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ON X–DECOMPOSABLE FINITE GROUPS

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Abstract. Let G be a finite group, A be a normal subgroup of G and $ncc(A)$ = the number of G–conjugacy classes of A, A is called n–decomposable, if $ncc(A) = n$. One can speak about decomposability of a group with respect to a non-empty subset X of positive integers. The present paper reports on the problem of finding all finite subsets X of positive integers, such that there is a X-decomposable finite group.

§1. INTRODUCTION

Let G be a finite group and let $\mathcal{N}_G$ be the set of normal subgroups of G. An element K of $\mathcal{N}_G$ is said to be **n-decomposable**, if K is a union of $n = ncc(K)$ distinct conjugacy classes of G. Let $\mathcal{K}_G = \{ncc(A) : A \in \mathcal{N}_G\}$ and X be a non-empty subset of positive integers. A group G is called X -decomposable, if $\mathcal{K}_G = X$.

In [19] Wujie Shi defined the notion of a complete normal subgroup of a finite group, which we call 2-decomposable. He proved that if G is a group and N a complete normal subgroup of G, then N is a minimal normal subgroup of G and it is an elementary Abelian p-group. Moreover, $N \subseteq Z(O_p(G))$, where $O_p(G)$ is a maximal normal p-subgroup of G, and $|N|(|N| - 1)$ divides $|G|$ and in particular, $|G|$ is even. Next, Wang Jing, a student of Wujie Shi, defined the notion of a sub-complete normal subgroup of a group G [20], which we call 3-decomposable. She proved that if N is a sub-complete normal subgroup of a finite group G, then N is a group in which every element has prime power order. Throughout this paper, as usual, G' denotes the derived subgroup of G, Z_n denotes the cyclic group of order n, $E(p^n)$ denotes an elementary Abelian p–group of order p^n, for a prime p, and $Z(G)$ is the center of G.

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A group G is called non-perfect, if $G' \neq G$. Also, $D(n)$ denotes the set of positive divisors of n and $SmallGroup(n,i)$ is the i-th group of order n in the small group library of GAP, [16]. All groups considered are assumed to be finite. Our notation is standard and taken mainly from [8], [9] and [14].

§2. MAIN RESULTS

In this section we first present some examples of X-decomposable finite groups and consider open questions. Then we consider the structure of non-perfect X-decomposable finite groups, for some finite subset X of positive integers. We first begin with the finite Abelian groups.

Lemma 1. *Let G be a finite Abelian group. Set $X = D(n)$, where $n = |G|$. Then G is $X-$decomposable.*

Proof follows from the fundamental theorem of finite Abelian groups.

The following lemmas consider the normal subgroups of some non-Abelian finite groups.

Lemma 2. *Let p be a prime, $q|p-1$ and $F_{p,q}$ be a group presented by :*

$$F_{p,q} = \langle a, b : a^p = b^q = 1,\ b^{-1}ab = a^u\rangle,$$

where u is an element of order q in Z_p^*, the multiplicative group of integers modulo p. Then the group $F_{p,q}$ is $\left\{1, 1+\frac{p-1}{q}, q+\frac{p-1}{q}\right\}-$decomposable.

Proof. It is not hard to show that, up to isomorphism, $F_{p,q}$ does not depend on which integer u of order q we choose and that $F_{p,q}$ is a Frobenius group. Let S be the subgroup of Z_p^* consisting of the powers of u, so $|S| = q$. Write $r = \frac{p-1}{q}$, and choose coset representatives $v_1, \cdots, v_r$ in Z_p^*. Then by [10], Proposition 25.9, the conjugacy classes of $F_{p,q}$ are

$$\{1\}, \qquad (a^{v_i})^{F_{p,q}} = \{a^{v_i s} : s \in S\}\ (1 \le i \le r),$$
$$(b^n)^{F_{p,q}} = \{a^m b^n : 0 \le m \le p-1\}\ (1 \le n \le q-1).$$

Suppose H is the unique Sylow $p-$subgroup of G. Then the mentioned information on conjugacy classes of $F_{p,q}$ show that H is $\left(1+\frac{p-1}{q}\right)$-decomposable. Finally, we can see that H is the unique non-trivial proper normal subgroup of $F_{p,q}$. Therefore, $F_{p,q}$ is $\left\{1, 1+\frac{p-1}{q}, q+\frac{p-1}{q}\right\}$ $-$decomposable. Lemma 2 is proved.

Lemma 3. *Every non-Abelian $p-$group of order p^3 is $\{1, p, 2p-1, p^2+p+1\}$-decomposable.*

Proof. It is a well-known fact that every non-Abelian $p-$group of order p^3 has p^2+p-1 conjugacy classes. Since the length of every conjugacy class of G is p, G is $\{1, p, 2p-1, p^2+p+1\}$-decomposable.

