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UNIVERSAL POWER SERIES WITH POISSON GAPS

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Abstract. Holomorphic functions on **D** and **C** that are universal with respect to both translates and derivatives are constructed. The corresponding universal functions are determined by lacunary power series with gaps of positive lower Poisson density.

§1. INTRODUCTION

For a compact set K in the complex plane $\mathbf{C}$ we denote by $A(K)$ the set of all complex valued functions, that are continuous on K and holomorphic in its interior K^0. Supplied with the uniform norm, $A(K)$ becomes a Banach space. By $\mathcal{M}$ we denote the family of all compact sets which have connected complement.

The problem of "universal approximation" of functions by so-called "universal elements" are classical and there exists an extended literature on the theory of functions which are universal in different respects. The first example (which we cite in a slightly modified form) was given by Fekete in 1914 (see Pál [20]), who proved the existence of a universal real power series $\sum\limits_{\nu=0}^{\infty} a_\nu x^\nu$ with the property that for every interval $[a,b]$ with $0 \notin [a,b]$ and every continuous function f on $[a,b]$ there exists a sequence $\{n_k\}$ such that $\{\sum\limits_{\nu=0}^{n_k} a_\nu x^\nu\}$ converges uniformly to $f(x)$ on $[a,b]$. Obviously this power series has radius of convergence $r = 0$.

Universal power series (with respect to overconvergence) in the complex plane with positive radii of convergence were constructed by W. Luh in 1970 [8] and

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independently by Chui and Parnes in 1971 [3].

Perhaps the best known example of a universal function was obtained by Birkhoff in 1929 [1], who proved the existence of an entire function ϕ with universal translates, meaning that for every entire function f there exists a subsequence $\{n_k\}$ of the set of natural numbers $\mathbb{N}$ such that $\{\phi(z+n_k)\}$ converges to $f(z)$ compactly on $\mathbf{C}$.

The approximation theorems of Runge and Mergelian are in general the basic tools for construction of the elements that are universal in a certain specified sense. For details we refer to the article of Grosse-Erdmann [7], where a survey of the various universalities as well as a complete bibliography of relevant contributions updated till 1998 can be found.

We mention some further results which are of interest for our investigations.

In 1953 MacLane [18] has constructed an entire function ϕ with universal derivatives, by proving that for every entire function f there exists a subsequence $\{n_k\}$ of $\mathbb{N}$ such that $\{\phi^{(n_k)}(z)\}$ converges to $f(z)$ compactly on $\mathbf{C}$.

Motivated by Birkhoff's result in a series of papers (see [9] – [14]) W. Luh has investigated holomorphic functions in more general open sets, universal under certain translations. For instance in [11] was proved that there exists a holomorphic function in the unit disk $\mathbf{D} := \{z : |z| < 1\}$ possessing the property that for every $\zeta \in \partial\mathbf{D}$, every $K \in \mathcal{M}$ and every $f \in A(K)$ there exist sequences $\{a_n\}$, $\{b_n\}$ such that $a_n \to 0$, $b_n \to \zeta$ and $\{\phi(a_n z + b_n)\}$ converges to $f(z)$ uniformly on K.

Some recent papers fixed their attentions to functions which together with universalities possess additional properties (Schneider [23] and Tenthoff [24]). In several articles (see [15] – [17]), Luh, Martirosian and Müller have dealt with functions which have lacunary power series expansions and are universal with respect to translates.

In this note we construct holomorphic functions possessing several different universalities. In Theorems 1 and 2 we prove the existence of holomorphic functions on $\mathbf{D}$ and $\mathbf{C}$ respectively with lacunary power series, that are universal with respect to both translates and derivatives. In addition it is shown that the "paths of approximation" can be prescribed in an arbitrary way. Our basic tool is a result from [5] (Lemma 1, stated below) on the approximation by lacunary polynomials with gaps of positive lower Poisson density. Finally we show that the functions which were obtained in Theorems 1 and 2 possess additional universal properties with respect to almost everywhere approximation of Lebesgue measurable functions.

§2. REMARKS ON DENSITY PROPERTIES

For a subsequence $Q = \{q_\nu\}_{\nu\in\mathbb{N}_0}$ of $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ various notions of densities have been introduced. We denote by $n(t)$ the number of elements of Q in the interval $[0, t]$.

