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SPECIFIC INDEX AND CURVATURE OF RANDOM SIMPLICIAL COMPLEXES

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Abstract. The main result is an ergodic theorem for infinitely extended stationary random simplicial complexes, which states the existence of limits almost surely and in $\mathcal{L}^1$. As a consequence we obtain the critical point theorem as well as the Gauss–Bonnet theorem. The existence of the specific Lipschitz–Killing curvature and the specific Einstein–Hilbert energy is shown as well.

§1. INTRODUCTION

The basic objects of a theory of pure gravity formulated in terms of Euclidean path integrals are random Riemannian spaces (M, g), where g denotes the metric of the manifold M. They are realized at random according to some Gibbs measure

$$P(d(M,g)) = \frac{1}{Z} \cdot \exp(-\beta \cdot H(M,g)) \; "d(M,g)", \tag{1.1}$$

where $\beta > 0$ and $"d(M,g)"$ is the analogon of the Lebesgue measure on the space of admissible spaces (M, g). The so-called Einstein–Hilbert action $H(M,g)$ is defined in terms of invariants, such as the volume $Vol(M,g)$ and the curvature $C(M,g)$, which in this model are random variables. The partition function Z is the normalizing constant.

The main difficulties of such a theory are, that H is unbounded, moreover it is not clear how to define the reference measure $"d(M,g)"$ in such a way that $\exp(-\beta H)$ becomes integrable.

Taking the point of view of modern statistical mechanics, as we do, the realizations (M, g) should be non-bounded. Then it is even more difficult to construct P.

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Furthermore, one of the first questions is how to define the volume or the curvature for such infinitely extended random spaces (M, g). One approach is to start from a lattice description and then to perform some scaling limit, as done by Regge [13]. In the present paper we give a formulation in terms of stationary random simplicial complexes μ embedded in a Euclidean space, including for instance simplicial surfaces (see [16]) or networks (see [11]). This concept has been formalized already in a much more general framework in the fundamental paper of M. Zähle [15]. We concentrate on the existence of specific energies, in particular specific Einstein–Hilbert action, specific curvature and Euler characteristic. This means the existence in some sense of ergodic limits of the form

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n|} \cdot E^{\phi}_{\Lambda_n}(\mu), \quad \mu \in \Gamma, \tag{1.2}$$

for suitable sequences $(\Lambda_n)_n$ of bounded Borel exhausting the space in which the random simplicial complex lives. Here $E^{\phi}_{\Lambda}(\mu)$ defines the energy of the complex μ in Λ, given by the interaction ϕ. Now for the first time these notions are made precise in analogy to classical statistical mechanics.

The main result is an ergodic theorem, which states the existence of the limits (1.2) almost surely and in $\mathcal{L}^1$. As a consequence, the critical point theorem and the Gauss–Bonnet theorem for the infinitely extended random simplicial complex, that were known in the mean sense (see [11], [16]), are now proved to be valid almost surely, and for much more general classes of stationary random simplicial complexes.

Finally, the mean specific Einstein–Hilbert energy is expressed by the mean specific volume and the mean specific Lipschitz–Killing curvature.

§2. RANDOM SIMPLICIAL COMPLEXES IN $\mathbb{R}^d$

We consider always cell complexes embedded in $E = \mathbb{R}^d$, $d > 1$, the d-dimensional Euclidean space. Let X denote the set of finite subsets x of E, which are **affinely independent**, i.e. the **affine hull** of x

$$aff\, x = \left\{ \sum_{a\in x} \lambda_a \cdot a : \sum_{a\in x} \lambda_a = 1,\ \lambda_a \in \mathbb{R} \right\}$$

is different from the affine hull of every proper subset of x. A k**-simplex** is given by the convex hull $\langle x \rangle$ of an $x \in X$ with $card\, x = k + 1$. k denotes here the **dimension** of x (resp. $\langle x\rangle$) given by $\dim x = \dim\langle x\rangle = card\, x - 1$. Denote by $\mathcal{M}^{\cdot}(X)$ the set of all **configurations μ of elements $x \in X$**, which are locally finite. This means that μ is of the form

$$\mu = \sum_{x\in D} \delta_x, \tag{2.1}$$

where $D \subseteq X$ is countable and locally finite in the sense, that

$$\zeta_B(D) := card\{z \in D : \langle z\rangle \cap B \neq \phi\} < +\infty \quad \text{for any } B \in \mathcal{B}_0(E). \tag{2.2}$$

Here $\mathcal{B}_0(E)$ denotes the set of open, bounded Borel sets in E. (In the following we always identify μ and its support D, thus considering μ as a measure in X as well as a subset of X.)

A configuration $\mu \in \mathcal{M}^{\cdot}(X)$ is called **simplicial complex in E**, if

$$(x \in \mu, y \subseteq x \Rightarrow y \in \mu), \tag{2.2}$$

$$(x, y \in \mu, \langle x\rangle \cap \langle y\rangle \neq \phi \Rightarrow \exists z \in X, z \subseteq x \cap y : \langle z\rangle = \langle x\rangle \cap \langle y\rangle). \tag{2.3}$$

Denote by $\mathcal{S}^{\cdot}(E)$ the set of all simplicial complexes in E. The **dimension of μ** is $\dim\mu = \max_{x\in\mu}\dim x$. It is obvious that $\dim\mu \leq d$ for each $\mu \in \mathcal{M}^{\cdot}(X)$. The **$k$-skeleton of μ** is $\mu_k = \sum_{x\in\mu,\dim x = k}\delta_x$. μ_0 is the set of vertices a of μ; μ_1 the set of edges e of μ. Elements $x \in \mu$ are called **cells**. If $\dim x = k$, x is called a **k-cell**. A cell $x \in \mu$ is **maximal**, if $(y \in \mu, y \subseteq x \Rightarrow y = x)$. It is clear that each $\mu \in \mathcal{S}^{\cdot}(E)$ contains maximal elements. The **barycenter** is the mapping $b : X \to E$, defined by

$$b(x) = \frac{1}{card\, x}\sum_{a\in x} a, \quad x \in X. \tag{2.4}$$

b induces a transformation $b : \mathcal{S}^{\cdot}(E) \to \mathcal{M}^{\cdot}(E)$, $\mu \longmapsto b\mu$, where $b\mu$ is the image of μ under b. Here $\mathcal{M}^{\cdot}(E)$ denotes the set of all locally bounded point configurations in E. A subset $\nu \subseteq \mu, \mu \in \mathcal{S}^{\cdot}(E)$, is called a subcomplex of μ, if $(x \in \nu, y \subseteq x \Rightarrow y \in \nu)$. We denote this by $\nu \leq \mu$. Let $\mathcal{K}$ be a subset of $\mathcal{S}^{\cdot}(E)$. Take in $\mathcal{K}$ the σ-field $\mathcal{B}_{\mathcal{K}}$ generated by the mapping

$$\zeta : \mathcal{K} \to \cup_{\mu\in\mathcal{K}}\mathbb{N}_0^{\mathcal{B}_0(E)}, \mu \longmapsto (B \longmapsto \zeta_B(\mu)). \tag{2.5}$$

A **random simplicial complex** (RASC) **in E of type $\mathcal{K}$** is defined as a probability measure P on $(\mathcal{K}, \mathcal{B}_{\mathcal{K}})$. The probability space $(\mathcal{K}, \mathcal{B}_{\mathcal{K}}, P)$ is called **stationary** if $\mathcal{K}$ as well as P are invariant under the group of transformations, which are induced by the Euclidean translations. We denote them by $\mu \to \mu - a := \sum_{x\in\mu} \delta_{x-a}$, where $x - a$ is the translation of x by means of $a \in E$.

