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THE GAUSS–BONNET THEOREM FOR STATIONARY RANDOM SIMPLICIAL SURFACES

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Abstract. The paper uses some ideas due to Joseph Mecke to demonstrate that a recent result of Mecke/Stoyan showing, that the theorem of Gauss–Bonnet is true for stationary 1-dimensional networks in $\mathbb{R}^d$, remains true for simplicial surfaces with or without boundaries. Here the theorem of Gauss–Bonnet $\mathcal{K} = 2\pi \cdot \mathcal{X}$ is to be understood in the sense that $\mathcal{K}$ is the mean total Gaussian curvature per unit volume, and $\mathcal{X}$ = the mean Euler characteristic per unit volume. Higher dimensional stationary random simplicial pseudomanifolds are discussed.

INTRODUCTION

The point of departure is a recent result of Mecke/Stoyan [4], where the Gauss–Bonnet theorem is shown for random networks in $\mathbb{R}^d$ in the following sense :

$$\mathcal{K} = 2\pi \cdot \mathcal{X}, \tag{1.1}$$

where $\mathcal{K}$ = the mean total Gaussian curvature per unit volume and $\mathcal{X}$ = the mean Euler characteristic per unit volume. The mean is taken with respect to a stationary, and thus infinitely extended random network in $\mathbb{R}^d$.

In this paper we extend the above result to stationary random simplicial surfaces in $\mathbb{R}^d$ with or without boundaries. To get this result we use the Palm theory of stationary random measures (see [2]), in particular some ideas of Mecke [3] developed for stationary random tesselations of the plane. Our result thus shows that the Gauss–Bonnet theorem remains true for random simplicial surfaces with nontrivial curvatures. In a forthcoming paper [5] we show that this is even true for higher dimensional simplicial complexes and not only in the mean but even individually.

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We stress here the intrinsic character of formula (1.1).

§2. RANDOM COMPLEXES IN $\mathbb{R}^d$

In the following we are always working in $E = \mathbb{R}^d$, the d-dimensional Euclidean space. We denote by $\mathcal{B}_0(E)$ the collection of bounded Borel sets in E.

Simplicial complexes and surfaces in E. Let X be the set of finite subsets x of E which are **affinely independent**. This means that the **affine hull of** x

$$aff\ x = \left\{ \sum_{a \in x} \lambda_a \cdot a \mid \lambda_a \in \mathbb{R}, \quad \sum_{a \in x} \lambda_a = 1 \right\}$$

is different from the affine hull of every proper subset of x. A k-**simplex** is given by the convex hull $\langle x \rangle$ of an affinely independent x with $card\ \ x = k+1$. The **dimension** of x respectively $\langle x \rangle$ is defined by $\dim\ x = \dim\ \langle x \rangle = card\ \ x - 1$. A **configuration** μ of affinely independent elements $x \in X$ is a locally finite point measure in X, i.e. $\mu \in \mathcal{M}^{\cdot}(X)$ if and only if μ is of the form

$$\mu = \sum_{x \in D} \delta_x, \tag{2.1}$$

where $D \subseteq X$ is countable and locally finite in the sense that

$$\zeta_B(D) = card\{z \in D | z \cap B \neq \phi\} < +\infty, \quad B \in \mathcal{B}_0(E). \tag{2.2}$$

(In the following we identify μ and D, thus considering μ as a measure in X or as subset of X.)

A configuration μ in X is called Euclidean simplicial complex or **simplicial complex in** E, if

$$(x \in \mu, y \subseteq x \Rightarrow y \in \mu); \tag{2.3}$$

$$(x, y \in \mu, \langle x \rangle \cap \langle y \rangle \neq \phi \Rightarrow \exists\ z \in X, z \subseteq x \cap y : \langle z \rangle = \langle x \rangle \cap \langle y \rangle). \tag{2.4}$$

