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CONSTRUCTION OF UNIVERSAL LAURENT AND FABER SERIES

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Abstract. For a finitely connected domain Ω the paper constructively proves the existence of a function f holomorphic on Ω, whose Laurent series partial sums possess universal approximation properties in Ω^c. Existence of functions holomorphic on a Jordan domain G, whose Faber series partial sums possess universal approximation properties in G^c is demonstrated. The class of functions with these properties is a G_δ-set and is dense in the space of all functions holomorphic on G.

§1. INTRODUCTION

The existence of a Taylor series on the unit disc with universal approximation properties of the partial sums has been shown independently by Luh [5] in 1970 and Chui and Parnes [2] in 1971. This result has been generalized by Luh [7] in 1986 for open sets with simply connected components. Let O be an open set with simply connected components. We denote by $s_n^{(\zeta)}(f)(z)$ the n^{th} partial sum of the Taylor expansion of a function f with center at ζ. In [7] Luh proved that there exist a function Φ holomorphic on O and a sequence of natural numbers $\{p_n\}_n$ such that

(1) The sequence $\{s_{p_n}^{(\zeta)}(\Phi)\}_n$ converges locally uniformly on O to Φ for each $\zeta \in O$.

(2) For each compact set $B \subset \overline{O}^c$ with connected complement and each function f continuous on B and holomorphic on its interior there exists a subsequence $\{p'_n\}_n$ of $\{p_n\}_n$, such that $\{s_{p'_n}^{(\zeta)}(\Phi)\}_n$ converges uniformly on B to f for each $\zeta \in O$.

In 1996 Nestoridis [12] obtained a similar result using Baire's theorem. He proved that the set of all holomorphic on the unit disc functions f, that satisfy the condition (A) below, is G_δ and is dense in the space of all functions holomorphic on the unit disc.

Condition (A) : Let $\sum\limits_{n=0}^{\infty} a_n z^n$ be the Taylor series of a function f. For every compact set $K \subset \{z : |z| \geq 1\}$ with connected complement and every function h continuous on K and holomorphic on its interior there exists a subsequence

$$S_{\tau_m}(z) = \sum_{n=0}^{\tau_m} a_n z^n, \quad m \in \mathcal{N},$$

that converges to h uniformly on K.

In 2002 Vlachou [16] proved that for $B \subset O^c$ the above classes of universal functions, defined by Luh and by Nestoridis, coincide. Nestoridis [12] pose the question, whether a construction of a universal Taylor series would be possible for multiply connected domains. Gehlen, Luh and Müller [4] gave a negative answer to this question in the case of a bounded multiply connected domain.

Although universal Taylor series have been constructed for more general domains (not necessarily simply connected, see, e.g., [10] and [15]), for multiply connected domains Laurent series appear to be more suitable. In this paper we give a constructive proof for the existence of universal Laurent series.

For universal Faber series Katsoprinakis, Nestoridis and Papadoperakis [6] reduced this problem via Faber mapping to the problem for universal Taylor series. However this method yields only the existence of universal Faber series for domains of bounded rotation. Our proof is constructive and can be applied to more general domains.

§2. UNIVERSAL LAURENT SERIES

2.1. Notation. Let Ω be a domain in the extended complex plane $\hat{\mathbb{C}}$ such that $\hat{\mathbb{C}} \setminus \Omega$ consists of finitely many components to be denoted $A_0, A_1, \ldots, A_k$ and assume that $\infty \in A_0$, and the points $a_j \in A_j$, $j = 1, \ldots, k$ are fixed. Then every function $f \in H(\Omega)$ (the space of all functions being holomorphic on Ω) possesses a unique decomposition $f = f_0 + f_1 + \ldots + f_k$, with $f_j \in H(A_j^c)$ $(j = 0, 1, \ldots, k)$ and $\lim_{z\to\infty} f_j(z) = 0$ $(j = 1, \ldots, k)$. Every f_j has a representation as a Laurent series

$$f_j(z) = \sum_{n=1}^{\infty} \frac{c_n(f_j, a_j)}{(z - a_j)^n}, \quad j = 1, \ldots, k,$$

where $c_n(f_j, a_j)$ are the Laurent coefficients of f_j with respect to a_j, if z is supposed to lie outside a large enough disc with center at a_j. Using this for $f \in H(\Omega)$ and $\zeta \in A_0^c$ we define the formal sums

$$M_N(f, \zeta)(z) = \sum_{n=0}^{N} \frac{f_0^{(n)}(\zeta)}{n!}(z - \zeta)^n + \sum_{n=1}^{N} \frac{c_n(f_1, a_1)}{(z - a_1)^n} + \ldots + \sum_{n=1}^{N} \frac{c_n(f_k, a_k)}{(z - a_k)^n}.$$