Lemma 4. *Suppose*

$$X=\left\{\frac{d+1}{2}:d|n\right\}\cup\left\{\frac{n+3}{2}\right\},\quad A=Y\cup\left\{\frac{n}{4}+2,\frac{n}{2}\right\},\quad B=Y\cup\left\{\frac{n}{2}+1,\frac{n}{2}\right\},$$

$$Y=\left\{\frac{d+2}{2}:d|n,\ 2|d\right\}\cup\left\{\frac{d+1}{2}:d|n,\ 2\not|d\right\}\cup\{n+3\},$$

and $n\geq 3$ is a positive integer. If n is odd, then the dihedral group D_{2n} of order $2n$ is X−decomposable. Moreover, if $4|n$, then D_{2n} is A−decomposable and if $4\not|n$ then D_{2n} is B−decomposable.

Proof. This group can be presented by $D_{2n}=\langle a,b:\ a^n=b^2=1,\ b^{-1}ab=a^{-1}\rangle$. We first assume that n is odd. In this case every proper normal subgroup of D_{2n} is contained in $\langle a\rangle$ and so D_{2n} is X−decomposable. Next we assume that n is even. In this case, we can see that D_{2n} has exactly two other normal subgroups $H=\langle a^2,b\rangle$ and $K=\langle a^2,ab\rangle$. To complete the example, we must compute $ncc(H)=ncc(K)$. If $4|n$, then $ncc(H)=\frac{n}{4}+2$ and so it is enough to consider the case that $4\not|n$. In this case, we can see that $ncc(H)=\frac{n+6}{4}$. So, if $4|n$, then D_{2n} is A−decomposable and if $4\not|n$, then the dihedral group D_{2n} is B−decomposable. Lemma 4 is proved.

Lemma 5. *Suppose*

$$X=\left\{\frac{d+1}{2}:d|n,\ 2\not|d\right\}\cup\left\{\frac{d+2}{2}:d|2n,\ 2|d\right\}\cup\{n+3\},\quad Y=X\cup\left\{\frac{n+4}{2},n+3\right\}$$

and $n\geq 2$ is a positive integer. If n is odd, then the generalized quaternion group Q_{4n} of order $8n$ is X−decomposable, and if n is even, then Q_{4n} is Y−decomposable.

Proof. The generalized quaternion group Q_{4n}, $n\geq 2$, can be presented by

$$Q_{4n}=\langle a,b:\ a^{2n}=1,\ b^2=a^n,\ b^{-1}ab=a^{-1}\rangle.$$

It is a well-known fact that Q_{4n} has $n+3$ conjugacy classes, as follows :

$$\{1\},\quad \{a^n\},\quad \{a^r,a^{-r}\},\quad 1\leq r\leq n-1,$$

$$\{a^{2j}b:0\leq j\leq n-1\},\quad \{a^{2j+1}b:0\leq j\leq n-1\}.$$

We consider two separate cases of n odd or even. If n is odd then every normal subgroup of Q_{4n} is contained in the cyclic subgroup $\langle a\rangle$. Thus, in this case Q_{4n} is X−decomposable. If n is even, we have two other normal subgroups $\langle a^2,b\rangle$ and $\langle a^2,ab\rangle$ which are both $\frac{n+4}{2}$−decomposable. Therefore, Q_{4n} is Y−decomposable. Lemma 5 is proved. Now it is natural to generally ask about the set $\mathcal{K}_G=\{ncc(A):\ A\trianglelefteq G\}$. In fact we have the following open question.

Question 1. Suppose X is a finite subset of positive integers containing 1. Does an $X-$decomposable finite group G exist ?.

We continue our classification of $\{1, n, m\}-$decomposable, $1 < n < m$, finite group and first look at the solvable case.

Theorem 1. *Let G be a $\{1, n, m\}-$decomposable finite group, $1 < n < m$. If G is solvable then G' is Abelian, $\mathcal{K}_G = \{1, G', G\}$, $G' \cong E(p^r)$ is maximal in G, and G is a Frobenius group with kernel G' and its complement is a cyclic group of prime order q with $p^r - 1 = (n-1)q$. Moreover, if G is non-solvable and non-perfect finite group with $n \leq 8$, then $n \geq 5$ and G' is simple.*

Proof. By Lemma 1, every finite Abelian group of order n is $D(n)-$ decomposable. Hence without loss of generality, we can assume that G is non-Abelian. Suppose G' is not Abelian. Then $1 < G'' < G' < G$ and so $|Ncc(G)| \geq 4$, i.e. a contradiction. Thus G' is Abelian. If N is a proper non-trivial normal subgroup of G different from G' then we must have $G = NG'$ and $N \cap G' = 1$. Therefore, $G \cong N \times G'$ and G is Abelian. But this is impossible. So $\mathcal{N}_G = \{G'\}$.

