq 1$,
3) $\limsup_{u\to 1}\rho(I - \theta(u)) = 2$,
4) $\limsup_{u\to 1}\rho(I - \theta(u)) = +\infty$.

For representations of some compact Abelian groups and of some dense additive subgroups of $\mathbb{R}$ the case 2 can occur for any $n \geq 1$. But if $\mathcal{U}$ admits "continuous division by 2" (which means the possibility to define square roots in a reasonable way near the unit element), then case 2 cannot occur and the "zero-2 law" holds for $\limsup_{u\to 1} \rho(I - \theta(u))$.

The second phenomenon, completely described below in the case of locally compact Abelian groups, concerns the behavior of $\limsup_{u\to 0}\|(I - \theta(u))\|$ for representations $\theta$ of Abelian groups satisfying condition 1), i.e, in additive notation, $\lim_{u\to 0}\rho(I - \theta(u)) = 0$. In this case we show in Section 3 that we have a "zero-$\infty$ law" : either $\lim_{u\to 0}\|I - \theta(u)\| = 0$, or $\limsup_{u\to 0}\|(I - \theta(u))\| = +\infty$. This result is based on a general structure theorem for locally compact Abelian groups and on a classical theorem of Gelfand [8], improved later by Hille [10] : if an invertible element $a$ in a Banach algebra $A$ satisfies $\text{spec}(a) = \{1\}$ and $\sup_{n\in Z} \|a^n\| < +\infty$, (or, more generally, $\|a^n\| = o(n)$ as $|n| \to \infty$), then $a$ is the unit element of $A$. This well-known result is not elementary : Gelfand's original version of this statement is equivalent to the fact that functions of zero exponential type which are bounded on the real line are constant, and to the fact that points are sets of synthesis for the algebra of absolutely convergent Fourier series.

Other results concerning the behavior of a one-parameter semigroup $(T(t))_{t>0}$ in a Banach algebra $A$ which imply the existence of an element $P \in A$ satisfying $\lim_{t\to 0^+} \|P - T(t)\| = 0$ appeared recently. It was shown in [6] that such an element $P$ exists if

$$\limsup_{t\to 0^+} \|T(t) - T((n+1)t)\| < \frac{n}{(n+1)^{1+\frac{1}{n}}}$$

for some $n \geq 1$, and more general results of this type involving the behavior of $\|T(s) - T(t)\|$ near the origin are obtained in [4] in the case of strongly continuous semigroups.

## §2. BEHAVIOR OF THE SPECTRAL RADIUS OF A GROUP REPRESENTATION

**Definition 2.1.** Let $(\mathcal{U}, \cdot)$ be a topological group. We will say that $\mathcal{U}$ admits continuous division by 2 if there exists an open subset $U$ of $\mathcal{U}$ containing the unit element 1 and a map $\phi : U \to \mathcal{U}$, which is continuous at 1 and satisfies $\phi(1) = 1$ and $\phi^2(u) = u$ for every $u \in U$.

Of course if $\mathcal{U}$ is an additive Abelian group the conditions $\phi(1) = 1, \phi^2(u) = u$ are to be replaced by the conditions $\phi(0) = 0$, $2\phi(u) = u$.

**Proposition 2.2.** *Let $\mathcal{U}$ be a topological group, and let $\theta : \mathcal{U} \to A$ be a locally spectrally bounded unitary representation of $\mathcal{U}$ on a Banach algebra. Set*

$$\Gamma(\theta) = \{\lambda \in \mathbf{C} \mid \liminf_{u\to 1} dist(\lambda, spec(\theta(u)) = 0\}.$$

*Then either $\Gamma(\theta) = \mathbf{T}$, or there exists a finite family $p_1, ..., p_k$ of positive integers such that $\Gamma(\theta) = \cup_{1\leq j\leq k} \Gamma_{p_j}$, where $\Gamma_{p_j} := \{z \in \ 1\}$. If, further, $\mathcal{U}$ admits continuous division by 2, then either $\Gamma_\theta = \{1\}$ or $\Gamma_\theta = \mathbf{T}$.*