In the sequel we consider only stationary RASCs. The $0 - \infty$ law of stochastic geometry implies for them that a typical configuration consists of an infinity of cells and thus is unbounded in E because it is locally finite.

Remark. Associated with a RASC P of type $\mathcal{K}$ is the notion of **underlying random piecewise linear space in** E defined by the mapping $\langle\cdot\rangle : \mathcal{K} \to \mathcal{F}(E)$, $\mu \longmapsto \bigcup_{\substack{x\in\mu \\ maximal}} \langle x\rangle$. $\mathcal{F}(E)$ denotes the set of closed subsets of E. Thus, $\langle\mu\rangle = \bigcup_{x\in\mu\ maximal}\langle x\rangle$ is considered always as a topological space, its topology being inherited from E. In $\mathcal{F}(E)$ we consider the σ-field $\mathcal{B}_{\mathcal{F}(E)} = \{F \subseteq \mathcal{F}(E) : \langle\cdot\rangle^{-1}(F) \in \mathcal{B}_{\mathcal{K}}\}$. We call the random variable $\langle\cdot\rangle$, defined on $(\mathcal{K}, \mathcal{B}_{\mathcal{K}}, P)$ **random piecewise linear space of** P as well as the image $P_{\langle\cdot\rangle}$ of P under $\langle\cdot\rangle$.

§3. POTENTIAL AND ENERGY

Consider the set

$$\mathcal{D} = \{(\nu, \mu) : \mu \in \mathcal{S}^{\cdot}(E),\ \nu \le \mu \text{ finite}\} \tag{3.1}$$

provided with the σ-field $\mathcal{B}_{\mathcal{D}}$ induced by $\mathcal{B}_{\mathcal{S}^{\cdot}(E)}$ in a natural way. A measurable function $\phi : \mathcal{D} \to \mathbb{R} \cup \{+\infty\}$ is called (interaction) **potential**, if ϕ is stationary with respect to translations, i.e.

$$\phi(\nu - a, \mu - a) = \phi(\nu, \mu) \text{ for each } a \in E \text{ and } (\nu, \mu) \in \mathcal{D}. \tag{3.2}$$

$\phi(\nu, \mu)$ is a real number (or $+\infty$), associated to the finite subcomplex ν of μ, which represents an intrinsic metric characteristic of ν in μ. It depends only on μ and not on the space E into which μ is embedded.

We now take the point of view of classical statistical mechanics. There the potential ϕ is the point of departure. All properties of the system have to be deduced from ϕ. What are the additional properties of ϕ which allow such an analysis ? First of all one needs that the **energy of a typical cell** converges for a large class Γ of configurations is a suitable sense. This energy is formally defined by the series

$$E^{\phi}(o|\mu) = \sum_{\substack{\nu\le\mu \\ o\in b\nu}}^{\bullet} \phi(\nu, \mu), \mu \in \mathcal{M}^0, \tag{3.3}$$

where $\mathcal{M}^{\circ} = \{\mu \in \mathcal{S}^{\cdot}(E) | 0 \in b\mu\}$. ($\sum^{\bullet}_{\nu\le\mu}$ means the sum over all finite subcomplexes ν in μ.) In general $E^{\phi}(o, \mu), \mu \in \mathcal{M}^0$, will be divergent for a given ϕ. Therefore, we will consider spaces of configurations of the following type : $\Gamma \subseteq \mathcal{S}^{\cdot}(E)$, measurable, translation invariant, satisfying the following two properties :

$$||\phi|| = \sup_{\mu \in \Gamma \cap \mathcal{M}^{\circ}} \dot{\sum_{\nu \leq \mu, 0 \in b\mu}} |\phi(\nu, \mu)| < +\infty; \tag{3.4}$$

for each $\varepsilon > 0$ there exists $\Lambda \in \mathcal{B}_0(E)$, such that

$$\sup_{\mu \in \Gamma \cap \mathcal{M}^{\circ}} \dot{\sum_{\substack{\nu \leq \mu, 0 \in b\nu \\ b\nu \cap \Lambda^c \neq \phi}}} |\phi(\nu, \mu)| < \varepsilon. \tag{3.5}$$

Within the class of sets Γ having these properties we then choose Γ sufficiently large and take the trace $\mathcal{B}_\Gamma = \Gamma \cap \mathcal{B}_{\mathcal{S}^{\cdot}}$ as a σ-field. We then say that Γ **has been chosen tempered for** ϕ.

Definition. *If ϕ is a potential and Γ is chosen tempered for ϕ, the* **energy** *of μ in $\Lambda \in \mathcal{B}_0(E)$ is*

$$E_\Lambda = E_\Lambda^\phi(\mu) = \begin{cases} \dot{\sum}_{\nu \leq \mu, b\nu \cap \Lambda \neq \phi} \phi(\nu, \mu), & \text{if } \mu \in \Gamma, \\ +\infty, & \text{if } \mu \notin \Gamma. \end{cases} \tag{3.6}$$

Because of (3.4) E_Λ is well defined and finite on Γ, because

$$|E_\Lambda(\mu)| \leq \sum_{a \in b\mu \cap \Lambda} \dot{\sum_{\substack{\nu \leq \mu, \\ a \in b\nu}}} |\phi(\nu, \mu)| \leq b\mu(\Lambda)||\phi|| < +\infty, \quad \mu \in \Gamma.$$

In particular, E_Λ is **stable** on Γ, i.e. there exists $B \geq 0$ such that for each $\mu \in \Gamma$ $E_\Lambda(\mu) \geq -B \cdot b\mu(\Lambda)$. This also shows that E_Λ is P-integrable if $P \in \mathcal{P}\Gamma$ (i.e. P is a probability on $(\Gamma, \mathcal{B}_\Gamma)$) and P **is of first order**, i.e. the image P_b of P under b has the property that the associated moment measure

$$\nu_b^1(B) = \int_\Gamma b\mu(B)\, P(d\mu), \quad B \in \mathcal{B}_E \tag{3.7}$$

is locally finite, i.e. finite on $\mathcal{B}_0(E)$.