For later use we define an explicit metric on the space $H(\Omega)$ that describes locally uniform convergence. We choose an exhausting sequence $\{K_n\}_n$ of Ω consisting of compact sets, and for $f, g \in H(\Omega)$ we set

$$p_n(f) = \max_{z \in K_n} |f(z)|, \quad n \in \mathcal{N},$$

$$p(f) = \sum_{n=1}^{\infty} \frac{p_n(f)}{2^n(1 + p_n(f))}, \quad d(f,g) = p(f - g).$$

It is easy to check that $(H(\Omega), d)$ is a complete metric space with the desired convergence properties, that is, a sequence of functions $\{f_n\}_n$ in $H(\Omega)$ converges to some function $f \in H(\Omega)$ with respect to the metric d if and only if it does so in the sense of uniform convergence on compact subsets of Ω.

Finally, for a compact set K the space of functions that are continuous on K and holomorphic in the interior of K we denote by A(K).

2.2. Construction of a Universal Laurent Series and its Properties. Now we will prove a result which already appears in [8] (we use the technics of [8], but not the results).

Theorem 2.1. *Let $\Omega \subset \hat{\mathbf{C}}$ be a domain such that $\hat{\mathbf{C}} \setminus \Omega = A_0 \cup \ldots \cup A_k$, $\infty \in A_0$, and let some points $a_j \in A_j$ $(j = 1, \ldots, k)$ be fixed. Then there exist a function $f \in H(\Omega)$ and a sequence of natural numbers $\{t_n\}_n$ with the following properties :*

(1) For each compact $K \subset \Omega$ and each point $\zeta \in A_0^c$ the sequence $\{M_{t_n}(f, \zeta)\}_n$ converges locally uniformly on Ω to f.

(2) For any compact $K \subset \Omega^c \setminus \{a_1, \ldots, a_k\}$ with connected complement and any function $g \in A(K)$ there exists a subsequence $\{t_{n_s}\}_s$ of $\{t_n\}_n$ such that $\{M_{t_{n_s}}(f, \zeta)\}_s$ converges to g uniformly on K for any choice of $\zeta \in A_0^c$.

Proof. 1) Preparatory. For every $j = 0, \ldots, k$ we choose an exhausting sequence $\{L_n^{(j)}\}_n$ of the set A_j^c such that $L_n^{(j)}$ is closed for $j = 1, \ldots, k$, $n \in \mathcal{N}$ and $L_n^{(0)}$ is compact for $n \in \mathcal{N}$. Without loss of generality we can assume that each $L_n^{(j)}$ has connected complement, since every A_j^c $(j = 0, \ldots, k)$ is simply connected in $\hat{\mathbf{C}}$. Furthermore, we choose a sequence of compact sets $\{K_n^*\}_n$ with each $K_n^* \subset \Omega^c \setminus \{a_1, \ldots, a_k\}$ possessing connected complement, in such a way that for every compact set K with the same properties there exists $N \in \mathcal{N}$ such that $K \subset K_N^*$. The existence of such a sequence can be proved by application of the technics of [12], Lemma 2.1. We consider an enumeration $\{(K_n, p_n)\}_n$ of the set

$$\{(K, p) : K \in \{K_n^* : n \in \mathcal{N}\}, \quad p \text{ is a polynomial with coefficients in } \mathcal{Q} + i\mathcal{Q}\},$$

in which every combination of K_n^* and p_n appears infinitely many times. Finally, we will use the notation $K_n^{(j)} = K_n \cap A_j$ $(j = 0, \dots, k, n \in \mathcal{N})$.