*Also lim $\sup_{u\to 1} \rho(p(\theta(u)) = \max_{z\in\Gamma(\theta)} |p(z)|$ for every polynomial $p$, and for every open subset $\Omega$ of $\mathbf{C}$ containing $\Gamma(\theta)$ there exists a neighborhood $V$ of 1 such that $spec(\theta(u)) \subset \Omega$ for every $u \in V$.*

**Proof :** It follows from the definition of $\Gamma(\theta)$ that $\Gamma(\theta)$ is closed, and $\lambda^n \in \Gamma(\theta)$ for $\lambda \in \Gamma(\theta)$, $n \in \mathbf{Z}$. If $|\lambda| \neq 1$ for some $\lambda \in \Gamma(\theta)$, then taking if necessary $\lambda^{-1}$ instead of $\lambda$ we can assume that $|\lambda| > 1$. We obtain

$$\limsup_{u\to 0} \rho(I - \theta(u)) \geq |\lambda|^n \qquad \text{for} \qquad n \geq 1,$$

which contradicts the fact that $\theta$ is locally spectrally bounded. Hence $\Gamma(\theta) \subset \mathbf{T}$. If there exists $\lambda \in \Gamma(\theta)$ such that $\lambda^n \neq 1$ for $n \geq 1$, then the set $\{\lambda^n\}_{n\in\mathbf{Z}}$ is dense in $\mathbf{T}$ and $\Gamma(\theta) = \mathbf{T}$. Otherwise there exists $F \subset \mathbb{N}^*$ such that $\Gamma(\theta) = \cup_{n\in F}\{z \in \mathbf{C} \mid z^n = 1\}$, and $F$ is clearly finite if $\Gamma(\theta) \neq \mathbf{T}$.

Assuming that $\mathcal{U}$ admits a continuous division by 2, let $\lambda \in \Gamma_\theta$, and let $\mu_1$ and $\mu_2 = -\mu_1$ be the square roots of $\lambda \in \mathbf{C}$. There exists a net $(u_\tau)_{\tau\in\mathcal{T}}$ in $\mathcal{U}$ such that $\lim_\tau u_\tau = 1$ and $\lim_\tau \alpha_\tau = \lambda$ for some net $(\alpha_\tau)_{\tau\in\mathcal{T}}$ of complex numbers satisfying $\alpha_\tau \in \operatorname{spec}(\theta(u_\tau))$ for $\tau \in \mathcal{T}$. We can assume that $u_\tau \in U$ for $\tau \in \mathcal{T}$, where $U$ is an open subset of $\mathcal{U}$ on which there exists a function $\phi : u \longmapsto u^{\frac{1}{2}}$ satisfying the conditions of Definition 2.1. There exists $\beta_\tau \in \operatorname{spec}(\theta(u_\tau^{1/2}))$ such that $\beta_\tau^2 = \alpha_\tau$. We have $\lim_\tau u_\tau^{1/2} = 1$ and

$$\limsup_\tau \min(|\beta_\tau - \mu_1|, |\beta_\tau - \mu_2|) \leq \lim_\tau |\alpha_\tau - \lambda|^{1/2} = 0.$$

This shows that there exists $i \in \{1,2\}$ such that $\mu_i$ is the limit of a subnet of the net $(\beta_\tau)_{\tau\in\mathcal{T}}$. Hence $\mu_i \in \Gamma_\theta$ and the equation $z^2 = \lambda$ admits at least a solution in $\Gamma_\theta$ for every $\lambda \in \Gamma_\theta$. This leaves only the possibilities $\Gamma_\theta = \{1\}$ or $\Gamma_\theta = \mathbf{T}$.
Now let $p \in \mathbf{C}[x]$, and let $\lambda \in \Gamma(\theta)$ such that $|p(\lambda)| = \max_{z\in\Gamma(\theta)} |p(z)|$. Let $\epsilon > 0$ and let $\delta > 0$ be such that $|p(z)| > |p(\lambda)| - \epsilon$ for $|z - \lambda| < \eta$. It follows from the definition of $\Gamma(\theta)$ that for every neighborhood $U$ of 1 there exists $u \in U$ such that $\operatorname{Spec}(\theta(u))$ contains some $z \in \mathbf{C}$ such that $|z - \lambda| < \eta$, and so $\rho(p(\theta) > |p(\lambda)| - \epsilon$, and