We now consider examples of potentials. They are given by counting-, volume-, and curvature measures and polynomials of these measures. The first two are based on the work of Banchoff [1], M. Zähle [15] and Cheeger et al. [4]. The third one is due to Regge [13].

Example 1. Index. **The index of a simplicial complex** $\mu \in \mathcal{S}^{\cdot}(E)$ **in a vertex** $a \in \mu_0$ **for the direction** $\mathbf{n} \in S^{d-1}$ is

$$\iota_{\mathbf{n}}(a,\mu) = \sum_{q=0}^{\dim \mu} (-1)^q \cdot \sum_{a \in x \in \mu_q} 1_{H_{\mathbf{n}}}(x-a), \tag{3.8}$$

where $H_{\mathbf{n}} = \{y \in X : \ 0 \in y, \mathbf{n} \cdot b < 0, \quad b \in y \setminus \{0\}\}$ denotes the half space in the set of all $y \in X$, $0 \in y$, determined by $\mathbf{n}$. ($\mathbf{n} \cdot b$ denotes the scalar product of $\mathbf{n}$ and b in E.)

If we define $\iota_{\mathbf{n}}(\nu,\mu) = 0$ for finite subcomplexes $\nu \subsetneq \mu$, which do not reduce to a vertex, $\iota_{\mathbf{n}}$ defines a potential. The energy of a typical vertex is given by $\iota_{\mathbf{n}}(0,\mu), \mu \in \mathcal{M}^0$; it is not bounded as a function of μ. A possible choice of a tempered $\Gamma \subseteq \mathcal{S}^{\cdot}(E)$ is the set of **Delone configurations** in $\mathcal{S}^{\cdot}(E)$, denoted by $\mathcal{S}^{\cdot}_{r,R}(E)$ for given parameters $0 < r < R < +\infty$. Here $\mu \in \mathcal{S}^{\cdot}_{r,R}(E)$ if and only if

$$\mu_0 \in \mathcal{M}_r^{\cdot}(E) = \{\eta \in \mathcal{M}^{\cdot}(E) : a, b \in \eta, a \neq b \Rightarrow |a-b| \geq \eta\},$$

and each maximal $x \in \mu$ is contained in a ball of fixed radius R. (The concept of a Delone configuartion goes back to Hilbert/Cohn–Vossen [7]).

We call the associated energy $E_\Lambda^{\iota_{\mathbf{n}}}(\mu)$ **the index of** μ **in** Λ **for the direction** $\mathbf{n}$. It is connected with the Euler characteristic by the following relation : For any pair $(\mathbf{n},\mu)$ for which $\mathbf{n}$ is in general position for μ (i.e. $\forall a \in \mu_0$, $\forall a \in x \in \mu$, $\forall b \in x \setminus \{a\}$: $\mathbf{n} \cdot (b-a) \neq 0$.)

$$E_\Lambda^{\iota_{\mathbf{n}}}(\mu) = \mathcal{X}(\mu_\Lambda) - \sum_{a \in (\mu_\Lambda)_0 \cap \Lambda^c} \iota_{\mathbf{n}}(a,\mu), \tag{3.9}$$

where $\mu_\Lambda = \sum_{x \cap \Lambda \neq \phi, x \in \mu,} \sum_{y \subseteq x} \delta_y$, and the Euler characteristic of this complex is

$$\mathcal{X}(\mu_\Lambda) = \sum_{q=0}^{\dim \mu} (-1)^q \cdot |\mu_{\Lambda,q}|. \tag{3.10}$$

Note that $\mathcal{X}_\Lambda : \mu \longmapsto \mathcal{X}(\mu_\Lambda)$ is an intrinsic quantity whereas $E_\Lambda^{\iota_{\mathbf{n}}}$ is not. Relation (3.9) says that $E_\Lambda^{\iota_{\mathbf{n}}}$ differs from $\mathcal{X}_\Lambda$ only by some boundary term. We observe also that $\mathcal{X}_\Lambda$ can be considered as an energy $E_\Lambda^\pi(\mu)$, for which the potential π is concentrated in the vertices and given by

$$\pi(a,\mu) = \sum_{q=0}^{\dim \mu} (-1)^q \cdot \frac{|\mu_q^a|}{|(\mu_q^a)_0 \cap \Lambda|}. \tag{3.11}$$

Here μ_q^a, $a \in \mu_0$ denotes the configuration of all q-cells $x \in \mu$ with $a \in x$.

Example 2. Curvature. Integration of the index ι_n with respect to the normalized surface measure $\sigma^{d-1}(dn)$ on the d-dimensional unit sphere S^{d-1} immediately gives another important example of a potential :

$$\phi_c(\nu,\mu) = \int_{S^{d-1}} \iota_n(\nu,\mu)\,\sigma^{d-1}(dn), \quad \nu \leq \mu \text{ finite.} \tag{3.12}$$

Explicitely : $\phi_c(\nu,\mu) = 0$ if ν is not a vertex of μ, and

$$\phi_c(a,\mu) = \sum_{q=0}^{\dim\mu} (-1)^q \cdot \sum_{a\in x\in\mu_q} \varepsilon(a,x), \quad a \in \mu_0, \tag{3.13}$$

where $\varepsilon(a,x) = \int_{S^{d-1}} 1_{H_n}(x-a)\,\sigma^{d-1}(dn)$. $\varepsilon(a,x)$ is the so called **normalized exterior angle of the q-cell x at its vertex a**. Observe that $\varepsilon(a,\{a\}) = 1$; moreover, if $q = 1$ and $a \in x \in \mu_1$, then $\varepsilon(a,x) = \frac{1}{2}$. Thus, in the case that μ is a **network**, i.e. if $\dim\mu = 1$, then $\phi_c(a,\mu) = 1 - \frac{1}{2}|\mu_1^a|$, if $a \in \mu_0$.

In the case where μ is a **simplicial surface with or without boundary** (for a definition see Zessin [16]), $\dim\mu = 2$ and each maximal cell $x \in \mu$ is 2-dimensional. If $a \in x \in \mu_2$, then $\varepsilon(a,x) = \frac{1}{2\pi}(\pi - \alpha(a,x))$. where $\alpha(a,x)$ denotes the angle of x in a. Thus

$$\phi_c(a,\mu) = 1 - \frac{1}{2}|\mu_1^a| + \left(\frac{1}{2}|\mu_2^a| - \frac{1}{2\pi}\sum_{a\in x\in\mu_2} \alpha(a,x)\right).$$

If a is not a boundary vertex of μ, then $|\mu_1^a| = |\mu_2^a|$. In this case

$$\phi_c(a,\mu) = \frac{1}{2\pi}\left[2\pi - \sum_{a\in x\in\mu_2} \alpha(a,x)\right].$$

If a is a boundary vertex of μ, we use the relation $|\mu_1^a| = |\mu_2^a| + 1$ to get

$$\phi_c(a,\mu) = \frac{1}{2\pi}\left[\pi - \sum_{a\in x\in\mu_2} \alpha(a,x)\right].$$

To summarize : up to the factor $\frac{1}{2\pi}$ the potential $\phi_c(a,\mu)$ is the deficit angle of μ in a. Thus, in the case of networks respectively simplicial surfaces ϕ_c measures the curvature of μ in a. Therefore, the associated energy $E_\Lambda^{\phi_c}(\mu)$ of μ in Λ, considered on the tempered set $\Gamma = \mathcal{S}_{r,R}^{\cdot}(E)$ of Delone configurations, is well defined and is called the **curvature of μ in Λ**. It is an intrinsic quantity since the normalized exterior angle has this property as shown in Banchoff [1].

