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INEQUALITIES IN THE SENSE OF BRUNN–MINKOWSKI–VITALE FOR RANDOM CONVEX BODIES

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Abstract. The well-known Brunn–Minkowski inequality concerning convex addition of measurable sets was generalized by R. A. Vitale for the case of random sets. The paper presents a new proof in the special case of random convex bodies, which does not employ the law of large numbers for random sets, but the mixed area measure. In this way, inequalities for mixed volumes and intrinsic volumes of random convex bodies are also obtained. Finally, consequences for stationary random hyperplane processes are discussed.

§1. INTRODUCTION

Nonempty compact convex subsets of $\mathbb{R}^d$ are called *convex bodies*. Let us denote by $K + L$ the Minkowski sum of the convex bodies $K, L \subset \mathbb{R}^d$ and by $V_d(K)$ the volume of a convex body K. The formula

$$V_d^{1/d}(pK + (1-p)L) \geq pV_d^{1/d}(K) + (1-p)V_d^{1/d}(L),$$

where $0 \leq p \leq 1$, is known as the Brunn–Minkowski inequality for convex bodies K, L [7]. By iteration, for convex bodies $K_1, ..., K_n$,

$$V_d^{1/d}(p_1K_1 + \cdots + p_nK_n) \geq p_1V_d^{1/d}(K_1) + \cdots + p_nV_d^{1/d}(K_n), \tag{1}$$

where $p_1, ..., p_n \geq 0$, $p_1 + ... + p_n = 1$.

This formula may be interpreted in a stochastic manner : Let $\mathbf{K}$ be a **random** convex body with range $\{K_1, ..., K_n\}$ and $Prob(\mathbf{K} = K_i) = p_i$, $i = 1, ..., n$. Then (1) can be written in the form

$$V_d^{1/d}(\mathbb{E}\mathbf{K}) \geq \mathbb{E}V_d^{1/d}(\mathbf{K}), \tag{2}$$

where $\mathbb{E}\mathbf{K}$ means the set-valued expectation of $\mathbf{K}$ [9], [10]. On the right we have the usual expectation of the real random variable $V_d^{1/d}(\mathbf{K})$.

The question arises, whether (2) is valid for arbitrary random convex bodies not necessarily discrete. R. Vitale [10] proved (2) for a large class of random sets $\mathbf{K}$ including random convex bodies. His proof relies on the strong law of large numbers for random sets [1], cf. [11].

The main results of the paper are as follows.

– Generalization of (2) for intrinsic volumes $V_m(\mathbf{K})$ of random convex bodies $\mathbf{K}$ in $\mathbb{R}^d$ (Theorem 2)

$$V_m^{1/m}(\mathbb{E}\mathbf{K}) \geq \mathbb{E}V_m^{1/m}(\mathbf{K}), \quad m = 2, ..., d-1. \tag{3}$$

– A new method of proof for (2) and (3), which makes no use of the law of large numbers for random sets.

– An inequality for mixed volumes of random convex bodies (Theorem 1), interesting in its own right. The inequalities (2) and (3) are obtained as special case.

– Consequences of the above results for the intersection densities of stationary Poisson hyperplane processes (their distribution can be described by a convex body, the so-called Steiner compact [3]).

§2. BASIC NOTIONS AND NOTATIONS

Let us denote the set of all non-empty compact convex subsets in $\mathbb{R}^d$, $d \geq 2$ by $\mathcal{K}$, and the σ-algebra of subsets of $\mathcal{K}$ as defined in [3] by $\mathbb{K}$. A **random convex body** is a random variable $\mathbf{K}$ with range $[\mathcal{K}, \mathbb{K}]$ (random element in $[\mathcal{K}, \mathbb{K}]$). Every $K \in \mathcal{K}$ is described by its support function $h(K, \cdot) : \mathbb{R}^d \longmapsto [0, \infty)$ defined by

$$h(K, \mathbf{u}) = \sup\{\langle \mathbf{x}, \mathbf{u} \rangle : \mathbf{x} \in K\},$$

where $\langle \mathbf{x}, \mathbf{u} \rangle$ denotes the inner product of $\mathbf{x}, \mathbf{u} \in \mathbb{R}^d$. The norm of an element $K \in \mathcal{K}$ is

$$||K|| = \max\{||x|| : \ x \in K\},$$

where $||x||$ denotes the usual norm in $\mathbb{R}^d$.

Let $\mathbf{K}$ be a random convex body. Assuming $\mathbb{E}||\mathbf{K}|| < \infty$, it can be shown [1], [9], [11], that there exists a convex body $\mathbb{E}\mathbf{K} \in \mathcal{K}$ with the support function

$$h(\mathbb{E}\mathbf{K}, \mathbf{u}) = \mathbb{E}h(\mathbf{K}, \mathbf{u}), \quad \mathbf{u} \in \mathbb{R}^d. \tag{4}$$

The convex body $\mathbb{E}\mathbf{K}$ is called **set-valued expectation** of the random convex body $\mathbf{K}$.