$$\limsup_{u\to 1} \rho(p(\theta(u))) \geq \max_{z\in\Gamma(\theta)} |p(z)|.$$

Let $\Omega$ be an open subset of $\mathbf{C}$ containing $\Gamma(\theta)$. There exists a neighborhood $U_0$ of 1 such that $\rho(\theta(u)) \leq M$ for every $u \in U_0$, and we can assume that $M > 1$ and $\Omega \subset D(0, M) := \{z \in \mathbf{C} \mid |z| \leq M\}$. Set $K := D(0, M) \setminus \Omega$. A routine compactness argument shows that there exists a finite open covering $W_1, ..., W_k$ of $K$ and a family $U_1, ..., U_k$ of neighborhoods of 1 contained in $U_0$ such that $\operatorname{spec}(\theta(u)) \cap W_j = \emptyset$ for every $u \in U_j$. Set $U = \cap_{1\leq j\leq k} U_j$. Then $\operatorname{spec}(\theta(u)) \subset \Omega$ for every $u \in U$. This shows in particular that

$$\limsup_{u\to 1} \rho(p(\theta(u)) \leq \max_{z\in\Gamma(\theta)} |p(z)|,$$

and so $\limsup_{u\to 1} \rho(p(\theta(u)) = \max_{z\in\Gamma(\theta)} |p(z)|$ for every polynomial $p \in \mathbf{C}[x]$.

**Corollary 2.3.** Let $\mathcal{U}$ be a topological group, and let $\theta : \mathcal{U} \to A$ be a locally spectrally bounded unitary representation of $\mathcal{U}$ on a Banach algebra. Then either $\lim_{u\to 1} \rho(\theta(u) - I) = 0$, or $\limsup_{u\to 1} \rho(\theta(u) - I) = 2\sin(\frac{n\pi}{2n+1}) \geq \sqrt{3}$ for some $n \geq 1$, or $\limsup_{u\to 1} \rho(\theta(u) - I) = 2$. If further, $\mathcal{U}$ admits continuous division by 2, then either $\lim_{u\to 1} \rho(\theta(u) - I) = 0$, or $\limsup_{u\to 1} \rho(\theta(u) - I) = 2$.

**Proof :** Set $p = x - 1$. Then

$$\limsup_{u\to 1} \rho(\theta(u) - I) = \limsup_{u\to 1} \rho(p(\theta(u))) = \max_{\lambda\in\Gamma(\theta)} |\lambda - 1|.$$

If $\Gamma(\theta) = \{1\}$, we get $\lim_{u\to 1}\rho(\theta(u) - I) = 0$. If $\Gamma(\theta) = \mathbf{T}$, or if $\Gamma(\theta)$ contains $\Gamma_{2n}$ for some $n \geq 1$, we get $\limsup_{u\to 1} \rho(\theta(u) - I) = 2$. Otherwise there exists a strictly increasing finite sequence $(n_1, ..., n_p)$ of positive integers such that $\Gamma(\theta) = \cup_{1\leq j\leq p}\Gamma_{2n_j+1}$. By a standard calculation

$$\max_{\lambda\in\Gamma_{2n+1}} |\lambda - 1| = 2\sin\left(\frac{n\pi}{2n+1}\right)$$

for $n \geq 1$, and so in this situation we have

$$\limsup_{u\to 1} \rho(\theta(u) - I) = 2\sin\left(\frac{n_1\pi}{2n_1+1}\right) \geq 2\sin\left(\frac{\pi}{3}\right) = \sqrt{3}.$$

Now if $\mathcal{U}$ admits a continuous division by 2 let $\lambda \in \Gamma_\theta$. Then either $\Gamma_\theta = \{1\}$, in which case $\lim_{u\to 1}\rho(\theta(u) - I) = 0$, or $\Gamma_\theta = \mathbf{T}$, in which case $\limsup_{u\to 1}\rho(\theta(u) - I) = 2$.
It is easy to see that all the situations described above can occur with representations of dense subgroups of $\mathbb{R}$ in finite dimensional algebras. The morphisms $\theta : \mathbb{R} \to \mathbf{T}$ for which $\Gamma(\theta) = \mathbf{T}$ are the well-known non-measurable characters of $\mathbb{R}$, i.e. the non-measurable morphisms from $(\mathbb{R}, +)$ onto $(\mathbf{T}, \cdot)$ (see [11]).
We also have the following example, which is a minor modification of an example of Borichev [3].