Example 3. Regge curvature and Einstein–Hilbert action.
We present these notions here in a special case and discuss the general situation in §6. Let Γ_2 denote the set $\mathcal{S}_{r,R,2}^{\cdot}(E)$ of Delone configurations $\mu \in \mathcal{S}_{r,R}^{\cdot}(E)$ satisfying the property ($x \in \mu$ maximal $\Rightarrow \dim x = 2$). The **Regge potential** is defined to be

$$\phi_R = -\left[\phi_c - \phi_v\right], \tag{3.14}$$

where $\phi_v(\nu,\mu)$ is the volume of the star of ν in μ. More precisely $\phi_v(\nu,\mu) = 0$ if ν is not a vertex of μ, and, if $a \in \mu_0$,

$$\phi_v(a,\mu) = \sum_{x \in \mu_2^a} \lambda^2(\langle x \rangle). \tag{3.15}$$

Γ_2 is tempered for ϕ_R. The associated energy $E_\Lambda^{\phi_R}(\mu)$ is Regge's simplicial version of the **Einstein–Hilbert action** (see Einstein [5], Hilbert [8] and Weyl [14]). It is an intrinsic quantity.

Example 4. Network curvature. In analogy with example 3 one can consider the following network curvature on the tempered set $\Gamma = \mathcal{S}_{r,R;1}^{\cdot}(E)$:

$$\phi_{nc} = -\left[\phi_c - \phi_l\right], \tag{3.16}$$

where $\phi_l(\nu,\mu) = 0$ if ν is not a vertex of μ, and, if $a \in \mu_0$,

$$\phi_l(a,\mu) = \sum_{e \in \mu_1^a} \lambda^1(\langle e \rangle). \tag{3.17}$$

4. SPECIFIC ENERGY

Let ϕ be a potential and $\Gamma \subseteq \mathcal{S}^{\cdot}(E)$ tempered for ϕ. Furthermore let P be a stationary probability on $(\Gamma, \mathcal{B}_\Gamma)$; write then $P \in \mathcal{P}_0\Gamma$. On $(\Gamma, \mathcal{B}_\Gamma, P)$ the energy E_Λ^ϕ is a random variable indexed by $\Lambda \in \mathcal{B}_0(E)$. The following main result of the present paper is an ergodic theorem for the energy. Its proof uses the same idea as Nguyen, Zessin [12], Satz 4 (which in turn is based on an idea of Föllmer [6]). In its formulation $(\Lambda_n)_n$ denotes an increasing sequence Λ_n of centered open balls in E satisfying $\Lambda_n \nearrow E$ if $n \to +\infty$.

Theorem. *Under the additional assumption*

$$P_b(\mathcal{M}_r^0(E)) = 1 \quad \text{for some } r > 0, \tag{4.1}$$

the following limit

$$e = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \cdot E^{\phi}_{\Lambda_n} \tag{4.2}$$

exists P-a.s. as well as in $\mathcal{L}^1(P)$ and is given by

$$e = E_P\left(\int_D g(\mu - a)b(\mu)\,(da)|\mathcal{J}\right).$$

Here $D \in \mathcal{B}_0(E)$ is some fixed domain satisfying $|D| = 1$, and $\mathcal{J}$ denotes the sub-σ-field of $\mathcal{B}_\Gamma$ consisting of all events, which are translation-invariant. In particular, e is invariant under translations and

$$E_P(e) = P^0(g), \tag{4.3}$$

where P^0 denotes the **Palm measure** *of P with respect to the barycenters, and*

$$g(\mu) = \dot{\sum_{\nu\leq\mu, 0\in b\nu}} \frac{\phi(\nu,\mu)}{|\nu|}, \mu \in \Gamma\cap\mathcal{M}^0. \tag{4.4}$$

Proof. 1. Decomposition of $E_\Lambda = E^{\phi}_\Lambda$: Consider

$$E_\Lambda(\mu) = \dot{\sum_{\nu\leq\mu, b\nu\subseteq\Lambda}} \phi(\nu,\mu) + \dot{\sum_{\substack{b\nu\cap\Lambda\neq\phi, p\nu\cap\Lambda^c\neq\phi \\ \nu\leq\mu}}} \phi(\nu,\mu).$$

The first term on the right hand side can be represented as

$$E^1_\Lambda(\mu) = \sum_{a\in b\mu\cap\Lambda} \dot{\sum_{\nu\leq\mu, a\in b\nu\subseteq\Lambda}} \frac{\phi(\nu,\mu)}{|b\nu|}.$$

The idea now is to approximate E_Λ by means of

$$g_\Lambda(\mu) = E^{(1)}_\Lambda(\mu) + \sum_{a\in b\mu\cap\Lambda} \dot{\sum_{\substack{\nu\leq\mu, a\in b\nu \\ b\nu\cap\Lambda^c\neq\phi}}} \frac{\phi(\nu,\mu)}{|b\nu|}.$$

An important observation is that $g_\Lambda(\mu) = \int_\Lambda g(\mu - a)b\,\mu(da)$, where g is defined by (4.4).

2. The distance of g_Λ from E_Λ is

$$\sup_{\mu\in\Gamma} |E_\Lambda(\mu) - g_\Lambda(\mu)| \leq 2\Delta(\Lambda), \tag{4.5}$$

where

$$\Delta(\Lambda) = \sup_{\mu\in\Gamma} \sum_{a\in b\mu\cap\Lambda} \dot{\sum_{\substack{\nu\leq\mu, a\in b\nu, \\ b\nu\cap\Lambda^c\neq\phi}}} |\phi(\nu,\mu)|.$$

3. Asymptotic behaviour of $\frac{1}{|\Lambda|}\cdot\Delta(\Lambda)$ as $\Lambda = \Lambda_n \nearrow E$. For $\varepsilon > 0$ choose a ball $\mathcal{K}_h(0)$ of radius h centered at 0, such that

$$\sup_{\mu\in\Gamma\cap\mathcal{M}^0} \dot{\sum_{\substack{\nu\leq\mu, 0\in b\nu \\ b\nu\cap\mathcal{K}_h^c(0)\neq\phi}}} |\phi(\nu,\mu)| < \varepsilon.$$