The upper ($\overline{\Delta}(Q)$) and the lower ($\underline{\Delta}(Q)$) densities of Q are given by

$$\overline{\Delta}(Q) := \overline{\lim_{t\to\infty}} \frac{n(t)}{t}, \qquad \underline{\Delta}(Q) := \underline{\lim}_{t\to\infty} \frac{n(t)}{t}.$$

If $\overline{\Delta}(Q) = \underline{\Delta}(Q)$, then the common value is called the density of Q. The maximal and minimal densities of Q are defined by

$$\Delta_{\max}(Q) := \lim_{r\to 1-} \left(\overline{\lim_{t\to\infty}} \frac{n(t)-n(tr)}{(1-r)t}\right), \qquad \Delta_{\min}(Q) := \lim_{r\to 1-} \left(\underline{\lim}_{t\to\infty} \frac{n(t)-n(tr)}{(1-r)t}\right)$$

respectively. These notions were essentially utilized by Pólya [21].
In this article we deal with two density concepts, which were introduced by Poisson. The expressions

$$\overline{\Delta}_p(Q) := \frac{2}{\pi} \overline{\lim_{s\to\infty}} \int\limits_0^\infty \frac{n(t)}{t} \cdot \frac{s}{t^2+s^2} dt, \qquad \underline{\Delta}_p(Q) := \frac{2}{\pi} \underline{\lim}_{s\to\infty} \int\limits_0^\infty \frac{n(t)}{t} \cdot \frac{s}{t^2+s^2} dt$$

are called the upper and the lower Poisson densities of Q respectively. The following chain of inequalities is well known :

$$\Delta_{\min}(Q) \le \underline{\Delta}(Q) \le \underline{\Delta}_p(Q) \le \overline{\Delta}_p(Q) \le \overline{\Delta}(Q) \le \Delta_{\max}(Q).$$

For further properties of the various notions of densities and their interdependences we refer to [22] (see also [2]).

§3. AUXILIARY RESULTS

In this section we state two lemmas from [13] and [5] which will be used in the proofs of our main results. We consider a subsequence $Q = \{q_\nu\}_{\nu\in\mathbb{N}_0}$ of $\mathbb{N}_0$ and denote by $\mathcal{P}_Q$ the set of all polynomials of the form $p(z) = \sum\limits_{\nu=0}^{m} p_\nu z^{q_\nu}$. The first lemma describes the possibilities of approximation of functions by polynomials from $\mathcal{P}_Q$, when Q has positive lower Poisson density.

Lemma 1 (see [5]). *Consider for $r > 0$ and $s > 0$ the disks*

$$\mathbf{D}_r := \{z : |z| < r\}, \qquad \mathbf{D}_s(a) := \{z : |z-a| < s\},$$

assume that $r + s < |a|$ *and* $\overline{\mathbf{D}_r} \cap \overline{\mathbf{D}_s(a)} = \phi$ *and define* $K := \overline{\mathbf{D}_r} \cup \overline{\mathbf{D}_s(a)}$. *Let* $Q = \{q_\nu\}_{\nu \in \mathbb{N}_0}$ *be a subsequence of* $\mathbb{N}_0$ *with* $\underline{\Delta}_p(Q) > \frac{1}{\pi} \arcsin \frac{s}{|a|}$, *and let* $f \in A(K)$ *be a function which in a neighborhood of the origin has a representation of the form* $f(z) = \sum_{\nu=0}^{\infty} f_\nu z^{q_\nu}$. *Then for every* $\varepsilon > 0$ *there exists a polynomial* $p \in \mathcal{P}_Q$ *satisfying*

$$\max_K |f(z) - p(z)| < \varepsilon.$$

The next Lemma is a generalization of MacLane's result which was mentioned in the introduction.

Lemma 2 (see [13]). *Let* $\lambda = \{\lambda_n\}_{n \in \mathbb{N}}$ *be a prescribed subsequence of* $\mathbb{N}_0$. *Then there exists an entire function* $\varphi = \varphi_\lambda$ *with the following property : for any entire function* f *there exists a subsequence* $\{m_k\}_{k \in \mathbb{N}}$ *of* $\mathbb{N}$, *such that the corresponding sequence of derivatives* $\{\varphi^{(\lambda_{m_k})}(z)\}$ *converges to* $f(z)$ *compactly on* $\mathbf{C}$.

§4. UNIVERSAL ENTIRE FUNCTIONS

Let $Q = \{q_\nu\}_{\nu \in \mathbb{N}_0}$ be a subsequence of $\mathbb{N}_0$. We denote by $\mathcal{E}_Q$ the set of all entire functions with a power series representation $f(z) = \sum_{\nu=0}^{\infty} f_\nu z^{q_\nu}$. First we prove the existence of an entire function which is universal with respect to both translates and derivatives.

Theorem 1. *Let be prescribed :*

- *a subsequence* Q *of* $\mathbb{N}_0$ *with* $\underline{\Delta}_p(Q) > 0$,
- *an unbounded sequence* $\{z_n\}$ *in* $\mathbf{C}$,
- *a subsequence* $\{\lambda_n\}$ *of* $\mathbb{N}_0$.