The volume $V_d(K)$ of a convex body K and the $(d-1)$-content $O_d(K)$ of its boundary ∂K are invariant under Euclidean motions and, as functions on $\mathcal{K}$, have several properties of continuity and additivity. In convex geometry they are considered as special cases of the so-called **intrinsic volumes** $V_m : \mathcal{K} \longmapsto [0, \infty)$, possessing similar properties ($m = 0, 1, ..., d$). They are closely related to the Minkowski-functionals [7]. A more general notion is that of **mixed volume** $V(K_1, ..., K_d)$ of convex bodies $K_1, ..., K_d \in \mathcal{K}$ [7]. Let B be the unit ball in $\mathbb{R}^d$, and α_k be the volume of the k-dimensional unit ball, i.e.

$$\alpha_k = \frac{\pi^{k/2}}{\Gamma(1 + k/2)}, \quad k = 0, 1, ..., d.$$

The intrinsic volume $V_m(K)$ of $K \in \mathcal{K}$ can be expressed as a special mixed volume [7] :

$$V_m(K) = \binom{d}{m} (\alpha_{d-m})^{-1} V \left(\underbrace{K, ..., K}_{m}, \underbrace{B, ..., B}_{d-m} \right), \quad m = 0, 1, ..., d. \tag{5}$$

Then, $V_d(K)$ is the usual volume of K, $V_{d-1}(K) = O_d(K)/2$, $V_1(K)$ is proportional to the mean width of K, and $V_0(K) = 1$.

§3. INEQUALITIES FOR MIXED VOLUMES

Let $\mathbf{S}$ be the σ-algebra of Borel subsets of the unit sphere S^{d-1} in $\mathbb{R}^d$, $d \geq 2$ let be fixed. There exists a function $S : \mathcal{K}^{d-1} \times \mathbf{S} \longmapsto [0, \infty)$, the so-called mixed area measure [7], with the properties

(i) For fixed $K_2, ..., K_d \in \mathcal{K}$, $S(K_2, ..., K_d, \cdot)$ is a finite measure on $\mathbf{S}$.

(ii) For all $K_1, ..., K_d \in \mathcal{K}$ the equation

$$V(K_1, ..., K_d) = \frac{1}{d} \int_{S^{d-1}} h(K_1, \mathbf{u}) S(K_2, ..., K_d, d\mathbf{u}) \tag{6}$$

holds.

The following formula plays a key role in the proofs of our main results.

Proposition 1. *Let* $\mathbf{K}$ *be a random convex body with* $\mathbb{E}||\mathbf{K}|| < \infty$. *If* $K_2, ..., K_d \in \mathcal{K}$, *then*

$$V(\mathbb{E}\mathbf{K}, K_2, ..., K_d) = \mathbb{E}V(\mathbf{K}, K_2, ..., K_d).$$

Proof. Applying formulae (4), (6) and Fubini's theorem we get

$$dV(\mathbb{E}\mathbf{K}, K_2, ..., K_d) = \int h(\mathbb{E}\mathbf{K}, \mathbf{u}) S(K_2, ..., K_d, d\mathbf{u}) =$$

$$= \mathbb{E} \int h(\mathbf{K}, \mathbf{u}) S(K_2, ..., K_d, d\mathbf{u}).$$

Finally, (6) implies

$$\int h(\mathbf{K}, \mathbf{u}) S(K_2, ..., K_d, d\mathbf{u}) = dV(\mathbf{K}, K_2, ..., K_d).$$

Proposition 1 is proved.

We learn from Schneider [7] the following special case of a generalized Alexandrov–Fenchel inequality (formula (6.4.5) in [7] for $i = 0$, $j = m-1$ and $k = m$) :

Proposition 2. *For $2 \le m \le d$ and $Y, Z, K_{m+1}, ..., K_d \in \mathcal{K}$, we have*

$$V\left(Y, \underbrace{Z, ..., Z}_{m-1}, K_{m+1}, ..., K_d\right) \ge V^{1-1/m}\left(\underbrace{Z, ..., Z}_{m}, K_{m+1}, ..., K_d\right) \times$$
$$\times V^{1/m}\left(\underbrace{Y, ..., Y}_{m}, K_{m+1}, ..., K_d\right).$$

(Cf. also [2], exercise p. 321.) That particular inequality can also be verified by repeated application of the Alexandrov–Fenchel inequality.

The next theorem contains our main result.