Let $\mathcal{S}^{\cdot} = \mathcal{S}^{\cdot}(E)$ denote the set of simplicial complexes in E. The dimension of μ is defined by $\dim \mu = \max\{\dim x | x \in \mu\}$. A general method to construct new complexes from a given one is the following : Given $\mu \in \mathcal{S}^{\cdot}$, let $\nu \subseteq \mu$ be some subset. Then the **closure of** ν is given by

$$cl\ \ \nu = \sum_{x \in \nu} \sum_{y \subseteq x} \delta_y. \tag{2.5}$$

Note that in general $cl\ \ \nu$ is not an element of $\mathcal{M}^{\cdot}(X)$, but from $\mathcal{M}^{\cdot\cdot}(X)$, where $\eta \in \mathcal{M}^{\cdot\cdot}(X)$ if and only if

$$\eta = \sum_{x \in D} n(x) \cdot \delta_x, \tag{2.6}$$

where $D \subseteq X$ is locally finite as above and $n : D \longmapsto \mathbb{N} \setminus \{0\}$ is a function with values being integers.
For instance if $\nu = \{x\}$, $x \in \mu$, then

$$cl\{x\} = \sum_{y \subseteq x} \delta_y. \tag{2.7}$$

The **faces of** $x \in \mu$ are defined by $cl\{y\}, y \subseteq x$. Call an element $x \in \mu$ **maximal** if $(y \in \mu, x \subseteq y \Rightarrow x = y)$. It is evident that each $\mu \in S^{\cdot}$ contains maximal elements x. Consider the subset $\nu = \{x \in \mu : x \text{ maximal}\}$. Then we can consider

$$cl\ \nu = \sum_{\substack{x \in \mu \\ maximal}} cl\{x\}.$$

It is also evident, that in this case $\mu = supp\ cl\ \nu$, where the support of $\eta \in \mathcal{M}^{\cdot\cdot}(X)$ is defined by $supp\ \eta = \{y \in X | \eta\{y\} \geq 1\}$. Another example is the k**-skeleton** μ_k **of** μ, given by

$$\mu_k = \sum_{\substack{x \in \mu \\ \dim x = k}} cl\{x\}. \tag{2.8}$$

The **star and link** of $z \in \mu$ are

$$\mu^z = \sum_{\substack{x \in \mu \\ x \supseteq z}} \delta_x; \tag{2.9}$$

$$lk(z, \mu) = supp\ cl\mu^z \setminus \mu^z. \tag{2.10}$$

In this paper we consider for simplicity the following class $\Gamma \subseteq S^{\cdot}(E)$ of **simplicial surfaces** μ **in** E. Here $\mu \in \Gamma$ if and only if $\mu \in S^{\cdot}(E)$ and μ satisfies the following conditions :

for each $x \in \mu$ maximal $\dim x = 2$;
each edge is contained in one or in two maximal $x \in \mu$;
for each $a \in \mu_0$ $\langle lk(a, \mu)\rangle$ is homomorphic to S^1 or to an interval in $\mathbb{R}^1$.

Note that μ can have a boundary or not.

Random simplicial complexes. We consider in Γ the σ-field $\mathcal{B}_\Gamma$ which is generated by the mapping

$$\zeta : \Gamma \longmapsto \mathbb{N}_0^{\mathcal{B}_0(E)}, \quad \mu \to (B \to \zeta_B(\mu)). \tag{2.11}$$

A probability P on $(\Gamma, \mathcal{B}_\Gamma)$ is called a **random simplicial surface in** E. In the following we consider only **stationary** random simplicial surfaces P in E. This means that P is invariant under the group of transformations in Γ which is induced by the Euclidean translations. We denote these transformations by : $\mu \longmapsto \mu - a$, $a \in E$. In such a situation Mecke's theory of Palm measures can and will be used (see [2]).

§3. PALM MEASURES

Without going into the details we recall briefly some notations and results for later use (see [2], [3]) : We are given a stationary random simplicial surface in E described by the probability space $(\Gamma, \mathcal{B}_\Gamma, P)$. Denote by $b : X \longmapsto E$ the **barycenter** defined by

$$b(x) = \frac{1}{card\ x} \sum_{a \in x} a, \tag{3.1}$$

b is measurable with respect to Matheron's σ-algebra for systems of closed subsets of E (see [1]) and induces a measurable transformation $b : \Gamma \longmapsto \mathcal{M}^{\cdot}(E)$, where $b\mu$, $\mu \in \Gamma$ is the image of μ under b. We use also

$$\mathcal{M}_k^0 = \{\mu \in \Gamma | 0 \in b\mu_{(k)}\}, \quad k = 0, 1, 2. \tag{3.2}$$