Then there exists an entire function ϕ *with the following properties :*
For any set $K \in \mathcal{M}$ *and any function* $f \in A(K)$ *there exist subsequences* $\{p_k\}$ *and* $\{q_k\}$ *of* $\mathbb{N}$ *such that* $\{\phi(z + z_{p_k})\}$ *and* $\{\phi^{(\lambda_{q_k})}(z)\}$ *converge to* $f(z)$ *uniformly on* K.
The function ϕ *has the form* $\phi = \Psi + \varphi$ *where* $\Psi \in \mathcal{E}_Q$ *and* φ *is an entire function.*

Proof. 1. Let $d \geq 1$ satisfy

$$\underline{\Delta}_p(Q) > \frac{1}{\pi} \arcsin \frac{1}{2d}.$$

We define

$$d_0 := 1, \qquad d_n := \frac{|z_n| - |z_{n-1}|}{2d}, \quad (n \in \mathbb{N}).$$

Without loss of generality, we assume that $\{d_n\}$ is strictly increasing and $\lim_{n\to\infty} d_n = \infty$ (otherwise we can choose a suitable subsequence of $\{z_n\}$ which has the desired property). By $\{\Omega_n\}_{n\in\mathbb{N}}$ we denote an enumeration of all polynomials whose coefficients have rational real and imaginary parts. Let φ be the entire function from Lemma 2 with respect to $\{\lambda_n\}$.

2. We construct a sequence of polynomials $P_n \in \mathcal{P}_Q$ and a sequence of numbers $m_n \in \mathbb{N}_0$ by induction. We take $P_0(z) \equiv 0$, $m_0 = 0$ and suppose that for $n \in \mathbb{N}$ $P_0, P_1, ..., P_{n-1}$; $m_0, m_1, ..., m_{n-1}$ have already been determined.

According to Lemma 2 we can choose a sufficiently large natural number $m_n := \lambda_{j_n}$ to have $m_n > m_{n-1} + n$, $m_n > deg(P_{n-1})$ and

$$\max_{|z|\le|z_{n-1}|} \left|\varphi^{(m_n)}(z) - \Omega_n(z)\right| < \frac{1}{n}. \tag{1}$$

By Lemma 1 we find a polynomial $P_n \in \mathcal{P}_Q$ satisfying

$$\max_{|w|\le|z_{n-1}|+d_{n-1}} \left|P_n(w) - P_{n-1}(w)\right| < \varepsilon_n := \frac{1}{(n+1)^2 \cdot m_n! \cdot (|z_{n-1}| + d_{n-1})}, \tag{2}$$

$$\max_{|w-z_n|\le d_n} \left|P_n(w) - \Omega_n(w - z_n) + \varphi(w)\right| < \frac{1}{n}. \tag{3}$$

By induction we get $\{P_n(w)\}$ and $\{m_n\}$.

It follows from (2) that

$$\Psi(w) := \sum_{\nu=1}^{\infty} \{P_\nu(w) - P_{\nu-1}(w)\}$$

is an entire function which belongs to $\mathcal{E}_Q$. We show that $\phi(w) := \Psi(w) + \varphi(w)$ has the desired universal properties.

3. We obtain for all $\nu \ge n$

$$\max_{|z|\le|z_{\nu-1}|} \left|P_\nu^{(m_n)}(z) - P_{\nu-1}^{(m_n)}(z)\right| =$$

$$= \max_{|z|\le|z_{\nu-1}|} \left|\frac{m_n!}{2\pi i} \int\limits_{|w|=|z_{\nu-1}|+d_{\nu-1}} \frac{P_\nu(w) - P_{\nu-1}(w)}{(w-z)^{m_n+1}}\, dw\right| \le$$

$$\le m_n! \cdot (|z_{\nu-1}| + d_{\nu-1}) \cdot \varepsilon_\nu \cdot \frac{1}{(d_{\nu-1})^{m_n+1}} \le m_\nu! \cdot (|z_{\nu-1}| + d_{\nu-1}) \cdot \varepsilon_\nu < \frac{1}{(\nu+1)^2}.$$

It follows

$$\max_{|z|\leq|z_{n-1}|}\left|\Psi^{(m_n)}(z)\right| \leq \sum_{\nu=n}^{\infty}\max_{|z|\leq|z_{\nu-1}|}\left|P_{\nu}^{(m_n)}(z)-P_{\nu-1}^{(m_n)}(z)\right| \leq \sum_{\nu=n}^{\infty}\frac{1}{(\nu+1)^2}<\frac{1}{n}.$$