**Example 2.4.** Let $(a_\tau)_{\tau\in\mathbb{R}}$ be a Hamel basis of $\mathbb{R}$ viewed as a vector space over the field $\mathbf{Q}$ of rational numbers such that $a_\tau > 0$ for $\tau \in \mathbb{R}$ and $\inf_{\tau\in\mathbb{R}} a_\tau = 0$. Set $G = \oplus_{\tau\in\mathbb{R}}\mathbf{Z}a_\tau$.
Then for every strictly increasing family $(p_1, ..., p_k)$ of integers $\geq 2$ there exists a bounded representation $\theta : G \to \mathbf{C}^k$ such that $\Gamma(\theta) = \cup_{1\leq j\leq k}\Gamma_{p_j}$.
To see this consider a strictly increasing family $(p_1, ..., p_k)$ of positive integers. For $1 \leq i \leq k$, we define $\theta_j : G \to \mathbf{C}$ by the formula

$$\theta_j\left(\sum_{\tau\in\mathbb{R}} n(\tau)a_\tau\right) = e^{2\left(\sum_{\tau\in\mathbb{R}} n(\tau)\right)\frac{i\pi}{p_j}}.$$

Let $\theta : G \to \mathbf{C}^k$ be the map $u \longmapsto (\theta_1(u), ..., \theta_k(u))$. Then $\theta_j(a_\tau) = e^{\frac{2i\pi}{p_j}}$ for $\tau \in \mathbb{R}$, and it follows immediately that $\Gamma(\theta) = \cup_{1\leq j\leq k}\Gamma_{p_j}$. In particular the additive group

$G$ constructed above admits for $n \geq 1$ a one- dimensional representation $\theta_{(n)}$ such that

$$\limsup_{u\to 0} \|I - \theta_{(n)}(u)\| = \limsup_{u\to 0} \rho(I - \theta_{(n)}(u)) = \sin(\frac{n\pi}{2n+1}).$$

We conclude this section by an example which shows that the set $\Gamma(\theta)$ can take all the forms described in Proposition 2.2 for bounded representations of compact Abelian groups.

**Example 2.5.** For a strictly increasing sequence of positive integers $p_1, ..., p_k$ with $p_1 \geq 2$, equip $G_j = \Gamma_{p_j}^{\mathbb{N}}$ with the product topology, and set $G = G_1 \times ... \times G_k$. Then the compact group $G$ admits a representation $\theta$ on $\mathbf{C}^k$ for which $\Gamma(\theta) = \cup_{1\leq j\leq k}\Gamma_{p_j}$. To see this pick a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and set $\chi_j(u_j) = \lim_{\mathcal{U}} u_{j,n}$ for $u_j = (u_{j,n})_{n\geq 0}$ in $G_j$, and $\theta(u) = (\chi_1(u_1), ..., \chi_k(u_k))$ for $u = (u_1, ..., u_k) \in G$. Looking at sequences $u_j^{(m)} = (u_{j,n}^{(m)})_{n\geq 0} \in G_j$ which are constant for $n > m$ and satisfy $u_{j,n}^{(m)} = 1$ for $n \leq m$, we see immediately that the bounded representation $\theta$ of $G$ satisfies $\Gamma(\theta) = \cup_{1\leq j\leq k}\Gamma_{p_j}$. Also for $1 \leq j \leq k$ the group $G$ admits a one-dimensional representation $\theta_j$ for which

$$\limsup_{u\to 0} \|I - \theta_j(u)\| = \limsup_{u\to 0} \rho(I - \theta_j(u)) = \sin(\frac{p_j\pi}{2p_j+1}).$$