2) Construction of the Universal Function. First we set

$$\Theta_0^{(0)}(z) = z - a_1, \qquad \Theta_0^{(j)}(z) = \frac{1}{z - a_j}, \quad j = 1, \dots, k, \quad \lambda_0 = 1.$$

We assume that for a number $n \in \mathcal{N}$ the functions $\Theta_0^{(m)}, \dots, \Theta_k^{(m)}$ and the numbers λ_m have been already determined for $m = 0, \dots, n-1$ in such a way that

(i) $\Theta_m^{(0)}$ is a polynomial,

(ii) $\Theta_m^{(j)}$ is a rational function with one pole at a_j for $j = 1, \dots, k$. Hence we have

$$\Theta_m^{(j)} = \frac{P_m^{(j)}(z)}{(z - a_j)^{q_m^{(j)}}}, \quad \deg\left(P_m^{(j)}\right) < q_m^{(j)}.$$

To perform the induction we set $g_{n-1} = \max\left\{\deg\left(\Theta_{n-1}^{(0)}\right); q_{n-1}^{(j)}, j = 1, \dots, k\right\}$, and choose a natural number λ_n such that $\lambda_n > (n-1)(\lambda_{n-1} + g_{n-1})$. By Runge's theorem on polynomial approximation, we find a polynomial $\Theta_n^{(0)}$ satisfying

$$\max_{z \in L_n^{(0)}} |\Theta_n^{(0)}(z)| < \frac{1}{2^n \max\limits_{z \in L_n^{(0)}} |z - a_1|^{\lambda_n}}, \tag{1}$$

$$\max_{z \in K_n^{(0)}} \left| \Theta_n^{(0)}(z) - \frac{p_n(z) - \sum\limits_{\nu=0}^{n-1} (z - a_1)^{\lambda_\nu} \Theta_\nu^{(0)}(z)}{(z - a_1)^{\lambda_\nu}} \right| < \frac{1}{n \max\limits_{z \in K_n^{(0)}} |z - a_1|^{\lambda_n}}. \tag{2}$$

For $j = 1, \dots, k$ we find a rational function $\Theta_n^{(j)}$ with a single pole at a_j, satisfying

$$\sup_{z \in L_n^{(j)}} |\Theta_n^{(j)}(z)| < \frac{1}{2^n \max\limits_{z \in L_n^{(j)}} \frac{1}{|z - a_j|^{\lambda_n}}}, \tag{3}$$

$$\max_{z \in K_n^{(j)}} \left| \Theta_n^{(j)}(z) - \frac{p_n(z) - \sum\limits_{\nu=0}^{n-1} \frac{1}{(z - a_j)^{\lambda_\nu}} \Theta_\nu^{(j)}(z)}{\frac{1}{(z - a_1)^{\lambda_\nu}}} \right| < \frac{1}{n \max\limits_{z \in K_n^{(0)}} \frac{1}{|z - a_j|^{\lambda_n}}}. \tag{4}$$

The existence of such a function follows from Runge's theorem on rational approximation (see, for instance, [13], Theorem 13.9). Note that the set $K_n^{(j)} \cup L_n^{(j)}$ has connected complement, which enables to shift the pole. Finally we consider

$$Q_m^{(0)}(z) = (z-a_1)^{\lambda_m} \Theta_m^{(0)}(z), \quad Q_m^{(j)}(z) = \frac{1}{(z-a_j)^{\lambda_m}} \Theta_m^{(j)}(z) \ (j = 1, \ldots, k),$$

and define

$$f_0(z) = \sum_{\nu=0}^{\infty} Q_\nu^{(0)}(z), \tag{5}$$

$$f_j(z) = \sum_{\nu=0}^{\infty} Q_\nu^{(j)}(z), \quad j = 1, \ldots, k. \tag{6}$$

Then the desired function is $f = f_0 + f_1 + \ldots + f_k$. Since $\{L_n^{(j)}\}_n$ are exhausting sequences for each A_j^c, the inequalities (1) and (2) assure the holomorphy of f on $\Omega = \bigcap_{j=0}^{k} A_j^c$.

Fix $j \in \{1, \ldots, k\}$. We consider the powers of $\frac{1}{z-a_j}$ in $Q_m^{(j)}$ and observe :

- The highest power of $\frac{1}{z-a_j}$ in $Q_m^{(j)}$ is less than or equal to $\lambda_m + g_m$.
- The lowest power of $\frac{1}{z-a_j}$ in $Q_{m+1}^{(j)}$ is greater than λ_{m+1}.

