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APPROXIMATION BY OVERCONVERGENT POWER SERIES

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Abstract. The paper gives a short survey of results on approximation by overconvergent power series and discusses the question of possible lacunas of power series representing the universal functions.

INTRODUCTION

For a compact set K in the complex plane $\mathbb{C}$ we denote by $A(K)$ the set of all complex valued functions, that are continuous on K and holomorphic in its interior K°. The family of all compact sets with connected complement will be denoted by $\mathcal{M}$. By $H(\mathcal{O})$ we denote the family of all functions that are holomorphic on $\mathcal{O}$.

Let $f(z) = \sum\limits_{\nu=0}^{\infty} a_\nu (z - z_0)^\nu$ be a power series with radius of convergence $R > 0$ and with partial sums $s_n(z) = \sum\limits_{\nu=0}^{n} a_\nu (z - z_0)^\nu$. Recall that such a series is said to be overconvergent, if there exists a subsequence $\{s_{n_k}(z)\}$ of $\{s_n(z)\}$ which converges, while $\{s_n(z)\}$ itself diverges.

If a sequence of functions $\{f_n\}$ converges to a function f uniformly on a set S, then we write

$$f_n(z) \underset{S}{\Longrightarrow} f(z) .$$

If $\{f_n\}$ converges compactly to f on an open set S (i.e. uniformly on all compact subsets of S), then we write

$$f_n(z) \underset{S}{\to} f(z) .$$

In 1921 Ostrowski started a thorough investigation of the phenomen of overconvergence. The main results in this direction are summarized in Peyerimhoff [10] (see, also, [2]). In 1970, Luh [4] and independently in 1971, Chui and Parnes [1] proved the existence of a holomorphic function in the unit disc $\mathbf{D} = \{z : |z| < 1\}$ possessing a universal approximation property with respect to overconvergence.

Theorem 1 (see [1]). *There exists a holomorphic function φ in $\mathbf{D}$ with the following property : For every compact set $B \subset \mathbf{C} \setminus \overline{\mathbf{D}}$ with connected complement and every function f, continuous on B and holomorphic in its interior, there exists a sequence $\{n_k\}$ such that $\{s_{n_k}(z)\}$ converges to $f(z)$ uniformly on B.*

This was the starting point for several investigations on universal functions. In this paper we give a short survey of the results in this direction, and discuss the question of possible lacunas of power series representing the universal functions.

§1. PRELIMINARIES

In 1986, Luh [5] proved the existence of a holomorphic function φ on an open set $\mathcal{O}$ with simply connected components, where the function φ has several universal approximation properties with respect to overconvergence.

Theorem 2 (see [5]). *Let $\mathcal{O} \subset \mathbf{C}$ be an open set with simply connected components. Suppose $\zeta \in \mathcal{O}$ and denote by $\{s_n^{(\zeta)}(z)\}$ the sequence of partial sums of the power series expansion of φ around ζ.*

There exists a sequence $\{p_n\}$ of natural numbers such that for all $\zeta \in \mathcal{O}$ the following properties hold :

(1) $\{s_{p_n}^{(\zeta)}(z)\}$ converges to $\varphi(z)$ compactly on $\mathcal{O}$;

(2) for any compact set $B \subset \overline{\mathcal{O}}^c$ with connected complement and any function $f \in A(B)$ there exists a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ such that $\{s_{p_{n_k}}^{(\zeta)}(z)\} \overrightarrow{\overrightarrow{\scriptstyle B}} f(z)$;

(3) for any open set $U \subset \overline{\mathcal{O}}^c$ with simply connected components and any function $g \in H(U)$ there exists a subsequence $\{p'_{n_k}\}$ of $\{p_n\}$ such that $\{s_{p'_{n_k}}^{(\zeta)}(z)\} \underset{U}{\rightarrow} f(z)$;

(4) for any measurable set $E \subset \mathbf{C} \setminus \overline{\mathbf{D}}$ and any measurable function h on E there exists a subsequence $\{p''_{n_k}\}$ of $\{p_n\}$ such that $\{s_{p''_n}^{(\zeta)}(z)\} \overrightarrow{\overrightarrow{\scriptstyle B}} f(z)$.