## §3. THE ZERO-$\sqrt{3}$ AND ZERO-2 LAWS FOR REPRESENTATIONS OF LOCALLY COMPACT ABELIAN GROUPS

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a discontinues additive map, and let $\chi = e^{\phi}$. Equip $A := \mathbf{C}^2$ with the norm $(u, v) \longmapsto |u| + |v|$ and product $(u_1, v_1)(u_2, v_2) = (u_1u_2, u_1v_2 + v_1u_2)$. We obtain a two dimensional Banach algebra and $I = (1, 0)$ is the unit element of $A$. Now set $\theta(x) = (1, \chi(x))$ for $x \in \mathbb{R}$. This gives a representation of $(\mathbb{R}, +)$ on $A$ such that $\rho((I - \theta(x)) = 0$ for $x \in \mathbb{R}$, while $\limsup_{x\to 0^+} \|\theta(x)\| = +\infty$. We will see that this phenomenon disappears for representations $\theta$ of locally compact Abelian groups $(G, +)$ which are locally bounded in the sense that there exists a neighborhood $U$ of 0 in $G$ and $M > 0$ such that $\|\theta(g)\| \leq M$ for every $g \in U$. For such representations the condition $\lim_{g\to 0} \rho(I - \theta(g)) = 0$ implies $\lim_{g\to 0} \|I - \theta(g)\| = 0$.

We begin with a preliminary observation. A well-known theorem of Gelfand [8] shows that if an invertible element of a unitary Banach algebra $A$ satisfies $\sup_{n\in\mathbf{Z}} \|a^n\| < +\infty$, and if $\mathrm{spec}(a) = \{1\}$, then $a$ is the unit element of $A$. As indicated in the Introduction, this result is equivalent to the fact that points are sets of synthesis for the algebra of absolutely convergent Fourier series, and also is equivalent to the fact that an entire function of zero exponential type which is bounded on the real line is constant.

Let $A$ be a Banach algebra with unit element $I$, and let $a \in A$ be such that $\rho(I-A) < 1$. We can define $\log(a)$ by the usual holomorphic functional calculus :

$$\log(a) = \log(1+a-1) = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n}(a-1)^n.$$

It follows from the standard properties of the holomorphic functional calculus that $e^{\log(a)} = a$ and that $\chi(\log(a)) = \log(\chi(a))$ and $\chi(a) = e^{\chi(\log(a))}$ for every character $\chi$ on $A$. Also if two commuting elements $a, b \in A$ satisfy $\rho(a-1) < 1$, $\rho(b-1) < 1$, $\rho(ab-1) < 1$, then we have $\log(ab) = \log(a) + \log(b)$. For a proof of this standard fact set $x = \log(ab), y = \log(a) + \log(b)$. We have $0 = e^x - e^y = (x-y)F(x,y)$, where $F$ is the entire function defined on $\mathbf{C}^2$ by the formula $F(z_1, z_2) = \frac{e^{z_1} - e^{z_2}}{z_1 - z_2}$ if $z_1 \neq z_2$, $F(z,z) = e^z$ for $z \in \mathbf{C}$. Hence $F(z_1, z_2) \neq 0$ unless there exists $k \in \mathbf{Z}$ such that $z_1 - z_2 = 2ik\pi$. Now if $\chi$ is a character on $A$, we have

$$\Im(\chi(x)) \in (-\frac{\pi}{2}, \frac{\pi}{2}), \qquad \Im(\chi(y)) \in (-\pi, \pi), \qquad |\chi(x) - \chi(y)| < \frac{3\pi}{2},$$

and so $F(x,y)$ is invertible and $x = y$.