Here we use (3.5). On the other hand, we estimate $\Delta(\Lambda)$, using the decomposition

$$= \sum_{a\in b\nu\cap\Lambda} \left[\dot{\sum_{\substack{a\in b\nu, \nu\leq\mu \\ b\nu\cap\Lambda^c\neq\phi, b\nu\subseteq\mathcal{K}_h(a)}}} |\phi(\nu,\mu)| + \dot{\sum_{\substack{\nu\leq\mu, a\in b\nu \\ b\nu\cap\mathcal{K}_h^c(a)\neq\phi, b\nu\cap\Lambda^c\neq\phi}}} |\phi(\nu,\mu)| \right],$$

where $\mathcal{K}_h(a) = \mathcal{K}_h(0) + a$. This yields

$$\Delta(\Lambda) \leq \sup_{\mu\in\Gamma} \left[b\mu(\partial_h\Lambda)\cdot||\phi|| + b\mu(\Lambda)\cdot\varepsilon\right],$$

where $\partial_h\Lambda = \{a \in E | d(a,\partial\Lambda) \leq h\}$. Using assumption (4.1) this can be further estimated from above by

$$\leq \frac{1}{c_0}\cdot\left[\partial_{2r+h}\Lambda\cdot||\phi|| + |\Lambda\cup\partial_r\Lambda|\cdot\varepsilon\right].$$

Here $c_0 = |\mathcal{K}_{r/2}(0)|$. Since the $\Lambda = \Lambda_n$ are balls exhausting E, we have

$$\frac{|\partial_{2r+h}\Lambda_n|}{|\Lambda_n|} \longrightarrow 0, \quad \frac{|\Lambda_n\cup\partial_r\Lambda_n|}{|\Lambda_n|} \longrightarrow 1, \quad n\to\infty.$$

Thus $\frac{1}{|\Lambda_n|}\Delta(\Lambda_n) \to 0$.

On account of (4.5) this implies that $\frac{1}{|\Lambda_n|}E_{\Lambda_n}$ and $\frac{1}{|\Lambda_n|}g_{\Lambda_n}$ have the same limiting behaviour, if one of them is convergent. It is therefore enough to analyse $\frac{1}{|\Lambda_n|}g_{\Lambda_n}$.

4. The limiting behaviour of $\frac{1}{|\Lambda_n|} g_{\Lambda_n}$.
This is well known and can be found for instance in [12] : Under the condition that $g \in \mathcal{L}^1(P^0)$ the limit

$$e = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} g_{\Lambda_n}$$

exists P-a.s. and in $\mathcal{L}^1(P)$ and is given by $e = E_P(\int_D g(\mu - a) b(\mu)(da) | \mathcal{J})$. Moreover, e is translation invariant and $E_P(e) = P^0(g)$. Thus, it remains to show that $g \in \mathcal{L}^1(P^0)$. But this follows from the fact that P^0 is a finite measure on Γ (on account of (4.1)), combined with the fact that $|g| \leq ||\phi|| < +\infty$.
We remark, that if in the situation of the theorem P is also ergodic with respect to the group of translations (i.e. $\mathcal{J}$ contains only events of probability 0 and 1), then the specific energy e is P-a.s. constant and equals $P^0(g)$.

§5. THE GAUSS–BONNET THEOREM AND THE CRITICAL POINT THEOREM

Here we discuss some consequences of Theorem. Consider the potentials $\iota_{\mathbf{n}}$ and ϕ_c for the tempered set $\Gamma = \mathcal{S}_{r,R}(E)$ of Delone configurations. If $P \in \mathcal{P}^0\Gamma$ is the underlying law, the theorem immediately yields the existence of the **specific index of μ in the direction n**

$$\iota(\mathbf{n}, \mu) = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \cdot E_{\Lambda_n}^{\iota_{\mathbf{n}}}(\mu), \mu \in \Gamma, \mathbf{n} \in S^{d-1}, \tag{5.1}$$

as well as the existence of the **specific curvature**

$$\kappa(\mu) = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \cdot E_{\Lambda_n}^{\phi_c}(\mu), \mu \in \Gamma. \tag{5.2}$$

Both limits exist P-a.s. and in $\mathcal{L}^1(P)$ and are invariant under translations. We discuss the relation of these two quantities to the Euler characteristic. To do this we assume in addition that the set $\mathcal{C} = \{(\mathbf{n}, \mu) |\ \mathbf{n}$ is in general position with respect to $\mu\}$ is full with respect to $\sigma^{d-1} \otimes P$. In this case (3.9) is valid $\sigma^{d-1} \otimes P$-a. s. in $(\mathbf{n}, \mu)$, and therefore

$$|E_\Lambda^{\iota_{\mathbf{n}}}(\mu) - \mathcal{X}_\Lambda(\mu)| \leq \bar{\Delta}(\Lambda) := \sup_{(\mathbf{n},\mu)\in \mathcal{C}} \sum_{a \in (\mu_\Lambda)_0 \cap \Lambda^c} |\iota_{\mathbf{n}}(a, \mu)| \tag{5.3}$$

is true $\sigma^{d-1} \otimes P$-a.s.. Integration with respect to σ^{d-1} yields immediately

$$|E_\Lambda^{\phi_c}(\mu) - \mathcal{X}_\Lambda(\mu)| \subseteq \bar{\Delta}(\Lambda) \ \ P - a.s.[\mu]. \tag{5.4}$$

On the other hand, it is clear that $\lim_{n\to\infty} \frac{1}{|\Lambda_n|} \bar{\Delta}(\Lambda_n) = 0$ because this is a boundary term. Therefore, the theorem implies the existence of the **specific Euler characteristic** of μ :

$$\mathcal{X}(\mu) = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \cdot \mathcal{X}_{\Lambda_n}(\mu) \quad P-a.s. \text{ and in } \mathcal{L}^1(P), \tag{5.5}$$

as well as the relations

$$\mathcal{X} = \iota \ \ \sigma^{d-1} \otimes P-a.s. \quad \text{and} \tag{5.6}$$

$$\mathcal{X} = \kappa \quad P-a.s.. \tag{5.7}$$

In particular we have $\kappa = \iota \ \sigma^{d-1} \otimes P$-a.s.. Relation (5.6) is an individual version for infinitely extended random simplicial complexes of the so called **critical point theorem** in global differential geometry, and (5.7) of the **Gauss–Bonnet theorem**. The existence of the mean specific Euler characteristic can be found already in Leistritz/Zähle [9]. Moreover, under the additional condition that $P\{\mu \in \Gamma : x \in \mu$ maximal $\Rightarrow \dim x = p\} = 1$ we get the mean value relations

$$E_P(\mathcal{X}) = \sum_{q=0}^{p} (-1)^q P^0_{\mathcal{M}^0_q} \left(\sum_{0 \in x \in \mu_q} 1_{H_n}(x-a) \right) \quad \sigma^{d-1} - a.s.[\mathrm{n}], \quad \text{and} \tag{5.8}$$

$$E_P(\chi) = \sum_{q=0}^{p} (-1)^q P^0_{\mathcal{M}^0_q} \left(\sum_{0 \in x \in \mu_q} \varepsilon(0,x) \right). \tag{5.9}$$