It is clear, that there is no function φ with the above properties, if $\mathcal{O}$ has a non-simply connected component. It is also easy to see that the topological assumptions imposed on the sets B, U and E and the analytical assumptions imposed on the functions f, g and h respectively are best possible and cannot be weakened.

§2. CONSTRUCTION OF UNIVERSAL HOLOMORPHIC FUNCTION WITH LACUNARY POWER SERIES REPRESENTATION

To prove the existence of a holomorphic function φ with lacunary power series, possessing universal approximation properties with respect to overconvergence, we will need the following lemma.

Lemma 1 (see [6]). *Let K be a compact set in $\mathcal{M}$ with $0 \in K^\circ$ and suppose that K_0 (= the component of K containing 0) is starlike with respect to 0. Let Q be a subsequence of $\mathbb{N}_0$ with upper density $\overline{d}(Q) = 1$, where $\overline{d}(Q) = \overline{\lim\limits_{n\to\infty}} \frac{\nu_Q(n)}{n}$ and $\nu_Q(n)$ is the number of $m \in Q$ with $m \leq n$. Assume that $f \in H(K)$ admits (near the origin) a power series representation*

$$f(z) = \sum_{n=0}^{\infty} f_n z^n \quad \text{with } f_n = 0 \text{ for } n \notin Q.$$

Then for every $\varepsilon > 0$ there exists a polynomial P of the form

$$P(z) = \sum_{n=0}^{\infty} p_n z^n \quad \text{with } p_n = 0 \text{ for } n \notin Q,$$

such that

$$\max_K |f(z) - P(z)| < \varepsilon.$$

Now we are able to prove the following theorem.

Theorem 3. *Let $R > 1$ and Q be a subsequence of $\mathbb{N}_0$ with upper density $\overline{d}(Q) = 1$. Then there exist a function $\varphi \in H(\mathbb{D})$ with*

$$\varphi(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu \quad \text{with } a_\nu = 0 \text{ for } \nu \notin Q$$

and a sequence $\{p_n\}$ of natural numbers, such that for $s_n(z) = \sum\limits_{\nu=0}^{n} a_\nu z^\nu$ the following properties hold :

(1) For every point z_0 ($|z_0| > R$) we have $s_{p_n}(z_0) \to \infty$ as $n \to \infty$.

(2) For every compact set $K \subset \{z : 1 \leq |z| \leq R\}$ with connected complement and every function $f \in A(K)$ there exists a subsequence $\{n_k\}$ satisfying

$$s_{p_{n_k}}(z) \underset{K}{\Longrightarrow} f(z)$$

Proof. 1. Preconsiderations. Following Nestoridis [9], we choose a sequence of compact sets $\{K_n^{**}\}$ in $\mathbf{D}^c$ with $K_n^{**} \in \mathcal{M}$ and define $K_n^* = K_n^{**} \cap \{z : |z| \leq R\}$. For every $n \in \mathbf{N}$, K_n^* is a compact set in $\mathbf{D}^c$ with connected complement, possessing the property that for every non-empty compact set $K \in \mathcal{M}$ with $K \subset \mathbf{D}^c$ there exists an integer $N = N(K)$ such that $K \subset K_N^*$.

Let $\{Q_n^*\}$ be an enumeration of all polynomials with coefficients, whose real and imaginary parts are rational. Any $n \in \mathbf{N}$ has a unique representation of the type

$$n = \binom{m}{2} + j \quad \text{with } m \in \mathbf{N},\ 1 \leq j \leq m.$$

We define

$$K_n = K_{\binom{m}{2}+j} = K_j^*, \quad Q_n = Q_{\binom{m}{2}+j} = Q_{m-j+1}^*,$$

and consider the sequence of pairs $\{(K_n, Q_n)\}$, where in (K_n, Q_n) any combination (K_ν^*, Q_μ^*) appears infinitely often.

2. Construction of the sequence of polynomials $\{P_n(z)\}$. Let $p_0 \in Q$; we define $P_0(z) = z^{p_0}$, and suppose that for $n \in \mathbf{N}$ the polynomials $P_0(z), \ldots, P_{n-1}(z)$ are already determined. Each of these polynomials contains only z^ν with $\nu \in Q$. Denote by p_{n-1} the degree of the polynomial P_{n-1} and choose $q_{n-1} \in Q$ such that $q_{n-1} > n \cdot p_{n-1}$.