**Lemma 3.1.** *Let $t = (t_1, ..., t_k) \longmapsto T(t)$ be a unitary representation of $\mathbb{R}^k$ on a Banach algebra $A$. If $\lim_{t\to 0} \rho(T(t) - I) = 0$, then either $\limsup_{t\to 0} \|(T(t)-I)\| = +\infty$, or the representation is continuous, so that $\lim_{t\to 0}\|(T(t)-I)\| = 0$.*

**Proof :** Set $|t| = max_{1\leq j\leq k}|t_j|$ for $t \in \mathbb{R}^k$. There exists $\delta > 0$ such that $\rho(I - T(t)) < 1$ for $|t| \leq \delta$. In this situation we can set as above $U(t) = \log(T(t))$ for $|t| \leq \delta$, and we have $U(s+t) = U(s) + U(t)$ if $|s| \leq \frac{\delta}{2}$, $|t| \leq \frac{\delta}{2}$. Also if $n \in \mathbb{N}$, we have $(U(2^n t)) = 2^n U(t)$ for $|t| \leq 2^{-n}\delta$. Let $t \in \mathbb{R}^k$ and let $n, m \in \mathbb{N}$ satisfying $2^n\delta > |t|$, $2^m\delta > |t|$. Then $2^n U(2^{-n}t) = 2^{m+n}U(2^{-m-n}t) = 2^{-m}U(2^{-m}t)$. So we can set $U(t) = 2^n U(2^{-n}t)$ for $t \in \mathbb{R}^k$, where $n \geq 0$ is any integer such that $2^n\delta \geq |t|$, and the map $t \longmapsto T(t)$ is well defined on $\mathbb{R}^k$. Let $s, t \in \mathbb{R}^k$, and let $n \in \mathbb{N}$ satisfying $2^{n-1}\delta \geq \max(|s|, |t|)$. Then $U(s+t) = 2^n U(2^{-n}(s+t)) = 2^{-n}U(s) + 2^{-n}U(t) = U(s) + U(t)$.

We can assume that $A$ is commutative. Let $\chi$ be a character on $A$. The map $t \longmapsto \chi(U(t)) = \log(\chi(T(t))$ is continuous for $|t| \leq \delta$; being additive on $\mathbb{R}^k$, it is continuous on $\mathbb{R}^k$. Set $e_j = (\delta_{j,n})_{1\leq n\leq k}$ for $1 \leq j \leq k$, where $\delta_{j,n}$ denotes the usual Kronecker symbol. Then $\chi(U(t_1, ..., t_n)) = \sum_{j=1}^k t_j \chi(U(e_j))$ for $t = (t_1, ..., t_k) = \mathbf{Q}^k$. By continuity, this equality holds for every $t = (t_1, ..., t_k) \in \mathbb{R}^k$.

Now set $V(t) = \sum_{j=1}^k t_j U(e_j)$ for $t = (t_1, ..., t_k) \in \mathbb{R}^k$ and $R(t) = T(t)e^{-V(t)} = e^{W(t)}$, where $W(t) = U(t) - V(t)$. We have $\chi(W(t)) = 0$ for every character $\chi$ on

$A$, and so spec$(R(t)) = \{1\}$ for $t \in \mathbb{R}^k$. Assume that $\sup_{t\in U} \|T(t)\| < +\infty$ for some neighborhood $U$ of the origin. Then $M := \sup_{|t|\le 1} \|T(t)\| < +\infty$. By construction, $R(e_j) = I$ for $1 \le j \le k$, and so $R(m) = I$ for every $m = (m_1, ..., m_k) \in \mathbf{Z}^k$. This shows that $\|R(t)\| \le M$ for $t \in \mathbb{R}^k$. Now let $t \in \mathbb{R}^k$. We have

$$\sup_{n\in\mathbf{Z}} \|R(t)^n\| = \sup_{n\in\mathbf{Z}} \|R(nt)\| \le M.$$

Since spec$(R(t)) = \{1\}$, it follows from Gelfand's theorem that $R(t) = I$. Hence $T(t) = e^{V(t)}$ for $t \in \mathbb{R}^k$, which completes the proof of the lemma.
A similar phenomenon holds for representations of compact Abelian groups.