Here $\mu_{(k)} = \sum_{x \in \mu, \dim x = k} \delta_x$. Given $\mu \in \mathcal{M}_k^0$ we can consider the **typical k-cell** $x_0^k(\mu)$, i.e. the unique $x \in \mu_{(k)}$ with $0 \in \langle x \rangle$. The **Palm measure of P** is defined by

$$P^\circ(\varphi) = \int_\Gamma \int_E g(a) \cdot \varphi(\mu - a) \ b\mu(da) P(d\mu), \tag{3.3}$$

where $\varphi : \Gamma \to \mathbb{R}_+ \cup \{+\infty\}$ is any measurable numerical function and $g : E \longmapsto \mathbb{R}_+$ is a measurable function such that $\int_E g(a) \ da = 1$. By definition P^0 is a σ-finite measure on Γ which is concentrated on $\mathcal{M}_0^0 \cup \mathcal{M}_1^0 \cup \mathcal{M}_2^0 =: \mathcal{M}^0$, the set of simplicial surfaces having 0 as a barycenter of some of its elements x.

We are mainly interested in **intrinsic properties** of the random simplicial surface. These are properties which depend only on the inner metric of the random surface μ which is induced by the Euclidean metric of E.

Such properties are deduced from the basic invariance property of the measure $b\mu(da)P^0(d\mu)$ under the transformation $(a, \mu) \longmapsto (-a, \mu - a)$. More formally thus is expressed in the

Lemma 1. *For each $f \geq 0$ measurable*

$$\int_\Gamma \int_E f(a, \mu) \ b\mu(da) P^0(d\mu) = \int_\Gamma \int_E f(-a, \mu - a) b\mu(da) P^0(d\mu). \tag{3.4}$$

An application of this invariance property to the function

$$f(a, \mu) = 1_{\mathcal{M}_2^0}(\mu) \cdot 1_{b\mu_{(1)} \cap \langle x_0^2(\mu) \rangle}(a) \cdot \varphi(\mu - a)$$

($\varphi \geq 0$ measurable) yields the following corollary.

Corollary 1. For all $\varphi \geq 0$ measurable

$$\int_{\mathcal{M}_2^0} \int_{\langle x_0^2(\mu)\rangle} \varphi(\mu - a) b\, \mu_{(1)}(da)\, P^0(d\mu) =$$

$$= \int_{\mathcal{M}_1^0 \cap \{n_0^{1,2}=1\}} \varphi(\mu)\, P^0(d\mu) + 2 \cdot \int_{\mathcal{M}_1^0 \cap \{n_0^{1,2}=2\}} \varphi(\mu)\, P^0(d\mu). \tag{3.5}$$

Here for $\mu \in \mathcal{M}_1^0$, $n_0^{1,2}(\mu) = \zeta_{x_0^1(\mu)}(\mu_2)$ denotes the cardinality of 2-cells of μ, which meet the typical edge $x_0^1(\mu)$.

The proof is the same in Mecke's proof of theorem 5.1 in [3]. Since the number of edges of a typical 2-cell $\langle x_0^2(\mu)\rangle, \mu \in \Gamma$, is 3, we obtain directly

$$3 \cdot P^0(\mathcal{M}_2^0) = 2 \cdot P^0(\mathcal{M}_1^0 \cap \{n_0^{1,2} = 2\}) + P^0(\mathcal{M}_1^0 \cap \{n_0^{1,2} = 1\}). \tag{3.6}$$

In particular, if P is a stationary random simplicial surface without boundary, i.e. $P^0(\mathcal{M}_1^0 \cap \{n_0^{1,2} = 1\}) = 0$, then

$$3 \cdot P^0(\mathcal{M}_2^0) = 2 \cdot P^0(\mathcal{M}_1^0). \tag{3.7}$$

The following corollary will be of fundamental importance in the sequel. We need the following notation : If $\mu \in \mathcal{M}_0^0$ denote by μ_2^0 the set of all 2-cells of μ, which have 0 as a vertex ; i.e. $\mu_2^0 = \sum_{0 \in x \in \mu_{(2)}} \delta_x$.