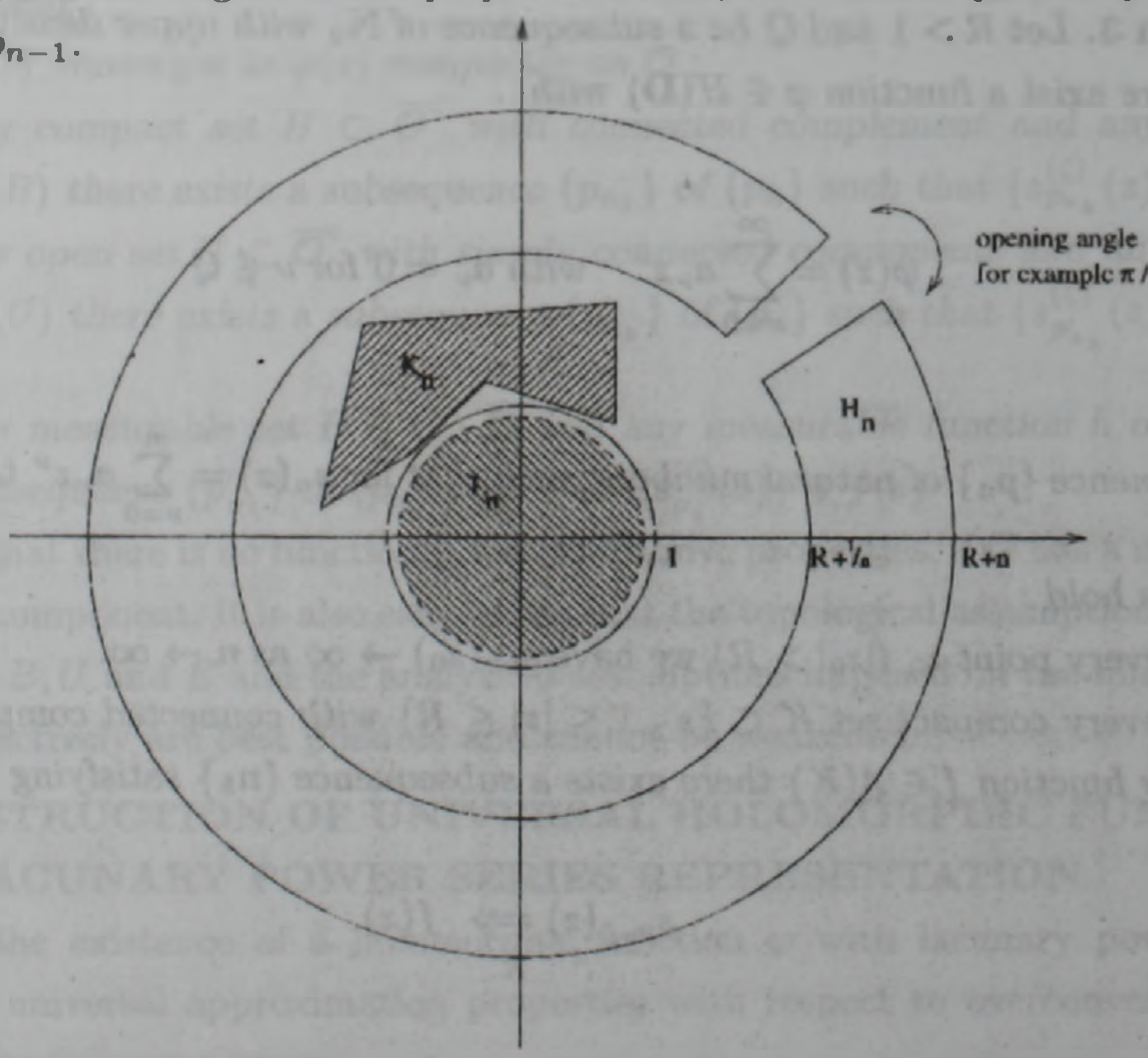

Fig. 1.