Together with (1) this yields

$$\max_{|z|\leq|z_{n-1}|}\left|\phi^{(m_n)}(z)-\Omega_n(z)\right|<\frac{2}{n}. \tag{4}$$

4. From (2) and (3) we get

$$\max_{|w-z_n|\leq d_n}|\Phi(w)-\Omega_n(w-z_n)|\leq$$

$$\leq \max_{|w-z_n|\leq d_n}|\Psi(w)-P_n(w)|+\max_{|w-z_n|\leq d_n}|P_n(w)-\Omega_n(w-z_n)+\varphi(w)|\leq$$

$$\leq \sum_{\nu=n+1}^{\infty}\max_{|w|\leq|z_{\nu-1}|+d_{\nu-1}}|P_\nu(w)-P_{\nu-1}(w)|+\frac{1}{n}\leq\sum_{\nu=n+1}^{\infty}\frac{1}{(\nu+1)^2}+\frac{1}{n}<\frac{2}{n},$$

or equivalently

$$\max_{|z|\leq d_n}|\phi(z+z_n)-\Omega_n(z)|<\frac{2}{n}. \tag{5}$$

5. Given a set $K\in\mathcal{M}$ and a function $f\in A(K)$, by Mergelian's theorem [19] (see, also [4]), there exists a subsequence $\{n_k\}$ of $\mathbb{N}$ with $\Omega_{n_k}(z)\underset{K}{\rightrightarrows}f(z)$.
There exists a k_0 such that $K\subset\{z:|z|\leq|z_{n_k-1}|\}$ for all $k>k_0$, and it follows from (4) that

$$\phi^{(m_{n_k})}(z)\underset{K}{\rightrightarrows}f(z).$$

On the other hand there exists a $\bar{k}_0$ such that $K\subset\{z:|z|\leq d_{n_k}\}$ for all $k\geq\bar{k}_0$, and it follows from (5) that also

$$\phi(z+z_{n_k})\underset{K}{\rightrightarrows}f(z).$$

This completes the proof of Theorem 1.

Remark 1. The proof shows that the function $\phi = \Psi + \varphi$ was constructed in such a way that

- the function $\Psi \in \mathcal{E}_Q$ has universal translates with respect to the prescribed sequence $\{z_n\}$,
- the entire function φ has universal derivatives with respect to the prescribed sequence $\{\lambda_n\}$ and ϕ has both universality properties.

§5. UNIVERSAL HOLOMORPHIC FUNCTIONS IN THE UNIT DISK

Let again $Q = \{q_\nu\}_{\nu \in \mathbb{N}_0}$ be a subsequence of $\mathbb{N}_0$. We now denote by $\mathcal{U}_Q$ the set of all functions f which are holomorphic in the unit disk $\mathbf{D}$ with a power series expansion $f(z) = \sum_{\nu=0}^{\infty} f_\nu z^{q_\nu}$. For a sequence $\{w_n\}$ of complex numbers we denote by $V(\{w_n\})$ the set of all its accumulation points.

Theorem 2. *Let be prescribed :*

- *a subsequence Q of $\mathbb{N}_0$ with $\underline{\Delta}_p(Q) > 0$,*
- *a sequence $\{a_n\}$ in $\mathbf{C} \setminus \{0\}$ with $0 \in V(\{a_n\})$,*
- *a sequence $\{b_n\}$ in $\mathbf{D}$ with $V(\{b_n\}) = \partial\mathbf{D}$,*
- *a subsequence $\{\lambda_n\}$ of $\mathbb{N}_0$.*

Then there exists a holomorphic function ϕ in $\mathbf{D}$ with the following properties : For any set $K \in \mathcal{M}$, any function $f \in A(K)$ and any $\zeta \in \partial\mathbf{D}$, subsequences $\{p_k\}$ and $\{q_k\}$ of $\mathbb{N}$ exist such that

$$\lim_{k\to\infty} a_{p_k} = 0, \qquad \lim_{k\to\infty} b_{q_k} = \zeta,$$

$$a_{p_k} z + b_{q_k} \in \mathbf{D} \quad \text{for all} \quad z \in K,$$

$$\{\phi(a_{p_k} z + b_{q_k})\} \quad \text{converges to } f(z) \text{ uniformly on } K.$$

If $K \subset \mathbf{D}$ then in addition there exists a subsequence $\{r_k\}$ of $\mathbb{N}$ such that

$$\{\phi^{(\lambda_{r_k})}(z)\} \quad \text{converges to} \quad f(z) \quad \text{uniformly on} \quad K.$$

The function ϕ has the form $\phi = \Psi + \varphi$ where $\Psi \in \mathcal{U}_Q$ and φ is an entire function.