An application of Lemma 1 to the function

$$f(a, \mu) = 1_{\mathcal{M}_2^0}(\mu \cdot 1_{\mu_0 \cap \langle x_0^2(\mu)\rangle}(a) \cdot h(a, \mu),$$

($h \geq 0$ measurable) yields by Mecke's reasoning in [3] the following result.

Corollary 2. For each $h \geq 0$ measurable

$$\int_{\mathcal{M}_2^0} \sum_{a \in x_0^2(\mu)} h(a, \mu)\, P^0(d\mu) = \int_{\mathcal{M}_0^0} \int_E h(-a, \mu - a)\, b\mu_2^0(da)\, P^0(d\mu). \tag{3.8}$$

This immediately implies (take $h \equiv 1$)

$$3 \cdot P^0(\mathcal{M}_2^0) = \int_{\mathcal{M}_0^0} \mu_2^0(E)\, P^0(d\mu), \tag{3.9}$$

i.e. the mean number of 2-cells in the random surface with respect to P^0, which have 0 as a vertex, is 3 times the mean number of 2-cells per unit volume, taken with respect to P^0.

§4. THE GAUSS–BONNET FORMULA FOR STATIONARY RANDOM INFINITELY EXTENDED SURFACES

The main theorem of the present paper proves that a recent theorem of Mecke/Stoyan in [4] given for stationary random 1-dimensional simplicial complexes can be extended to stationary random surfaces with or without boundaries, thus indicating, that this theorem is true more generally for stationary random pseudomanifolds. (This will be done in a forthcoming paper [4]). Its proof is based on the Palm methods presented above.

Let P be a stationary random simplicial surface in E. Consider for $\mu \in \Gamma$ and a given vertex $a \in \mu_0$ the **curvature of** μ **in** a, defined by

$$\mathcal{K}(a,\mu) = \sum_{a \in x \in \mu_{(2)}} \beta(a, x, \mu_2), \tag{4.1}$$

where β is the deficit angle

$$\beta(a,x,\mu_2) = \begin{cases} \frac{2\pi}{card\ \mu_2^a} - \alpha(a,x), & \text{if } \langle lk(a,\mu)\rangle \text{ is a circle} \\ \frac{\pi}{card\ \mu_2^a} - \alpha(a,x), & \text{if } \langle lk(a,\mu)\rangle \text{ is an interval.} \end{cases} \tag{4.2}$$

Here $\alpha(a,x), a \in x$, denotes the (intrinsic positive) angles of x at a. Roughly speaking $\mathcal{K}(a,\mu)$ measures the deviation of μ from the Euclidean surface (tesselation).

An important observation is the following lemma.

Lemma 2. *For $\mu \in \mathcal{M}_0^0$ one has*

$$\mathcal{K}(0,\mu) = \begin{cases} 2\pi - \int_E \gamma(-a,\mu-a)\ b\mu_2^0(da), & \text{if } \mu \in \Gamma \cap \mathcal{M}_0^0 \cap \{\langle lk(0,.)\rangle = circle\}, \\ \pi - \int_E \gamma(-a,\mu-a)\ b\mu_2^0(da), & \text{if } \mu \in \Gamma \cap \mathcal{M}_0^0 \cap \{\langle lk(0,.)\rangle = \ interval\}. \end{cases} \tag{4.3}$$

Here $\gamma(a,\mu)$ denotes the angle of $x_0^2(\mu)$ at a, if $\mu \in \mathcal{M}_2^0$ with $a \in \mu_0 \cap x_0^2(\mu)$.

Proof : follows from the fact that the angle α is translation invariant (in the sense that $\alpha(a+b, x+b) = \alpha(a,x)$ for each $b \in E$), so that one can write

$$\sum_{0 \in x \in \mu_2} \alpha(0,x) = \int_E \gamma(-a,\mu-a)\ b\mu_2^0(da). \tag{4.4}$$

Moreover, from Corollary 2 we have the following lemma.

Lemma 3.

$$\int_{\mathcal{M}_0^0} \int_E \gamma(-a,\mu-a)\ b\mu_2^0(da) P^0(d\mu) = \pi P^0(\mathcal{M}_2^0). \tag{4.5}$$

Proof : We have to show that $\sum_{a \in x_0^2(\mu)} \gamma(a,\mu) = \pi$, if $\mu \in \mathcal{M}_2^0$, which is evident.