Since G' is a maximal subgroup of G, $|G : G'| = q$ with q prime. Since G' is a minimal normal subgroup of G, G' is an elementary Abelian subgroup of order, say p^r. Thus, $|G| = p^r q$. Since G is non-Abelian, $q \neq p$ and $C_G(x) = G'$, for any $1 \neq x \in G'$. Therefore, by [11], Theorem 1.2, p. 1136, G is a Frobenius group with kernel G'. Since G' is Abelian, by [11], Theorem 5.1, p. 1160, $n-1 = \frac{|G'|-1}{q}$. Therefore, $p^r - 1 = (n-1)q$, as desired. If G is non-solvable and non-perfect, then by [2], Lemma 2 and [4], Lemma 2.1 $n \geq 5$ and G' is simple. Theorem 1 is proved.

Lemma 6. *Let G be a $n-$decomposable non-solvable non-perfect finite group and $|\mathcal{N}_G| \geq 2$. Then $|\mathcal{N}_G| = 2$, n is a prime number and $G \cong Z_n \times B$, where B is a non-Abelian simple group with exactly n conjugacy classes.*

Proof. Let A and B be elements of $\mathcal{N}_G$. Then by [1], Theorem 2, $G \cong A \times B$. It is easy to see that A and B are simple groups. By [14], p. 88, A and B are the only proper non-trivial normal subgroups of G. So $|\mathcal{N}_G| = 2$. If A and B are non-Abelian simple groups then $G' = G$, i.e. a contradiction. Therefore, one of A or B, say A, is Abelian. Since A is simple, n is a prime number and $A \cong Z_n$, proving the lemma.

Theorem 2. *Let G be a non-perfect non-solvable $\{1, n, m\}-$decomposable finite group. Then G is isomorphic to $Z_5 \times A_5$, S_5, S_6, $A_6.2_2$, $A_6.2_3$, Aut $(PSL(2, q))$, for $q = 7, 8$. Here, $A_6.2_2$ and $A_6.2_3$ are non-isomorphic split extensions of the alternating group A_6, in the small group library of GAP [17].*

Proof. If $n = 2$ then according to [17], Theorem 2.1, G' is the unique non-trivial proper normal subgroup of G and is elementary Abelian. This implies that G is solvable, a contradiction. Suppose $n = 3$ and H is a non-trivial proper normal subgroup of G. As H is 3-decomposable, it follows from [18] that H is solvable, but this is impossible. Thus $n \geq 4$ and the theorem follows from [1], Theorem 5, [2], Theorems 5 and 6, and [4], Theorems 2.5 and 2.6. Theorem 2 is proved.

Lemma 7. *Suppose G is a X-decomposable finite group with $X = \{1, 2, ..., n\}$. Then $n \leq 3$ and $G \cong 1, Z_2$ or S_3.*

Proof. If G is Abelian then by fundamental theorem of finite Abelian groups, $|G| \leq 2$. Suppose G is non-Abelian. Then by [12], Proposition 2.1, G is a Frobenius group with kernel N of odd order $\frac{|G|}{2}$. On the other hand, by [12], Theorem 2.2 if $G \ncong S_3$ then G has a normal or Abelian 2-complement M. Suppose M is normal. Then $G \cong Z_2 \times M$. This imlies that $n \neq 4$. Suppose $n \geq 5$ and H is a $(n-2)$-decomposable subgroup of G. Then H is a normal subgroup of M. Since M is $(n-1)$-decomposable, by [12], Proposition 2.1 $|M|$ must be even, but this is impossible. Lemma 7 is proved.

Theorem 3. *Suppose G is a non-perfect $\{1, 2, 3, n\}$-, $\{1, 3, 4, m\}$- or $\{1, 2, 4, m\}$-decomposable finite group, for $n > 3$ and $m > 4$. Then in the first case $n = 5, 6, 7$ and G is isomorphic to Z_6, D_8, Q_8, S_4, SmallGroup$(20, 3)$ or SmallGroup$(24, 3)$. In the second case, G is isomorphic to Small Group$(36, 9)$, a metAbelian group of order $2^n(2^{\frac{n-1}{2}} - 1)$, in which n is odd positive integer and $2^{\frac{n-1}{2}} - 1$ is a Mersenne prime or a metAbelian group of order $2^n(2^{\frac{n}{3}} - 1)$, where $3|n$ and $2^{\frac{n}{3}} - 1$ is a Mersenne prime. Finally, in the third case, G is isomorphic to an Abelian group of order 8, SmallGroup$(12, 1)$, SmallGroup$(24, 13)$, SmallGroup$(168, 43)$, a metAbelian group of order 2^{n+1} $(2^n - 1)$ or a metAbelian group of order $2^{2n}(2^n - 1)$, where $2^n - 1$ is a Mersenne prime.*

Proof. The proof follows from the main results of [5] - [7] and some elementary calculation with GAP, [16]. We end this paper with the following question :

Question 2 : Is there any classification of perfect X-decomposable finite groups ?

Резюме. Пусть G – конечная группа, а A – нормальная подгруппа группы G и $ncc(A)$ = число G-сопряжённых классов A. Подгруппа A называется n-представимой, если $ncc(A) = n$. Можно говорить о представимости группы относительно непустого подмножества X положительных целых чисел. В настоящей статье решается задача нахождения всех конечных подмножеств X положительных целых чисел таких, что существует X-представимая конечная группа.

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