Proof. 1. We consider any sequence $\{\zeta^{(k)}\}_{k\in\mathbb{N}_0}$ of points $\zeta^{(k)} \in \partial\mathbf{D}$ with $V(\{\zeta^{(k)}\}) = \partial\mathbf{D}$. For each $k \in \mathbb{N}_0$ we choose a subsequence $\{z_\nu^{(k)}\}_{\nu\in\mathbb{N}}$ of $\{b_n\}$ with the properties $\lim_{\nu\to\infty} z_\nu^{(k)} = \zeta^{(k)}$, $|z_\nu^{(0)}| < |z_\nu^{(1)}| < \ldots < |z_\nu^{(\nu)}|$ for each $\nu \in \mathbb{N}$ and that there exists a sequence $\{G_\nu\}_{\nu\in\mathbb{N}}$ of disks $G_\nu := \{z : |z| < \varrho_\nu\}$ such that $\varrho_1 < \varrho_2 < \ldots < \varrho_\nu \to 1$ and $z_\nu^{(k)} \in G_{\nu+1} \setminus \overline{G_\nu}$ for $k = 0, 1, ..., \nu$.

Next, we choose ℓ_ν large enough, to have $s_\nu := \sqrt{|a_{\ell_\nu}|} \to 0$ for $\nu \to \infty$, as well as

$$|z_\nu^{(k)}| - |z_\nu^{(k-1)}| > 2s_\nu \quad \text{for } k = 0, ..., \nu,$$

$$\{z : |z - z_\nu^{(k)}| < s_\nu\} \subset G_{\nu+1} \setminus \overline{G_\nu} \quad \text{for } k = 1, ..., \nu,$$

$$\Delta_p(Q) > \frac{1}{\pi} \arcsin \frac{s_\nu}{|z_\nu^{(k)}|} \quad \text{for } k = 1, ..., \nu; \ \nu = 1, 2, ...$$

Let again $\{\Omega_n\}_{n\in\mathbb{N}}$ be an enumeration of all polynomials with coefficients whose real and imaginary parts are rational and let φ be the entire function from Lemma 2 with respect to $\{\lambda_n\}$.

2. We construct polynomials $P_{n\mu} \in \mathcal{P}_Q$ ($\mu = 0, ..., n$; $n = 1, 2, ...$) and integers m_n by induction, where in the n-th step the polynomials $P_{n1}, P_{n2}, ..., P_{nn}$ and an integer m_n are defined by an approximation process, using Lemma 1.

We start with $P_{10}(z) \equiv P_{11}(z) \equiv 0$ and $m_1 := 0$. Suppose that $n \geq 2$ and that the groups of polynomials $P_{\nu 1}, P_{\nu 2}, ..., P_{\nu\nu}$ and the numbers $m_\nu \in \mathbb{N}$ have already been determined for all $\nu = 1, ..., n-1$. Since in any case we put $P_{\nu+1,0}(z) := P_{\nu\nu}(z)$, so at the start the polynomial P_{n0} is known. According to Lemma 2 we can choose $m_n := \lambda_{j_n}$ large enough to have $m_n > m_{n-1} + n$, $m_n > deg(P_{n0})$ and

$$\max_{\overline{G_n}} |\varphi^{(m_n)}(z) - \Omega_n(z)| < \frac{1}{n}. \tag{6}$$

If now for a μ with $1 \leq \mu \leq n$ the polynomial $P_{n,\mu-1}$ has already been chosen, then we find according to Lemma 1 a polynomial $P_{n\mu} \in \mathcal{P}_Q$ satisfying

$$\max_{|w| \leq |z_n^{(\mu-1)}| + s_n} \left| P_{n\mu}(w) - P_{n,\mu-1}(w) \right| < \varepsilon_n := \frac{(s_n)^{m_n+1}}{(n+1)^3 \cdot m_n!}, \tag{7}$$

$$\max_{|w - z_n^{(\mu)}| \leq s_n} \left| P_{n\mu}(w) - \Omega_n\Big(\frac{w - z_n^{(\mu)}}{a_{\ell_n}}\Big) + \varphi(w) \right| < \frac{1}{n}. \tag{8}$$

By induction we get all polynomials $P_{n\mu}$ and all numbers m_n.

It follows easily from (7) that the function

$$\Psi(w) := \sum_{\nu=1}^{\infty} \sum_{\mu=1}^{\nu} \{P_{\nu\mu}(w) - P_{\nu,\mu-1}(w)\}$$

belongs to $\mathcal{U}_Q$. We will show that the function $\phi(w) := \Psi(w) + \varphi(w)$ possesses the asserted universal properties.