By the definition of λ_n the powers of $\frac{1}{z-a_j}$ do not overlap within the $Q_m^{(j)}$ and the functional series on the right hand side of (6) is indeed a Laurent series. For similar reasons the functional series on the right hand side of (5) turn to be a Taylor series. Hence, setting $t_n = \lambda_n + g_n$ and $q_n = \lambda_{n+1}$, we obtain the equations

$$S_{t_n}(f_0, a_1)(z) = \sum_{\nu=0}^{t_n} \frac{f_0^{(\nu)}(a_1)}{\nu!}(z-a_1)^\nu = \sum_{\nu=0}^{n} Q_\nu^{(0)}(z),$$

$$\sigma_{t_n}(f_j, a_j)(z) = \sum_{\nu=1}^{t_n} \frac{c_\nu(f_j, a_j)}{(z-a_j)^\nu} = \sum_{\nu=0}^{n} Q_\nu^{(j)}(z).$$

Thus the first statement of the theorem holds for $\zeta = a_1$. Moreover, the Taylor series of f_0 with respect to the center a_1 possesses pure Ostrowski gaps $\{t_n, q_n\}$ with $\frac{q_n}{t_n} \to \infty$ (i.e. $a_n(f, a_1) = 0$ for $n \in \bigcup_{n=1}^{\infty} \{t_n + 1, \ldots, q_n\}$). Theorem 1 in [7] states that the difference $M_{t_n}(f, \zeta) - M_{t_n}(f, a_1)$ converges locally uniformly to zero on Ω. This proves the first statement.

3) Proof of the Universal Properties. Let K be an arbitrary compact set with the properties of the second statement and an arbitrary function $g \in A(K)$. First we find an $n_0 \in \mathcal{N}$ with $K \in K_{n_0}$. By Mergelyan's theorem and the definition of the sequence $\{(K_n, p_n)\}_n$ we can choose a sequence $\{n_s\}_s$ of natural numbers with $n_s \geq s$ satisfying

$$\max_{K \cap A_j} \left| p_{n_s}(z) - \left(g(z) - \sum_{\substack{\mu \neq j \\ \mu = 0}}^{k} f_\mu(z) \right) \right| < \frac{1}{s}, \quad j = 0, \ldots, k, \quad s \in \mathcal{N},$$

and $K_{n_s} = K_{n_0}, (s \in \mathcal{N})$. The estimations (2) and (4) are equivalent to

$$\max_{z \in K_{n_s}^{(0)}} |S_{t_{n_s}}(f_0, a_1)(z) - p_{n_s}(z)| < \frac{1}{n_s} \leq \frac{1}{s}, \quad s \in \mathcal{N}, \tag{7}$$

$$\max_{z \in K_{n_s}^{(j)}} |\sigma_{t_{n_s}}(f_j, a_j)(z) - p_{n_s}(z)| < \frac{1}{n_s} \leq \frac{1}{s} \quad j = 1, \ldots, k, \quad s \in \mathcal{N}. \tag{8}$$

For $s \in \mathcal{N}$ the following inequalities hold :

$$\max_{z \in K \cap A_0} |M_{t_{n_s}}(f, a_1)(z) - g(z)| \leq \max_{z \in K_{n_s}^{(0)}} |S_{t_{n_s}}(f_0, a_1)(z) - p_{n_s}(z)| +$$

$$+ \max_{z \in K \cap A_0} \left| p_{n_s}(z) - \left(g(z) - \sum_{\mu=1}^{k} f_\mu(z) \right) \right| + \sum_{\mu=1}^{k} \max_{z \in K \cap A_0} |\sigma_{t_{n_s}}(f_\mu, a_\mu)(z) - f_\mu(z)|.$$

The second line of this estimation tends to zero as $s \to \infty$ by (7) and (8). The last term tends to zero, since $K \cap A_0$ is a compact subset of A_μ^c for each $\mu = 1, \ldots, k$. Similarly one can estimate the difference $\max_{z\in K\cap A_j} |M_{t_{n_s}}(f,a_1)(z) - g(z)|$ for $j = 1,\ldots,k$ to obtain

$$\max_{z\in K} |M_{t_{n_s}}(f,a_1)(z) - g(z)| \to 0, \quad s \to \infty.$$

Applying again Theorem 1 from [7], we get the second statement of the theorem. This completes the proof of Theorem 2.1.