**Lemma 3.2.** *Let $\theta$ be a unitary representation of a compact Abelian group $(G, +)$ on a Banach algebra $A$. If $\lim_{g\to 0}\rho(\theta(g) - I) = 0$, then either*

$$\limsup_{g\to 0} \|(\theta(g) - I)\| = +\infty,$$

*or the representation is continuous, so that $\lim_{g\to 0}\|(\theta(g) - I)\| = 0$.*

**Proof :** We can assume that $A$ is commutative and that span$\{\theta(g)\}_{g\in G}$ is dense in $A$. Let $\chi$ be a character on $A$. The map $g \longmapsto \chi(\theta(g))$ is continuous on $G$, and so $\chi(\theta(G))$ is a compact multiplicative subgroup of $\mathbf{C}\setminus\{0\}$. Hence $\chi(\theta(G)) \subset \mathbf{T}$. This shows that $\chi \circ \theta$ is a continuous character of the group $G$. It follows from the definition of the topologies of the character space $\hat{A}$ and of the dual group $\hat{G}$ that $\tilde{\theta} : \chi \longmapsto \chi\circ\theta$ is a continuous map from $\hat{A}$ into $\hat{G}$.
Hence $\tilde{\theta}(\hat{A})$ is a compact subset of the discrete group $\hat{G}$, which shows that $\tilde{\theta}(\hat{A})$ is finite. Also $\tilde{\theta}$ is one-to-one since span$\{\theta(g)\}_{g\in G}$ is dense in $A$, and $\hat{A}$ is finite. Let $\chi_1, ..., \chi_k$ be the elements of $\hat{A}$. A standard application of Shilov's idempotent theorem shows that there exists idempotents $e_1, ..., e_k$ in A such that $I = e_1 + ... + e_k$, $e_i e_j = 0$ for $i \ne j$ and $\chi_i(e_j) = \delta_{i,j}$ for $1 \le i \le k$, $1 \le j \le k$. Set $\psi(g) = \sum_{j=1}^k \chi_j(g)e_j$. Then $\psi : G \to A$ is a continuous representation of $G$ on $A$, and $\chi(\psi(g)) = \chi(\theta(g))$ for every $\chi \in \hat{A}$ and every $g \in G$.
Assume that $\theta$ is locally bounded. Clearly, $\theta\circ\psi^{-1}$ is a locally bounded representation of $G$ on $A$. Since $G$ is compact, $\theta\circ\psi^{-1}$ is in fact bounded, and there exists $M > 0$ such that $\theta\circ\psi^{-1}(g) \le M$ for every $u \in G$. Since spec$((\theta\circ\psi^{-1})(g)) = \{1\}$, it follows again from Gelfand's theorem that $(\theta\circ\psi^{-1})(g) = I$ for every $g \in G$. Hence $\theta = \psi$ and $\theta$ is continuous.

**Theorem 3.3.** *Let $H$ be a locally compact Abelian group, and let $\theta$ be a unitary representation of $H$ on a Banach algebra. If $\lim_{h\to 0}\rho(I - \theta(h)) = 0$, then either*

$$\lim_{h\to 0} \|I - \theta(h)\| = 0 \quad \text{or} \quad \limsup_{h\to 0} \|I - \theta(h)\| = +\infty.$$

**Proof :** It follows from a standard result of the theory of locally compact Abelian groups (see, for example, [15]) that $H$ possesses an open (hence also closed) subgroup $H_1$ which is isomorphic to a direct product $\mathbb{R}^k \times G$, where $G$ is a compact Abelian group. So we can assume that $H = R^k \times G$. Set $\theta_1(t) = \theta(t,0)$ for $t \in \mathbb{R}^k$, and set $\theta_2(g) = \theta(0,g)$ for $g \in G$. If $\limsup_{h\to 0}\|I - \theta(h)\| < +\infty$, then

$$\limsup_{t\to 0} \|I - \theta_1(t)\| < +\infty, \qquad \limsup_{g\to 0} \|I - \theta_2(g)\| < +\infty,$$

and it follows from Lemma 2.1 and Lemma 2.2 that

$$\lim_{h\to 0} \|I - \theta(h)\| = \lim_{(t,g)\to(0,0)} \|I - \theta_1(t)\theta_2(g)\| = 0.$$

**Corollary 3.4.** Let $H$ be a locally compact Abelian group, and let $\theta$ be a unitary representation of $H$ on a Banach algebra. Then either

$$\lim_{h\to 0} \|I - \theta(h)\| = 0 \quad \text{or} \quad \limsup_{h\to 0} \|I - \theta(h)\| \geq \sqrt{3}.$$

If further, $H$ admits continuous division by 2, then either $\lim_{h\to 0}\|I - \theta(h)\| = 0$, or $\limsup_{h\to 0}\|I - \theta(h)\| \geq 2$.