3. For all $\nu \geq n$ and $\mu = 1, ..., \nu$ we have

$$\max_{|z| \leq |z_\nu^{(\mu-1)}|} \left| P_{\nu\mu}^{(m_n)}(z) - P_{\nu,\mu-1}^{(m_n)}(z) \right| =$$

$$= \max_{|z| \leq |z_\nu^{(\nu-1)}|} \left| \frac{m_n!}{2\pi i} \int\limits_{|w| = |z_\nu^{(\nu-1)}| + s_\nu} \frac{P_{\nu\mu}(w) - P_{\nu,\mu-1}(w)}{(w-z)^{m_n+1}} dw \right| \leq$$

$$\leq m_n! \cdot \left(|z_\nu^{(\mu-1)}| + s_\nu \right) \cdot \varepsilon_\nu \cdot \frac{1}{(s_\nu)^{m_\nu+1}} \leq m_n! \cdot \varepsilon_\nu \cdot \frac{1}{(s_\nu)^{m_\nu+1}} = \frac{1}{(\nu+1)^3}.$$

Therefore

$$\max_{G_n} |\Psi^{(m_n)}(z)| \leq \sum_{\nu=n}^{\infty} \sum_{\mu=1}^{\nu} \max_{G_n} \left| P_{\nu\mu}^{(m_n)}(z) - P_{\nu,\mu-1}^{(m_n)}(z) \right| \leq$$

$$\leq \sum_{\nu=n}^{\infty} \sum_{\mu=1}^{\nu} \max_{|z| \leq |z_\nu^{(\mu-1)}|} \left| P_{\nu\mu}^{(m_n)}(z) - P_{\nu,\mu-1}^{(m_n)}(z) \right| \leq \sum_{\nu=n}^{\infty} \sum_{\mu=1}^{\nu} \frac{1}{(\nu+1)^3} < \frac{1}{n}.$$

Together with (6) this yields

$$\max_{G_n} |\phi^{(m_n)}(z) - \Omega_n(z)| < \frac{2}{n}. \tag{9}$$

4. For $n \in \mathbb{N}$ and $\mu = 1, ..., n$, (7) implies

$$\max_{|w - z_n^{(\mu)}| \leq s_n} |\Psi(w) - P_{n\mu}(w)| \leq \sum_{\nu=\mu+1}^{n} \max_{|w| \leq |z_n^{(\nu-1)}| + s_n} |P_{n\nu}(w) - P_{n,\nu-1}(w)| +$$

$$+ \sum_{m=n+1}^{\infty} \sum_{\nu=1}^{m} \max_{|w| \leq |z_m^{(\nu-1)}| + s_m} |P_{m\nu}(w) - P_{m,\nu-1}(w)| \leq$$

$$\leq \sum_{\nu=\mu+1}^{n} \frac{1}{(n+1)^3} + \sum_{m=n+1}^{\infty} \sum_{\nu=1}^{m} \frac{1}{(m+1)^3} < \frac{2}{n}.$$

We have

$$\phi(w) - \Omega_n\Big(\frac{w - z_n^{(\mu)}}{a_{\ell_n}}\Big) = \{\Psi(w) - P_{n\mu}(w)\} + \Big\{P_{n\mu}(w) - \Omega\Big(\frac{w - z_n^{(\mu)}}{a_{\ell_n}}\Big) + \varphi(w)\Big\}$$

and therefore we obtain by (8) for $n \in \mathbb{N}$ and $\mu = 1, ..., n$

$$\max_{|w - z_n^{(\mu)}| \le s_n} \Big|\phi(w) - \Omega_n\Big(\frac{w - z_n^{(\mu)}}{a_{\ell_n}}\Big)\Big| < \frac{3}{n},$$

or equivalently

$$\max_{|z| \le \frac{1}{s_n}} |\phi(a_{\ell_n} z + z_n^{(\mu)}) - \Omega_n(z)| < \frac{3}{n}. \tag{10}$$

5. Given a set $K \in \mathcal{M}$ and a function $f \in A(K)$, by Mergelian's theorem [19] (see also [4]), there exists a subsequence $\{n_k\} \subset \mathbb{N}$ with $\Omega_{n_k}(z) \underset{K}{\rightrightarrows} f(z)$.
a) If $K \subset \mathbf{D}$ then there exists a k_0 such that $K \subset G_{n_k}$ for all $k > k_0$, and (9) implies that $\{\phi^{(m_{n_k})}(z)\}$ converges to $f(z)$ uniformly on K.
b) For any $K \in \mathcal{M}$ there exists a $\bar{k}_0$ such that $K \subset \{z : |z| \le \frac{1}{s_{n_k}}\}$ for all $k > \bar{k}_0$. If any $\zeta \in \partial\mathbf{D}$ is given, then ζ is an accumulation point of the set $\{z_{n_k}^{(\mu)} : \mu = 1, ..., n_k\, ;\, k > \bar{k}_0\}$ and there exists $\mu_k, 1 \le \mu_k \le n_k$ such that $z_{n_k}^{(\mu_k)} \to \zeta$ for $k \to \infty$. Hence it follows from [11] that $\{\phi(a_{\ell_{n_k}} z + z_{n_k}^{(\mu_k)})\}$ converges to $f(z)$ uniformly on K. This completes the proof of Theorem 2.