A function satisfying both statements of the Theorem 2.1 will be called a *universal Laurent series.*

Theorem 2.2. *Let $\Omega \subset \hat{\mathbf{C}}$ be a domain such that $\hat{\mathbf{C}} \setminus \Omega = A_0 \cup \ldots \cup A_k$, $\infty \in A_0$, and let $a_j \in A_j$ $(j = 1,\ldots,k)$ be fixed points. Then the set of all universal Laurent series is dense in the space $(H(\Omega), d)$.*

Proof. For an arbitrary $g \in H(\Omega)$ and $\varepsilon > 0$ we choose a function f_0 possessing both properties of Theorem 2.1. Now we find a number $\delta > 0$ such that $p(\delta f_0) < \frac{\varepsilon}{2}$. By Runge's theorem on rational approximation we find a rational function R with poles at $\{a_1,\ldots,a_k\}$ satisfying $d(R,g) < \frac{\varepsilon}{2}$. For $f = \delta f_0 + R$ we have

$$d(f,g) = d(\delta f_0 + R, g) \leq p(\delta f_0) + d(R,g) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence f "lies near" to g. Next, for all $N \in \mathcal{N}$ greater than any power of $z - a_1$ or $\frac{1}{z-a_j}$ $(j = 1,\ldots,k)$ appearing in R (possibly after shifting the pole within R) we get

$$M_N(f,a_1) = \delta M_N(f_0,a_1) + R.$$

By the first statement of Theorem 2.1, there is a sequence $\{t_n\}_n$ such that $\{M_{t_n}(f_0,a_1)\}_n$ converges to f_0 with respect to the metric d. Hence $\{M_{t_n}(f,a_1)\}_n$ converges to f.

Let $K \subset \Omega^c \setminus \{a_1,\ldots,a_k\}$ be a compact set and $h \in A(K)$. By the second statement of Theorem 2.1 we can choose a subsequence $\{t_{n_s}\}_s$ of $\{t_n\}_n$ such that $\{M_{t_{n_s}}(f_0,a_1)\}_s$ converges uniformly on K to $\frac{1}{\delta}(h - R)$ (note that R also belongs to the class $A(K)$). Hence $\{M_{t_{n_s}}(f,a_1)\}_s$ converges uniformly on K to h. Thus f satisfies both properties of universal Laurent series. Theorem 2.2 is proved.

§3. UNIVERSAL FABER SERIES

For an introduction into Faber series the reader is referred to [3] and [14]. Following [14] we consider a bounded closed set $\overline{B}$ the complement $\overline{B}^c$ of which is a simply connected domain (in $\hat{\mathbf{C}}$). By the Riemann mapping theorem there exists a conformal mapping $\psi : \overline{D}^c \longmapsto \overline{B}^c$. The Faber polynomials with respect to ψ we denote by $\{p_n\}_n$.

For $R > 1$ by B_R we denote the bounded domain with the boundary $\{\psi(z) : |z| = R\}$. Then every function $f \in H(B_R)$ can be represented as a Faber series

$$f(z) = \sum_{n=0}^{\infty} c_n(f)p_n(z),$$

where

$$c_n(f) = \frac{1}{2\pi i}\int_{|s|=r} \frac{f(\psi(s))}{s^{n+1}}\,ds, \quad n \in \mathcal{N}_0, \quad 1 < r < R$$

are the Faber coefficients of f. The corresponding partial sums $F_m(f, z)$ are given by

$$F_m(f, z) = \sum_{n=0}^{m} c_n(f)p_n(z).$$

First we will show that the set of universal Faber series is a G_δ-set and is dense in the space $H(B_R)$ for any $R > 1$. To this end we fix a number $R > 1$, a set $G = B_R$ and define an abstract class of universal functions.

Definition 1. The set $\mathcal{U}_F(G)$ of *universal Faber series* is the class of all functions $f \in H(G)$ having the property that for any compact set $K \subset G^c$ with connected complement, any function $g \in A(K)$ and any $\varepsilon > 0$ there exists an index $n \in \mathcal{N}_0$ with

$$\max_{z \in K} |F_n(f, z) - g(z)| < \varepsilon.$$

Theorem 3.1. *The set $\mathcal{U}_F(G)$ of universal Faber series is a G_δ-set and is dense in the space $H(G)$.*

Proof : First we fix the following three sequences :

(1) Let $\{H_n\}_{n \in \mathcal{N}}$ be an exhausting sequence of G consisting of closed simply connected domains.

(2) Let $\{r_j; j \in \mathcal{N}\}$ be an enumeration of all polynomials with coefficients in $\mathcal{Q} + i\mathcal{Q}$.

(3) Let $\{K_n\}_{n \in \mathcal{N}}$ be an enumeration of all compact sets $K \subset G^c$ with connected complement such that for all sets $K \subseteq G^c$ having the same property there exists an $n_0 \in \mathcal{N}$ satisfying $K \subset K_{n_0}$.