Now we consider the sets L_n, K_n and H_n (see Fig. 1), where $L_n = \{z : |z| \leq 1 - \frac{1}{2n}\}$, H_n is as above and K_n was already defined. Then $L_n \cup K_n \cup H_n \in \mathcal{M}$. According to Lemma 1 there exists a polynomial

$$P_n(z) = \sum_{\substack{\nu \in Q \\ \nu \geq q_{n-1}}} a_\nu z^\nu$$

with the following properties :

$$\max_{L_n} |P_n(z)| < \frac{1}{n^2}, \tag{2.1}$$

$$\max_{K_n} \left| P_n(z) - \left\{ Q_n(z) - \sum_{\nu=0}^{n-1} P_\nu(z) \right\} \right| < \frac{1}{n}, \tag{2.2}$$

$$\max_{H_n} \left| P_n(z) - \left\{ (n+1) - \sum_{\nu=0}^{n-1} P_\nu(z) \right\} \right| < 1. \tag{2.3}$$

We define $\varphi(z) = \sum\limits_{\nu=0}^{\infty} P_\nu(z)$.

3. Properties of $\varphi(z)$. By (2.1) the series $\sum\limits_{\nu=0}^{\infty} P_\nu(z)$ converges compactly on $\mathbf{D}$. Thus, φ is at least holomorphic on the unit disc $\mathbf{D}$.

Consider the polynomials P_ν, which contain only the following powers :

$$P_n(z) : z^{q_{n-1}}, ..., z^{p_n}; \quad P_{n+1}(z) : z^{q_n}, ..., z^{p_{n+1}}.$$

Because of $q_n > (n+1)p_n$, we have no overlapping. Hence the power series $\sum\limits_{\nu=0}^{\infty} a_\nu z^\nu$ is obtained by formal arranging the series $\sum\limits_{\nu=0}^{\infty} P_\nu(z)$ by ascending powers of z. This power series contains only the terms z^ν with $\nu \in Q$. In particular, for the partial sums we have

$$\sum_{\nu=0}^{n} P_\nu(z) = \sum_{\nu=0}^{p_n} a_\nu z^\nu = s_{p_n}(z).$$

Furthermore $a_\nu = 0$ for $p_n < \nu < q_n$, this means that the power series $\varphi(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ has Ostrowski gaps $\{p_n, q_n\}$ with $\frac{q_n}{p_n} \geq n+1 \to \infty$.

4. Let K be a compact set such that $K \subset \{1 \leq |z| \leq R\}$ and $K \in \mathcal{M}$. Given a function $f \in A(K)$, by Mergelian's theorem there exist a number $N \in \mathbf{N}$ with $K \subset K_N^*$ and a sequence $\{m_k\}$ with $m_k \geq k$ satisfying

$$\max_K |f(z) - Q^*_{m_k}(z)| < \frac{1}{k}. \tag{2.4}$$

Setting

$$n_k = \binom{m_k + N - 1}{2} + N,$$

we obtain $K_{n_k} = K_N^*$, $Q_{n_k} = Q^*_{m_k}$ and $K \subset K_{n_k}$. From (2.2) we have

$$\max_{K_{n_k}} \left| \sum_{\nu=0}^{n_k} P_\nu(z) - Q_{n_k}(z) \right| < \frac{1}{n_k}.$$

Hence

$$\max_{K_{n_k}} \left| \sum_{\nu=0}^{p_{n_k}} a_\nu z^\nu - Q_{n_k}(z) \right| < \frac{1}{n_k}.$$

This implies

$$\max_{K_{n_k}} \left| s_{p_{n_k}}(z) - Q^*_{m_k}(z) \right| < \frac{1}{n_k}. \tag{2.5}$$

From (2.4) and (2.5)

$$\max_K |s_{p_{n_k}}(z) - f(z)| \leq \max_K |s_{p_{n_k}}(z) - Q^*_{m_k}(z)| + \max_K |Q^*_{m_k}(z) - f(z)| < \frac{1}{n_k} + \frac{1}{k}.$$

5. The function φ is holomorphic exactly in $\mathbf{D}$. Suppose the opposite, i.e. that φ admits analytical continuation outside $\mathbf{D}$. Then by Ostrowski's overconvergence theorem $\{s_{p_n}(z)\}$ would converge compactly to $\varphi(z)$ in a domain bigger than the unit disc $\mathbf{D}$. This is not possible because of the step 4 of the proof.