Remark 2. The proof shows that the function $\phi = \Psi + \varphi$ was constructed in such a way that

- the function $\Psi \in \mathcal{U}_Q$ has universal translates with respect to the prescribed pair of sequences $\{a_n\}$, $\{b_n\}$,
- the entire function φ has universal derivatives with respect to the prescribed sequence $\{\lambda_n\}$

and that ϕ has both of these universalities simultaneously. In addition, the power series expansion of ϕ around the origin

$$\phi(z) := \sum_{\nu=0}^{\infty} \phi_\nu z^\nu$$

has "quasi-gaps" according to the sequence Q (with positive lower Poisson density) in the sense that we have

$$\lim_{\substack{\nu\to\infty \\ \nu\in Q}} |\phi_\nu|^{1/\nu} = 0.$$

§6. UNIVERSAL APPROXIMATION OF MEASURABLE FUNCTIONS

We now show that the functions which were constructed in Theorem 1 and Theorem 2 possess some universal properties with respect to almost everywhere approximation of Lebesgue measurable functions.

For a Lebesgue measurable set $S \subset \mathbf{C}$ we denote by $\mu(S)$ its (two dimensional) Lebesgue measure.

Theorem 3. *Let the sequences Q, $\{z_n\}$ and $\{\lambda_n\}$ and the entire function ϕ be the same as in Theorem 1. Then ϕ has the property that for any measurable set $S \subset \mathbf{C}$ and any measurable function g on S there exist subsequences $\{r_k\}$ and $\{s_k\}$ of $\mathbb{N}$ such that $\{\phi(z + z_{r_k})\}$ and $\{\phi^{(\lambda_{s_k})}(z)\}$ converge to $g(z)$ almost everywhere on S.*

The proof follows immediately from the following more general lemma, which might be of independent interest.

Lemma 3. *Suppose that $\{f_n\}$ is a sequence of entire functions with the following properties :*

For any set $K \in \mathcal{M}$ with $K^0 = \emptyset$ and any continuous function φ on K there exists a subsequence $\{n_k\}$ of $\mathbb{N}$ such that $\{f_{n_k}(z)\}$ converges to $\varphi(z)$ uniformly on K.

Then the sequence $\{f_n\}$ has also the following property :

For any measurable set $S \subset \mathbf{C}$ and any measurable function g on S there exists a subsequence $\{m_k\}$ of $\mathbb{N}$ such that $\{f_{m_k}(z)\}$ converges to $g(z)$ almost everywhere on S.

Proof. 1. Given a measurable set $S \subset \mathbf{C}$ and a measurable function g on S, the function

$$g_S(z) := \begin{cases} g(z) & \text{if } z \in S \\ 0 & \text{if } z \notin S \end{cases}$$

is measurable on $\mathbf{C}$ and it suffices to approximate g_S almost everywhere on $\mathbf{C}$.

For $n \in \mathbb{N}$ we consider the sets $B_n := \{z : |z| \le n\}$ and the functions

$$g_n(z) := \begin{cases} g_S(z) & \text{if } g_S(z) \in \mathbf{C} \\ n & \text{if } g_S(z) = \infty. \end{cases}$$

Since g_n is measurable on B_n, Lusin's theorem implies the existence of a measurable set $L_n^* \subset B_n$ with $\mu(B_n \backslash L_n^*) \le \frac{1}{2^{n+1}}$ and a continuous function φ_n on L_n^* such that

$\varphi_n(z) = g_n(z)$ for all $z \in L_n^*$. The set $L_n := L_n^* \setminus \{\mathbb{Q} + i\mathbb{Q}\}$ ($\mathbb{Q}$ is the set of all rational numbers) satisfies $L_n^0 = \emptyset$ and $\mu(L_n) = \mu(L_n^*)$. Since L_n is measurable, we can find a compact set $M_n \subset L_n$ satisfying $\mu(L_n \setminus M_n) \leq \frac{1}{2^{n+2}}$.
The complement of M_n has the representation $M_n^c = \bigcup\limits_{j \in \mathcal{J}_n} G_n^{(j)}$ where $\mathcal{J}_n$ is at most countable set and the components $G_n^{(j)}$ are pairwise disjoint. We choose any point

$$z_n^{(j)} = r_n^{(j)} e^{i\Theta_n^{(j)}} \in G_n^{(j)} \quad \text{with} \quad r_n^{(j)} > 0, \quad \Theta_n^{(j)} \in \mathbb{R}$$

and consider the sector

$$H_n^{(j)} := \left\{ z = z_n^{(j)} + \varrho e^{i\Theta} : 0 < \varrho < 2n, |\Theta - \Theta_n^{(j)}| < \frac{1}{n^2 2^{n+j+4}} \right\}.$$