Now assume in addition that $0 < P^0(\mathcal{M}_2^0) < +\infty$. This combined with (3.9) implies first $0 < P^0(\mathcal{M}_0^0) < +\infty$. Then lemma 2 and lemma 3 imply that the integral $\mathcal{K}(P) := \int_{\mathcal{M}_0^0} \mathcal{K}(0,\mu)P^0(d\mu)$ is finite and satisfies the equation

$$\mathcal{K}(P) = 2\pi \cdot P^0(\mathcal{M}_0^0 \cap \{\langle lk(0,.)\rangle = \text{circle}\}) +$$
$$+\pi \cdot P^0(\mathcal{M}_0^0 \cap \{\langle lk(0,.)\rangle = \text{interval}\}) - \pi P^0(\mathcal{M}_2^0). \tag{4.6}$$

We call $\mathcal{K}(P)$ the **mean specific total curvature** of P. We now assume in addition that

$$P^0(\mathcal{M}_1^0 \cap \{n_0^{1,2} = 1\}) = P^0(\mathcal{M}_0^0 \cap \{\langle lk(0,.)\rangle = \text{interval}\}). \tag{4.7}$$

This means that the mean number of exterior edges per unit volume equals the mean number of exterior vertices per unit volume. This is the case for random simplicial surfaces without boundary and, more generally, for random simplicial surfaces μ, which are **Delone configurations**, i.e. μ_0 consists of hard balls, and each $x \in \mu_{(2)}$ is contained in a ball of fixed radius. Under the additional assumption (4.7) we obtain from (4.6) combined with (3.6) that

$$\mathcal{K}(P) = \pi[2 \cdot P^0(\mathcal{M}_0^0 \cap \{\langle lk(0,.)\rangle = \text{circle}\}) + P^0(\mathcal{M}_0^0 \cap \{\langle lk(0,.)\rangle = \text{interval}\}) -$$
$$-(2 \cdot P^0(\mathcal{M}_1^0 \cap \{n_0^{1,2} = 2\}) + P^0(\mathcal{M}_1^0 \cap \{n_0^{1,2} = 1\}) - 2 \cdot P^0(\mathcal{M}_2^0))] =$$
$$= 2\pi \cdot [P^0(\mathcal{M}_0^0 \cap \{\langle lk(0,.)\rangle = \text{circle}\}) - P^0(\mathcal{M}_1^0 \cap \{n_0^{1,2} = 2\}) + P^0(\mathcal{M}_2^0)].$$

We now call the alternating sum

$$\mathcal{X}(P) = P^0(\mathcal{M}_0^0 \cap \{\langle lk(0,.)\rangle = \text{circle}\}) - P^0(\mathcal{M}_1^0 \cap \{n_0^{1,2} = 2\}) + P^0(\mathcal{M}_2^0), \tag{4.8}$$

which is well defined for P under the above conditions, the **mean specific Euler characteristic** of P. $\mathcal{X}(P)$ associates to P the mean number of the following alternating cardinalities per unit volume : number of inner vertices – number of inner edges + number of 2-cells.
To summarize we have proven the following version of the theorem of Gauss–Bonnet.

Theorem. *Let P be a stationary random simplicial surface in $\mathbb{R}^d$, $d \geq 2$, satisfying $0 < P^0(\mathcal{M}_2^0) < +\infty$ as well as condition (4.7). Then*

$$\mathcal{K}(P) = 2\pi \cdot \mathcal{X}(P).$$

Резюме. В статье используются некоторые идеи Йозефа Мекке чтобы показать, что недавний результат Мекке/Штояна показывающий, что теорема Гаусса–Бонне верна для стационарных одномерных сетей в $\mathbb{R}^d$, остаётся верной для симплициальных поверхностей с границами или без границ. Здесь теорема Гаусса–Бонне $\mathcal{K} = 2\pi \cdot \mathcal{X}$ понимается в смысле, что $\mathcal{K}$ является средней тотальной гауссовской кривизной в единичном объёме, а $\mathcal{X}$ – средняя характеристика Эйлера в единичном объёме. В настоящей статье рассматриваются многомерные стационарные случайные симплициальные псевдомногообразия.

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