They are contained in Theorem 3.3.6 of Zähle [15]. In the special case of a **random network**, i.e. if $\dim \mu = p = 1$ P-a.s., equation (5.9), combined with the corresponding remarks in Example 2, reduces to

$$E_P(\mathcal{X}) = P^0(\mathcal{M}^0_0) - \frac{1}{2} \cdot P^0_{\mathcal{M}^0_0}(|\mu^0_1|). \tag{5.10}$$

Since $\kappa = \mathcal{X}$ P-a.s., we also have in this case that $E_P(\kappa) = P^0(\mathcal{M}^0_0) - \frac{1}{2} P^0_{\mathcal{M}^0_0}(|\mu^0_1|)$. This result has recently been obtained by Mecke/Stoyan [11].

In the case of random simplicial surfaces in the sense of [16], equation (5.9) yields

$$P^0_{\mathcal{M}^0_0}\ (\text{deficit angle of } \mu \text{ in its vertex } 0)\ = 2\pi \cdot E_P(\mathcal{X}), \tag{5.11}$$

if we take into account the observation of Example 2 above. The left hand side of (5.11) was called in [16] the mean specific total curvature of P and denoted there by $\mathcal{K}(P)$. Note also that in view of (3.11) combined with the theorem, $E_P(\mathcal{X})$ equals the mean specific Euler characteristic $\chi(P)$ of P as defined in [16]. Thus equation (5.7) and (5.9) generalize considerably the recent results in [11, 16].

§6. SPECIFIC EINSTEIN–HILBERT ENERGY

Here we consider the Regge potential ϕ_R for the tempered set $\Gamma_2 = \mathcal{S}_{r,R,2}(E)$ of Delone configurations $\mu \in \mathcal{S}_{r,R}(E)$ as defined above. If $P \in \mathcal{P}^0\Gamma$ is the given law then the theorem implies P-a.s. and in $\mathcal{L}^1(P)$ the existence of the **specific Einstein–Hilbert energy**

$$\eta = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} E^{\phi_R}_{\Lambda_n}. \tag{6.1}$$

η is invariant under translations, and its mean is given by

$$P^0_{\mathcal{M}^0_0}(\phi_R(0,.)) = -\left[P^0_{\mathcal{M}^0_0}(\phi_c(0,.)) - P^0_{\mathcal{M}^0_0}(\phi_v(0,.))\right]. \tag{6.2}$$

This expresses the mean specific Einstein–Hilbert energy by means of the mean specific volume and the mean specific curvature.

§7. GENERALIZATIONS

Until now we considered only curvatures which are concentrated in the vertices of a given complex. We now develop the general notion of index and curvature along the lines of Banchoff [1] (see also Cheeger et al. [4] and Budach [2]). Then we indicate shortly some generalizations of results obtained above.

The index of a simplicial complex $\mu \in \mathcal{S}(E)$ in a cell $x \in \mu$ for the direction n$\in S^{d-1}(E)$ is now defined by

$$I_{\mathbf{n}}(x,\mu) = \sum_{x\subseteq y\in\mu} (-1)^{dim\, y - dim\, x} \cdot 1_{H_{\mathbf{n}}}(y - b(x)), \tag{7.1}$$

where

$$H_{\mathbf{n}} = \{y \in X : 0 \in b(y), \mathbf{n}\cdot b = 0 \text{ for any } b \in x_0(y), \mathbf{n}\cdot b < 0 \text{ for any } b \in y \setminus x_0(y)\}.$$

Here $x_0(y)$ denotes the unique $z \subseteq y$ with $0 \in b(z)$. We then can define a potential $I_{\mathbf{n}}(\nu, \mu), \nu \leq \mu$ finite, as follows : If ν is not of the form $\bar{x} = \sum_{y \subseteq x} \delta_y, x \in X$, then $I_{\mathbf{n}}(\nu, \mu) = 0$. Otherwise $I_{\mathbf{n}}(\bar{x}, \mu) = I_{\mathbf{n}}(x, \mu)$. It is obvious that the set Γ of Delone configurations $S_{r,R}(E)$ is tempered for $I_{\mathbf{n}}$. Thus the associated energy $E_{\Lambda}^{I_{\mathbf{n}}}(\mu)$ is well defined for $\mu \in \Gamma, \Lambda \in \mathcal{B}_0(E)$.

We then obtain the notion of curvature if we integrate $I_{\mathbf{n}}(\nu, \mu)$ with respect to $\mathbf{n}$ in the right way :

If $\nu \leq \mu$ is a finite subcomplex of $\mu \in S^{\cdot}(E)$, which is not of the form $\bar{x}, x \in \mu$, then we set $\phi_{\mathcal{CK}}(\nu, \mu) = 0$. If $x \in \mu$ with $\dim x \geq 1$, we define the **Chern–Gauss–Bonnet potential** of $\bar{x}$ in μ by

$$\phi_{\mathcal{CK}}(\bar{x}, \mu) = \mathcal{H}^{\dim x}(\langle x \rangle) \cdot \sum_{x \subseteq y \in \mu} (-1)^{\dim y - \dim x} \cdot \varepsilon(x, y); \tag{7.2}$$

if $\dim x = 0$ we use also this definition with the convention $\mathcal{H}^0(\langle x \rangle) = 1$ for any x. (Here $\mathcal{H}^k$ is the k-dimensional Hausdorff measure in E.) $\varepsilon(x, y)$ denotes the **normalized exterior angle of the cell y at the face x**. It is defined as the ratio of the area of the set of normals to the support hyperplanes of y at x to the area of the entire surface generated by these normals. To be more precise : if $x \subseteq y$ and $r = \dim x$, $s = \dim y$,

$$\varepsilon(x, y) = \int_{S^{d-1}} 1_{H_{\mathbf{n}}}(y - b(x))\, \sigma^{s-r-1}(d\mathbf{n}). \tag{7.3}$$

This is an intrinsic quantity.