The proof follows immediately from Corollary 1.3 and Theorem 2.3.

In fact we have obtained a more precise result for locally bounded representations of a locally compact Abelian group $(H,+)$ which admits continuous division by 2 : if $\theta$ is a locally bounded unitary representation of $H$ into a Banach algebra, then either $\lim_{h\to 0}\|I - T(h)\| = 0$, which means that the representation is continuous with respect to the norm of $A$, or

$$\limsup_{h\to 0} \|p(T(h))\| \geq \limsup_{h\to 0} \rho(p(T(h)) = \max_{|z|=1} |p(z)|$$

for every polynomial $p$.

**Резюме.** Хорошо известный "закон 2-нуля" утверждает, что если $(T(t))_{t\in\mathbb{R}}$ строго непрерывная однопараметрическая группа ограниченных операторов в банаховом пространстве $X$, и если $\limsup_{t\to 0^+} \|I - T(t)\| < 2$, то $\lim_{t\to 0^+} \|I - T(t)\| = 0$. Мы обсуждаем здесь аналогичные задачи для унитарных представлений $\theta$ общей топологической группы $\mathcal{U}$ унитарной банаховой алгебры $A$.. Пусть 1 – единичный элемент группы $\mathcal{U}$, и пусть $I$ – единичный элемент группы $A$. Элементарные геометрические рассуждения показывают, что ситуация с спектральным радиусом $\rho(I - \theta(u))$ при $u \to 1$ является достаточно простой,

поскольку существуют только четыре возможности : $\lim_{u\to 1} \rho(I - \theta(u)) = 0$, $\limsup_{u\to 1} \rho(I - \theta(u)) = \sin(\frac{n\pi}{2n+1}) \geq \sqrt{3}$ для некоторого $n \geq 1$, $\limsup_{u\to 1} \rho(I - \theta(u)) = 2$ и $\limsup_{u\to 1} \rho(I - \theta(u)) = +\infty$. Если группа $\mathcal{U}$ допускает "непрерывное деление на 2," второй случай не возможен и "закон 2-нуля" выполняется для $\limsup_{u\to 1} \rho(I - \theta(u))$. Другой феномен имеет место для унитарных представлений абелевых локально компактных групп $(H, +)$ на банаховой алгебре $A$. Используя классический результат Гельфанда, эквивалентный факту, что точки являются множествами синтеза для алгебры абсолютно сходящихся рядов Фурье, в работе показано, что если $\lim_{h\to 0} \rho(I - \theta(h)) = 0$, то $\lim_{h\to 0} \|I - \theta(h)\| = 0$, т.е. представление непрерывно относительно нормы $A$, или $\limsup_{h\to 0} \|I - \theta(h)\| = +\infty$. Итак, если рассмотреть любое унитарное представление $(H, +)$, то $\lim_{h\to 0} \|I - \theta(h)\| = 0$, или $\limsup_{h\to 0} \|I - \theta(h)\| \geq \sqrt{3}$. Если локально компактная абелева группа допускает непрерывное деление на 2, то $\lim_{h\to 0} \|I - \theta(h)\| = 0$, или $\limsup_{h\to 0} \|I - \theta(h)\| \geq 2$. Если рассмотреть представления, ограниченные в некоторой окрестности точки 0, то получаем более точный результат : либо представление непрерывно относительно нормы алгебры $A$, либо $\limsup_{h\to 0} \|p(\theta(u))\| \geq \limsup_{h\to 0} \rho(p(\theta(u)) = \max_{|z|=1} |p(z)|$ для любого многочлена $p$.

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