The proof consists of three lemmas. The first two lemmas give a representation of $\mathcal{U}_F(G)$ as a G_δ-set and the third one shows that this set is dense in $H(G)$. For

$m, j, s \in \mathcal{N}$, and $n \in \mathcal{N}_0$ we set

$$O(G, m, j, s, n) = \left\{ f \in H(G) : \max_{z \in K_m} |F_n(f, z) - r_j(z)| < \frac{1}{s} \right\}. \tag{9}$$

Lemma 3.1. *The following equality holds :*

$$\mathcal{U}_F(G) = \bigcap_{m=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcap_{s=1}^{\infty} \bigcup_{n=0}^{\infty} O(G, m, j, s, n). \tag{10}$$

Proof. Let $f \in \mathcal{U}_F(G)$ and $m, j, s \in \mathcal{N}$ be given. Since K_m is a compact subset of G with connected complement by the definition of $\mathcal{U}_F(G)$ with $\varepsilon = \frac{1}{s}$ we obtain that f belongs to the right hand side of (10).
Now suppose that f belongs to the right hand side of (10). For a given compact set $K \subset G$ with connected complement, a function $g \in A(K)$ and a number $\varepsilon > 0$ there exist numbers $m, s \in \mathcal{N}$ such that $K \in K_m$ and $\frac{1}{s} < \frac{\varepsilon}{2}$. By Mergelyan's theorem we conclude the existence of an $j \in \mathcal{N}$ such that $\max_{z \in K_m} |r_j(z) - g(z)| < \frac{\varepsilon}{2}$. By (9), for the numbers m, j, s there is an $n \in \mathcal{N}_0$ satisfying $\max_{z \in K_m} |F_n(f, z) - r_j(z)| < \frac{1}{s}$. Therefore $\max_{z \in K_m} |F_n(f, z) - g(z)| < \frac{1}{s} + \frac{\varepsilon}{2} < \varepsilon$. This shows that f belongs to the left hand side of (10). Lemma 3.1 is proved.

Lemma 3.2. *For all $m, j, s \in \mathcal{N}$ and $n \in \mathcal{N}_0$ the set $O(G, m, j, s, n)$ defined by (9) is an open set in $H(G)$ (in the topology of locally uniform convergence).*
Proof. Fixing $m, j, s \in \mathcal{N}, n \in \mathcal{N}_0$ and $f \in O(G, m, j, s, n)$, for $r \in (1, R)$ we put $C_r = \{\varphi(s); |s| = r\}$. For $\delta > 0$ we define the set

$$U_\delta(f) = \{g \in H(G) : \max_{z \in C_r} |g(z) - f(z)| < \delta\},$$

and show that for an appropriate choice of δ this set is contained in $O(G, m, j, s, n)$. For a fixed $r \in (1, R)$ and any function $g \in U_\delta(f)$ we have

$$|c_\nu(f) - c_\nu(g)| = \left| \frac{1}{2\pi i} \int_{|s|=r} \frac{f(\psi(s)) - g(\psi(s))}{s^{\nu+1}} \, ds \right| < \frac{\delta}{r^\nu}, \quad \nu \in \mathcal{N}_0.$$

Now we set $M = \max\{|p_\nu(z)| : \ z \in K_m, \ 0 \le \nu \le n\}$, and choose

$$\delta = \frac{1 - r}{(\frac{1}{r})^n - r} \frac{1}{M} \left(\frac{1}{s} - \max_{z \in K_m} |F_n(f, z) - r_j(z)| \right).$$

It is clear that $\delta > 0$, and hence for all $g \in U_\delta(f)$

$$\max_{z\in K_m} |F_n(g,z) - F_n(f,z)| \le \sum_{\nu=0}^{n} |c_\nu(g) - c_\nu(f)| \max_{z\in K_m} |p_\nu(z)| <$$

$$< \frac{1}{s} - \max_{z\in K_m} |F_n(f,z) - r_j(z)|.$$

Therefore for all $g \in U_\delta(f)$

$$\max_{z\in K_m} |F_n(g,z) - r_j(z)| \le \max_{z\in K_m} |F_n(g,z) - F_n(f,z)| + \max_{z\in K_m} |F_n(f,z) - r_j(z)| <$$

$$< \frac{1}{s} - \max_{z\in K_m} |F_n(f,z) - r_j(z)| + \max_{z\in K_m} |F_n(f,z) - r_j(z)| = \frac{1}{s},$$

yielding $U_\delta(f) \subset O(G,m,j,s,n)$. Lemma 3.2 is proved.