By means of the Chern–Gauss–Bonnet potential we then define curvature as follows : Let Γ_p be the set of Delone complexes $\mu \in S_{r,R}(E)$ having the property that each maximal cell of μ has the same dimension p. Given $0 \leq k \leq p$ and $\Lambda \in \mathcal{B}_0(E)$, the **Lipschitz–Killing curvature** of μ **of order** (p, k) in Λ is the energy defined by the **Lipschitz–Killing potential** as follows :

$$\phi_{\mathcal{CK}}^{(p,k)}(\nu, \mu) = 1_{\{\dim = p-k\}}(\nu) \cdot \phi_{\mathcal{CK}}(\nu, \mu), \mu \in \Gamma_p \quad \text{and} \quad \nu \leq \mu \text{ finite.} \tag{7.4}$$

The **Regge potential of order** (p, k) is defined in this context by

$$\phi_R^{(p,k)} = -\left[\phi_{\mathcal{CK}}^{(p,k)} - \phi_v^{(p,k)}\right], \tag{7.5}$$

where $\phi_v^{(p,k)}$ is the **volume potential of order** (p,k) given by $\phi_v^{(p,k)}(\nu,\mu) = 0$ if ν is not of the form $\bar{x}$ for some $x \in \mu_{p-k}$, and otherwise by

$$\phi_v^{(p,k)}(\bar{x},\mu) = \mathcal{H}^{p-k}(\langle x\rangle) \cdot \sum_{x \subseteq y \in \mu_p} \mathcal{H}^p(\langle y\rangle). \tag{7.6}$$

The energy associated to $\phi_R^{(p,k)}$, if considered on Γ_p, is the **Einstein–Hilbert energy of order** (p,k).

Note that the set Γ_p of Delone complexes is tempered for the potentials $I_{\mathbf{n}}$, $\phi_{\mathcal{LK}}^{(p,k)}$, $\phi_v^{(p,k)}$ and $\phi_R^{(p,k)}$. If now $P \in \mathcal{P}^0\Gamma_p$ is an underlying stationary law on Γ_p the ergodic theorem for the energy implies the existence of the corresponding specific energies, i.e. the **specific index** $I(\mathbf{n},\cdot)$ for any $\mathbf{n} \in S^{d-1}$, the **specific volume of order** (p,k) denoted by $\lambda_{p,k}(\cdot)$, and the **specific Lipschitz–Killing curvature of order** (p,k), denoted by $\kappa_{p,k}(\cdot)$.

Recall that these quantities are given by conditional expectations of the form $e = E_P(\int_D g(\mu - a) b(\mu)(da)/\mathcal{J})$, where g is given by means of the underlying potential ϕ by (4.4). Thus they are translation-invariant and their expectation is

$$E_P(e) = P^0_{\mathcal{M}^0}\left(\sum_{\nu \le \mu, 0 \in b(\nu)} \frac{\phi(\nu,\mu)}{|\nu|}\right). \tag{7.7}$$

The existence of the specific Euler characteristic has already been shown above. But we give another argument to show that the old and new specific indices ι and I coincide $\sigma^{d-1} \otimes P$-almost surely. First of all we observe that $E_\Lambda^{I_{\mathbf{n}}}(\mu)$ is related to $\mathcal{X}(\mu_\Lambda)$ by the following analogon of (3.9). For any $\mathbf{n} \in S^{d-1}$, $E_\Lambda^{I_{\mathbf{n}}}(\mu)$ and $\mathcal{X}(\mu_\Lambda)$ differ from one another only by a boundary term, thereby implying an estimate of the form (5.3) with some error term $\tilde{\Delta}(\Lambda)$ which does not depend on μ and satisfies $\lim_{n\to\infty} \frac{1}{|\Lambda_n|}\tilde{\Delta}(\Lambda_n) = 0$. This follows from the critical point theorem 5 of Banchoff [1] for finite complexes. From this we then obtain for any $\mathbf{n} \in S^{d-1}$,

$$I(\mathbf{n},\cdot) = \mathcal{X}(\cdot) \quad P-a.s., \tag{7.8}$$

which is another version of the critical point theorem for stationary infinitely extended random simplicial complexes. If combined with (5.6) it yields that ι and $\mathcal{X}$ coincide $\sigma^{d-1} \otimes P$-almost surely.

Now consider the mean specific Lipschitz–Killing curvatures and the mean specific volumes. If we evaluate (7.7) for the curvature potential $\phi_{\mathcal{LK}}^{(p,k)}$ we obtain

$$E_P(\kappa_{p,k}) = \frac{1}{2^{p-k+1}-1}\times$$

$$\times P^0_{\mathcal{M}^0}\left[\sum_{\tilde{x}\in\mu, 0\in b(\tilde{x})\,\dim\tilde{x}=p-k} \mathcal{H}^{p-k}(\langle x\rangle)\sum_{x\subseteq y\in\mu}(-1)^{\dim y-\dim x}\cdot\varepsilon(x,y)\right].$$

This can be written as

$$E_P(\kappa_{p,k}) = \frac{1}{2^{p-k+1}-1}\times$$

$$\times P^0_{\mathcal{M}^0_{p-k}}\left[\mathcal{H}^{p-k}(\langle x_0(\mu_{p-k})\rangle)\sum_{x_0(\mu_{p-k})\subseteq y\in\mu}(-1)^{\dim y-p+k}\cdot\varepsilon(x_0(\mu_{p-k}),y)\right]. \tag{7.9}$$

Here $\mathcal{M}^0_{p-k} = \{\mu\in\Gamma_p | 0\in b(\mu_{p-k})\}$ and $P^0_{\mathcal{M}^0_{p-k}} = Res_{\mathcal{M}^0_{p-k}}P^0$. In the same way we obtain for the mean specific volume

$$E_P(\lambda_{p,k}) = \frac{1}{2^{p-k+1}-1}\cdot P^0_{\mathcal{M}^0_{p-k}}\left[\mathcal{H}^{p-k}(\langle x_0(\mu_{p-k})\rangle)\cdot\mathcal{H}^p(\langle\mu_p^{x_0(\mu_{p-k})}\rangle)\right], \tag{7.10}$$

where $\mu_p^{x_0(\mu_{p-k})} = \sum_{x_0(\mu_{p-k})\subseteq y\in\mu_p}\delta_y$.
Finally, the mean specific Einstein–Hilbert energy of order (p,k) i.e. $E_P(\eta_{p,k})$, where

$$\eta_{p,k} = \lim_{n\to\infty}\frac{1}{|\Lambda_n|}E_{\Lambda_n}^{\phi_R^{(p,k)}},$$

is explicitely given by

$$E_P(\eta_{p,k}) = \frac{-P^0_{\mathcal{M}^0_{p-k}}}{2^{p-k+1}-1}\left[\mathcal{H}^{p-k}(\langle x_0(\mu_{p-k})\rangle)\times\right.$$

$$\left.\times\sum_{x_0(\mu_{p-k})\subseteq y\in\mu_p}((-1)^{\dim y-p+k}\varepsilon(x_0(\mu_{p-k}),y)-\mathcal{H}^p(\langle z\rangle))\right]. \tag{7.11}$$

§8. CONCLUDING REMARKS

If we look at the mean specific Einstein–Hilbert energies of order (p,k) in (7.11) we can ask for their **ground states**. These states are given by the minima of the functional $P \to E_P(\eta_{p,k})$, which is well defined in Γ_p. It is now evident that the important problem of detailed description of the set of ground states for $\phi_R^{(p,k)}$ in the physically interesting case $p=4, k=2$ is far from obvious.