The set $K_n := B_n \cap \{ \bigcup\limits_{j \in \mathcal{J}_n} (G_n^{(j)} \cup H_n^{(j)}) \}^c$ satisfies $K_n \in \mathcal{M}$ and $K_n \subset M_n \subset L_n$.

Hence $K_n^0 = \emptyset$. We obtain

$$\mu(M_n \setminus K_n) \leq \mu\Big(\bigcup_{j \in \mathcal{J}_n} H_n^{(j)} \Big) \leq \frac{2}{2^{n+4}} \sum_{j=0}^{\infty} \frac{1}{2^j} = \frac{1}{2^{n+2}},$$

and hence $\mu(B_n \setminus K_n) \leq \mu(B_n \setminus L_n^*) + \mu(L_n \setminus M_n) + \mu(M_n \setminus K_n) \leq \frac{1}{2^n}$.

The measurable set $E := \bigcup\limits_{n=1}^{\infty} \bigcap\limits_{\nu=n}^{\infty} K_\nu$ satisfies

$$\mu(B_n \setminus E) \leq \mu\Big(\bigcup_{\nu=n}^{\infty} (B_\nu \setminus K_\nu) \Big) \leq \sum_{\nu=n}^{\infty} \frac{1}{2^\nu} = \frac{1}{2^{n-1}}.$$

Therefore $\mu(\mathbb{C} \setminus E) = \lim\limits_{n \to \infty} \mu(B_n \setminus E) = 0$.
2. The properties of $\{f_n\}$ imply that for every $n \in \mathbb{N}$ there exists an index $m_n \in \mathbb{N}$ satisfying

$$\max_{K_n} |f_{m_n}(z) - \varphi_n(z)| < \frac{1}{n}.$$

For any point $z_0 \in E$, there exists an N_0 such that $z_0 \in K_n$ for all $n \geq N_0$.

If $g_S(z_0) \in \mathbf{C}$ then for all $n \geq N_0$ we get

$$|f_{m_n}(z_0) - g_S(z_0)| \leq \max_{K_n} |f_{m_n}(z) - \varphi_n(z)| < \frac{1}{n}.$$

If $g_S(z_0) = \infty$ we get $|f_{m_n}(z_0)| \geq n - \max_{K_n} |f_{m_n}(z) - \varphi_n(z)| \geq n - \frac{1}{n}$. Therefore

$\{f_{m_n}(z)\}$ converges to $g_S(z)$ for all $z \in E$. Lemma 3 is proved.

Theorem 4. *Let the sequences Q, $\{a_n\}$, $\{b_n\}$, $\{\lambda_n\}$ and the holomorphic function ϕ on $\mathbf{D}$ be the same as in Theorem 2. Then ϕ has also the following properties : For any measurable set $S \subset \mathbf{C}$, any measurable function g on S and any $\zeta \in \partial\mathrm{D}$ there exist subsequences $\{r_k\}, \{s_k\}$ of $\mathbb{N}$ such that*

$$\lim_{k\to\infty} a_{r_k} = 0, \qquad \lim_{k\to\infty} b_{s_k} = \zeta,$$

$$a_{r_k} z + b_{s_k} \in \mathbf{D} \quad \textit{for almost all} \quad z \in S,$$

$$\{\phi(a_{r_k} z + b_{s_k})\} \quad \textit{converges to} \quad g(z) \quad \textit{almost everywhere on} \quad S.$$

If $S \subset \mathbf{D}$, then there exists in addition a subsequence $\{t_k\}$ of $\mathbb{N}$ such that

$$\{\phi^{(\lambda_{t_k})}(z)\} \quad \text{converges to} \quad g(z) \quad \text{almost everywhere on} \quad S.$$

The proof follows by the application of a lemma which is analogous to Lemma 3 and which is proved in a similar way.

Резюме. Построены голоморфные функции на $\mathbf{D}$ и $\mathbf{C}$, универсальные относительно как сдвигов так и производных. Соответствующие универсальные функции определяются лакунарными степенными рядами с пробелами положительной нижней пуассоновской плотностью.

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