Lemma 3.3. *For every choice of $m, j, s \in \mathcal{N}$ the set $\bigcup_{n=0}^{\infty} O(G,m,j,s,n)$ is dense in $H(G)$.*

Proof. Given a compact set $K \subset G$, an $\varepsilon > 0$ and numbers $m, j, s \in \mathcal{N}$, let $f \in H(G)$ be arbitrary. Since G is at least simply connected, without loss of generality we can assume that K has connected complement in G. Hence $K \cap K_m = \emptyset$ and $K \cup K_m$ has connected complement. So due to Runge's theorem on polynomial approximation there exists a polynomial P satisfying

$$\max_{z\in K} |P(z) - f(z)| < \varepsilon, \quad \max_{z\in K_m} |P(z) - r_j(z)| < \frac{1}{s}.$$

Let n be the degree of P. Since every Faber polynomial p_n is of full degree, i.e. $\deg(p_n) = n$, we have $P = F_n(P,\cdot)$. Therefore

$$\max_{z\in K_m} |F_n(P,z) - r_j(z)| < \frac{1}{s},$$

yielding $P \in O(G,m,j,s,n)$. Lemma 3.3 is proved.

Now Theorem 3.1 follows immediately from Lemmas 3.1 – 3.3 and Baire's category theorem.

A universal Faber series is constructed by the method already used in the preceding section. So we outline a sketch proof, containing only the construction of the universal Faber series.

Theorem 3.2. *There exists a function f holomorphic on the domain G (as defined above) and a sequence of natural numbers $\{t_n\}n$ satisfying the following properties :*
(1) The sequence$\{F_{t_n}(f,\cdot)\}_n$ converges locally uniformly on G to f.
(2) For each $K \subset G^c$ with connected complement and each $g \in A(K)$ there exists a subsequence $\{t_{n_s}\}_s$ of $\{t_n\}_n$, such that $\{F_{t_{n_s}}(f,\cdot)\}_s$ converges uniformly on K to g.

Proof. We use the same sequences as in the proof of Theorem 3.1. First we choose a sequence $\{(K_n^*, r_n^*)\}_n$, where every combination of K_n and r_n appears infinitely many times. Set $P_0 = 0$ and $\lambda_0 = 1$. Supposing that for an $n \in \mathcal{N}$ the polynomials $P_0, \ldots, P_{n-1}$ and the natural numbers $\lambda_0, \ldots, \lambda_{n-1}$ have already been defined, we set

$$\lambda_n = \lambda_{n-1} + \max\{\deg(P_\nu); 0 \leq \nu \leq n-1\} + 1,$$

and choose a polynomial P_n with the following properties

$$\max_{z \in H_n} |P_n(z)| < \frac{1}{n^2 \max\limits_{z \in H_n} |z|^{\lambda_n}}, \tag{11}$$

$$\max_{z \in K_n^*} \left| P_n(z) - \frac{r_n^*(z) - \sum\limits_{\nu=0}^{n-1} z^{\lambda_\nu} P_\nu(z)}{z^{\lambda_n}} \right| < \frac{1}{n \max\limits_{z \in K_n^*} |z|^{\lambda_n}}. \tag{12}$$

Finally we define

$$T_m(z) = \sum_{\nu=0}^{m} z^{\lambda_\nu} P_\nu(z); \qquad f(z) = \lim_{m \to \infty} T_m(z).$$

Similar considerations as in Theorem 2.1 allow to conclude that there is no overlapping of the powers of z in the sum of the right hand side of the first equation. This yields $F_{t_n}(f,\cdot) = T_n$. The universality of f can also be shown analogously as in Theorem 2.1.

Remark. Actually we have used two different definitions of universality (Definition 1 and Theorem 3.2). The general proof outlined in [16], Theorem 2.6 shows that these two classes of universal Faber series coincide.

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Резюме. For a finitely connected domain Ω the paper constructively proves the existence of a function f holomorphic on Ω, whose Laurent series partial sums possess universal approximation properties in Ω^c. Existence of functions holomorphic on a Jordan domain G, whose Faber series partial sums possess universal approximation properties in G^c is demonstrated. The class of functions with these properties is a G_δ-set and is dense in the space of all functions holomorphic on G.

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