We conclude this paper with a remark concerning the intrinsic character of the mean specific energies in (7.9), (7.10) and (7.11). These energies are represented by means of the non-normalized Palm measure P^0, which is not an intrinsic quantity. We obtain them in terms of intrinsic normalized Palm measures, if we suppose also that P is ergodic with respect to translations and if we normalize the energy not by the volume $|\Lambda|$ but by an intrinsic quantity. Consider for instance the sequence $b\mu_{p-k}(\Lambda_n), n \geq 1$, which counts the number of cells of μ_{p-k} with barycenter in Λ_n. By the ergodic theorem

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n|} b\mu_{p-k}(\Lambda_n) = P^0(\mathcal{M}^0_{p-k}).$$

If we assume that $0 < P^0(\mathcal{M}^0_{p-k}) < +\infty$, then we can divide the energy by $b\mu_{p-k}(\Lambda_n)$ and get for instance the following law of large numbers

$$\lim_{n\to\infty} \frac{1}{b\mu_{p-k}(\Lambda_n)} \cdot E_{\Lambda_n}^{\phi_R^{(p,k)}}(\mu) = -\frac{1}{2^{p-k+1}-1} \cdot \frac{1}{P^0(\mathcal{M}^0_{p-k})} \cdot P^0_{\mathcal{M}^0_{p-k}} \Bigg[\mathcal{H}^{p-k}(\langle x_0(\mu_{p-k})\rangle) \cdot \sum_{x_0(\mu_{p-k}) \subseteq y \in \mu_p} \left((-1)^{\dim y - p + k} \cdot \varepsilon(x_0(\mu_{p-k}), y) - \mathcal{H}^p(\langle y \rangle)\right) \Bigg] \tag{8.1}$$

P-a.s. and in $\mathcal{L}^1(P)$. Here the limit is an expectation taken with respect to the probability $\frac{1}{P^0(\mathcal{M}^0_{p-k})} \cdot P^0_{\mathcal{M}^0_{p-k}}$.

Instead of $b\mu_{p-k}(\Lambda_n)$, one could have taken also the volume of the random space $\langle \mu_{p-k} \rangle$ in Λ_n. It follows from the ergodic theorem that P-a.s. and in $\mathcal{L}^1(P)$

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n|} \mathcal{H}^{p-k}(\langle \mu_{p-k} \rangle \cap \Lambda_n) = \frac{1}{2^{p-k+1}-1} \cdot P^0_{\mathcal{M}^0_{p-k}}(\mathcal{H}^{p-k}(\langle x_0(\mu_{p-k}) \rangle)).$$

If we assume now that $0 < P^0_{\mathcal{M}^0_{p-k}}(\mathcal{H}^{p-k}(\langle x_0(\mu_{p-k})\rangle)) < +\infty$, then

$$\lim_{n\to\infty} \frac{1}{\mathcal{H}^{p-k}(\langle \mu_{p-k}\rangle \cap \Lambda_n)} \cdot E_{\Lambda_n}^{\phi_R^{(p-k)}}(\mu) = P^0_{\mathcal{M}^0_{p-k}}\left[\frac{1}{P^0_{\mathcal{M}^0_{p-k}}(\mathcal{H}^{p-k}(\langle x_0(\mu_{p-k})\rangle))}\times \right.$$

$$\left. \times \mathcal{H}^{p-k}(\langle x_0(\mu_{p-k})\rangle) \cdot \sum_{x_0(\mu_{p-k})\subseteq y\in\mu_p} ((-1)^{\dim y-p+k} \cdot \varepsilon(x_0(\mu_{p-k}), y) - \mathcal{H}^p(\langle y\rangle))\right]. \tag{8.2}$$

Again we obtain a law of large numbers, which is completely intrinsic.

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Резюме. Основным результатом статьи является эргодическая теорема для бесконечно расширяющихся стационарных случайных симплициальных комплексов, которая утверждает почти наверное существование пределов в $\mathcal{L}^1$. Как следствие, получается теорема о критической точке, а также аналог теоремы Гаусса–Боне. Показано также существование удельной кривизны Липшица–Киллинга и удельной энергии Эйнштейна–Гильберта.

REFERENCES

1. T. Banchoff, "Critical points and curvature for embedded polyhedra", J. Differential Geometry, vol. 1, pp. 245 – 256, 1967.
2. L. Budach, "Lipschitz–Killing curvatures of angular partially ordered sets", Advances in Mathematics, vol. 78, pp. 140 –167, 1989.
3. J. Cheeger, W. Müller, R. Schrader, "On the curvature of piecewise flat spaces", Commun. Math. Phys., vol. 92, pp. 405 – 454, 1984.
4. J. Cheeger, W. Müller, R. Schrader, "Kinematic and tube formulas for piecewise linear spaces", Indiana Univ. Math. Journal, vol. 35, pp. 737 – 754, 1986.
5. A. Einstein, "Hamiltonsches Prinzip und allgemeine Relativitätstheorie", Sitzungsbericht der Preušischen Akademie der Wissenschaften, vol. 42, pp. 1111 – 1116, 1916.
6. H. Föllmer, "On entropy and information gain in random fields", Z. Wahrscheinlichkeitstheorie verw. Geb., pp. 207 – 221, 1973.
7. D. Hilbert, S. Cohn–Vossen, Anschauliche Geometrie, Springer, 1932.
8. D. Hilbert, Göttinger Nachrichten, Sitzung vom 20. November, 1915.
9. L. Leistritz, M. Zähle, "Topological mean value relations for random cell complexes", Math. Nachrichten, vol. 155, pp. 57 – 72, 1992.
10. P. McMullen, "Non-linear angle-sum relations for polyhedral cones and polytopes", Math. Proc. Cambridge Philos. Soc., vol. 78, pp. 247 – 261, 1975.
11. K. Mecke, D. Stoyan, "The specific connectivity number of random networks", Adv. Appl. Prob., vol. 33, pp. 541 – 558, 2001.

12. X. X. Nguyen, H. Zessin, "Punktprozesse mit Wechselwirkung", Z. Wahrscheinlichkeitstheorie verw. Geb., vol. 37, pp. 91 – 126, 1976.
13. T. Regge, "General relativity without coordinates", Nuovo Cimento, vol. 19, pp. 558 – 571, 1961.
14. H. Weyl, "Zur Gravitationstheorie", Annalen der Physik, vol. 54, pp. 117 – 145, 1918.
15. M. Zähle, "Random cell complexes and generalised sets", The Annals of Probability, vol. 16, pp. 1742 – 1766, 1988.
16. H. Zessin, "The Gauss–Bonnet theorem for stationary random simplicial surfaces", Preprint 02–009 der Forschergruppe Bielefeld, University of Bielefeld, 2002.

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