6. Divergence outside the unit disc **D**. It follows from (2.3) that

$$\max_{H_n}\left|\sum_{\nu=0}^{n} P_\nu(z)-(n+1)\right|<1.$$

Hence

$$\max_{H_n}|s_{p_n}(z)-(n+1)|<1.$$

Let z_0 be a fixed point satisfying $|z_0|>R$. Then there exists a number N_0 such that $z_0\in H_n$ for all $n\geq N_0$. For these n we have

$$|s_{p_n}(z_0)-(n+1)|\leq\max_{H_n}|s_{p_n}(z)-(n+1)|<1.$$

This implies

$$|s_{p_n}(z_0)|=|s_{p_n}(z_0)-(n+1)+(n+1)|>n.$$

Therefore

$$|s_{p_n}(z)|\to\infty \quad \text{for all } z \quad \text{with } |z|>R.$$

This completes the proof of Theorem 3.

Remark 1. Using Lemma 1 we can achieve an analoguous result for $R=\infty$: Let Q be a subsequence of $\mathbf{N}_0$ with upper density $\overline{d}(Q)=1$, and let G be a simply connected starlike domain such that $\mathbf{D}\subset G$ and $\overline{\mathbf{D}}\not\subset G$. Then there exists a function φ, holomorphic exactly in G with the following properties :

(1) $\varphi(z)=\sum_{\nu\in Q}a_\nu z^\nu$ with radius of convergence equal to 1 ;

(2) there exists a subsequence $\{p_n\}$ such that $s_{p_n}(z)\overrightarrow{_G}\varphi(z)$;

(3) for every compact set $K\subset G^c$ with connected complement and every function $f\in A(K)$ there exists a subsequence $\{n_k\}$ satisfying $s_{p_{n_k}}(z)\underset{K}{\rightrightarrows}f(z)$.

§3. A GENERALIZATION OF MENSHOV'S THEOREM

Theorem 4. Let the power series of $\varphi\in H(\mathbf{D})$ have the universal approximation properties according to Theorem 3, u,v be real-valued Lebesgue-measurable functions on $\partial\mathbf{D}$. Then there exists a subsequence $\{n_k\}$ of the natural numbers, such that

$$Re\ s_{n_k}(z)\to u(z) \quad \textit{almost everywhere on } \partial\mathbf{D}$$

and

$$Im\ s_{n_k}(z) \to v(z) \quad almost\ everywhere\ on\ \partial\mathbf{D}.$$

Proof. Let u, v be real-valued, measurable functions on $\partial\mathbf{D}$. By Lusin's theorem (see, e.g., [11], [12]) for all $k \in \mathbf{N}$ there exist continuous, real-valued functions u_k, v_k on $\partial\mathbf{D}$ and a compact set $E \subset \partial\mathbf{D}$ satisfying $E \neq \partial\mathbf{D}$ and $\mu(E) = \mu(\partial\mathbf{D}) = 2\pi$ such that for all $z \in E$

$$u_k(z) \to u(z), \quad v_k(z) \to v(z), \quad \text{as } k \to \infty.$$

We define $h_k(z) = u_k(z) + iv_k(z)$ for $z \in E$. Then h_k is a continuous function on E. Consider the sets

$$E_k = E \setminus \left\{ z = e^{i\omega} : |\omega| < \frac{1}{4k},\ k \in \mathbf{N} \right\}.$$

It is clear that for all $k \in \mathbf{N}$ the set E_k is compact and $E_k \in \mathcal{M}$. Hence by Theorem 3 there exists a number $n_k \in \mathbf{N}$ satisfying

$$\max_{E_k} |s_{n_k}(z) - h_k(z)| < \frac{1}{k}.$$

Let $z_0 \in E$ be an arbitrary point with $z_0 \neq 1$. There exists a number k_0 such that z_0 is contained in E_k for all $k \geq k_0$, so we obtain

$$s_{n_k}(z_0) \to u(z_0) + iv(z_0) \quad \text{for } k \to \infty\ .$$

This implies

$$s_{n_k}(z) \to u(z) + iv(z) \quad \text{almost everywhere on } \partial\mathbf{D}.$$

Theorem 4 is proved.

If we consider the power series $\varphi(z) = \sum_{\nu \in Q} a_\nu z^\nu$ as above and set $z = e^{it}$ $(t \in \mathbf{R})$, then we obtain

$$\sum_{\nu \in Q} a_\nu e^{i\nu t} = \sum_{\nu \in Q} a_\nu (\cos \nu t + i \sin \nu t),$$

that is, a (formal) trigonometric series with gaps. So, we can say that Theorem 4 is in some sense a generalization of the Menšov's well known theorem [8] on the existence of universal trigonometric series.

§4. SOME REMARKS ON THE DENSITY

We have $\overline{d}(Q) \leq d_{\max}(Q)$, where $d_{\max}(Q)$ is the maximum density (in the sense of Pólya) of a subsequence Q of $\mathbf{N}_0$ (see [3]) :

$$d_{\max}(Q) = \lim_{\theta \to 1-} \left(\overline{\lim_{n \to \infty}} \frac{\nu_Q(n) - \nu_Q(\theta n)}{(1-\theta)n} \right).$$

The following example shows that we cannot require $d_{\max}(Q) < 1$ in Theorem 3 : Let $R > 1$ and Q be a subsequence of $\mathbf{N}_0$ with $d_{\max}(Q) < 1$. Suppose that there exists a function $\varphi \in H(\mathbf{D})$ with universal approximation properties according to Theorem 3.

Let K^* be the closed circular arc of $|z| = r$ with $1 < r < R$ and length $2\pi R d_{\max}$. Let G be a simply connected domain in $\{z : \ 1 < |z| < R\}$ which contains K^*. Further, we denote by K the compact set with connected complement in $\{z : \ 1 < |z| < R\}$ satisfying $G \supset K^\circ \supset K^*$ and $K \neq K^*$. By Theorem 3 there exists a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ such that

$$s_{p_{n_k}}(z) \underset{K}{\Longrightarrow} 0.$$

Therefore (see [7], p. 29)

$$s_{p_{n_k}}(z) \xrightarrow[\mathbf{D}_{r+\varepsilon}]{} 0 \quad \text{for some } \varepsilon > 0.$$

In particular $\varphi(z) \equiv 0$ for all $z \in \mathbf{D}$. We came to a contradiction.

Резюме. Статья даёт краткий обзор результатов касающихся аппроксимации по сверхсходящимся степенным рядам и обсуждает вопрос возможной лакунарности степенных рядов, представляющих универсальные функции.

REFERENCES

1. C. K. Chui, M. N. Parnes, "Approximation by overconvergence of power series", Journ. Math. Analysis and Applications vol. 36, pp. 693 – 696, 1971.
2. E. Hille, Analytic Function Theory, II, Chelsea, New York, 1987.
3. P. Koosis, The Logarithmic Integral, vol. I+II, Cambridge Univ. Press, Cambridge, 1988.
4. W. Luh, "Approximation analytischer Funktionen durch überkonvergente Potenzreihen und deren Matrix-Transformierten", Mitt. Math. Sem. Giessen, vol. 88, 1970.
5. W. Luh, "Universal approximation properties of overconvergent power series on open sets", Analysis vol. 6, pp. 191 – 207, 1986.

6. W. Luh, V. A. Martirosian, J. Müller, "Restricted T-universal functions", J. Approxim. Theory, vol. 114, pp. 201 – 213, 2002.
7. J. Müller, "Über analytische Fortsetzung mit Matrixverfahren", Mitt. Math. Sem. Giessen, vol. 199, 1990.
8. D. E. Menshov, "On the partial sums of trigonometric series" [in Russian], Math. Sb. (N.S.) vol. 20, pp. 197 – 238, 1947.
9. V. Nestoridis, "Universal Taylor series", Ann. Inst. Fourier, Grenoble, vol. 46, no. 5, 1293 – 1306, 1996.
10. A. Peyerimhoff, Lectures on Summability, Lecture Notes in Math., Springer Verlag, 1970.
11. W. Rudin, Real and Complex Analysis, 3rd ed., McGraw-Hill, New York, 1987.
12. K. R. Stromberg, An Introduction to Classical Real Analysis, Wadsworth Inc., Belmont, CA, 1981.

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