Theorem 1. *Let $\mathbf{K}$ be a random convex body with $\mathbb{E}||\mathbf{K}|| < \infty$. If $2 \le m \le d$ and $K_{m+1}, ..., K_d \in \mathcal{K}$, then*

$$V^{1/m}\left(\underbrace{\mathbb{E}\mathbf{K}, ..., \mathbb{E}\mathbf{K}}_{m}, K_{m+1}, ..., K_d\right) \ge \mathbb{E}V^{1/m}\left(\underbrace{\mathbf{K}, ..., \mathbf{K}}_{m}, K_{m+1}, ..., K_d\right).$$

Proof. Proposition 2 implies

$$V\left(\mathbf{K}, \mathbb{E}\mathbf{K}, ..., \mathbb{E}\mathbf{K}, K_{m+1}, ..., K_d\right) \ge V^{1-1/m}\left(\mathbb{E}\mathbf{K}, ..., \mathbb{E}\mathbf{K}, K_{m+1}, ..., K_d\right) \times$$
$$\times V^{1/m}\left(\mathbf{K}, ..., \mathbf{K}, K_{m+1}, ..., K_d\right).$$

Forming the expectation on both sides and applying Proposition 1, we obtain

$$V\left(\mathbb{E}\mathbf{K}, ..., \mathbb{E}\mathbf{K}, K_{m+1}, ..., K_d\right) \ge V^{1-1/m}\left(\mathbb{E}\mathbf{K}, ..., \mathbb{E}\mathbf{K}, K_{m+1}, ..., K_d\right) \times$$
$$\times \mathbb{E}V^{1/m}\left(\mathbf{K}, ..., \mathbf{K}, K_{m+1}, ..., K_d\right).$$

The assertion follows immediately.

§4. INEQUALITIES FOR INTRINSIC VOLUMES

The next theorem is derived from Theorem 1 and (5), putting $K_{m+1} = ... = K_d = B$.

Theorem 2. *Let $\mathbf{K}$ be a random convex body with $\mathbb{E}||\mathbf{K}|| < \infty$. If $2 \le m \le d$, then*

$$V_m^{1/m}(\mathbb{E}\mathbf{K}) \ge \mathbb{E}V_m^{1/m}(\mathbf{K}).$$

These formulae are called **generalized Brunn–Minkowski–Vitale inequalities**. In the case $m = d$ the result was already proved by R. Vitale in [10], and quoted by Weil and Weacker in [11].

If $K \in \mathcal{K}$ and D is a random rotation about the origin, then DK is a random convex body.

Corollary 1. *Let D be a random rotation about the origin. Then for all $K \in \mathcal{K}$ and $m = 2, ..., d$ the inequalities $V_m(\mathbb{E}DK) \ge V_m(K)$ are fulfilled.*

Proof. From Theorem 2 we obtain $V_m^{1/m}(\mathbb{E}DK) \ge \mathbb{E}V_m^{1/m}(DK)$. The rotation invariance of V_m implies $V_m(DK) = V_m(K)$.

§5. ALTERNATIVE PROOF FOR INTRINSIC VOLUMES

Theorem 2 might as well be proved in the same way Vitale [10] proved (2). We make use of a generalized Brunn–Minkowski inequality concerning intrinsic volumes [2], [7].

Proposition 3. *For $K_1, K_2 \in \mathcal{K}$, $0 \le \lambda \le 1$ and $m = 2, ..., d$ the inequality*

$$V_m^{1/m}(\lambda K_1 + (1-p)K_2) \ge \lambda V_m^{1/m}(K_1) + (1-\lambda)V_m^{1/m}(K_2) \tag{7}$$

holds.

Now, given a random convex body $\mathbf{K}$, such that $\mathbb{E}||\mathbf{K}|| < \infty$, we consider a sequence $X_1, X_2, ...$ of mutually independent random sets, each distributed like $\mathbf{K}$. By iteration (7) is transformed into

$$V_m^{1/m}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \ge \frac{1}{n}\sum_{i=1}^{n} V_m^{1/m}(X_i), \quad n \ge 2. \tag{8}$$

For a constant c depending only on m and d, we have $V_m^{1/m}(L) \le c||L||$ for any $L \in \mathcal{K}$. Since $\mathbb{E}||\mathbf{K}|| < \infty$, the Kolmogorov strong law of large numbers implies, that the right-hand side of (8) tends to $\mathbb{E}V_m^{1/m}(\mathbf{K})$ as $n \to \infty$.

Regarding the left-hand side of (8), there is the a.s. convergence of $\frac{1}{n}\sum\limits_{i=1}^{n} X_i$ to $\mathbb{E}\mathbf{K}$ by a strong law of large numbers for random convex sets (Artstein and Vitale [1], cf. [11]). Due to the continuity of V_m (cf. e.g. [2]), we have $V_m^{1/m}(\mathbb{E}\mathbf{K}) \ge \mathbb{E}V_m^{1/m}(\mathbf{K})$, which is the assertion of Theorem 2.

§6. SOME CONSEQUENCES FOR HYPERPLANE PROCESSES

Let Φ be a stationary Poisson hyperplane process (SPHP) in $\mathbb{R}^d$ [3], [4]. The mean $(d-1)$-content of Φ per unit volume is called **intensity** and denoted by λ. The direction of a hyperplane is described by the perpendicular line through the origin (1-subspace). We denote the set of all 1-subspaces by $\mathcal{H}$ and the σ-algebra of Borel subsets of $\mathcal{H}$ by $\mathcal{B}$ [3], [4].

Given $A \in \mathcal{B}$, let us denote by Φ_A the set of all hyperplanes from Φ with a direction in A. Then Φ_A is again a SPHP, the intensity of which is denoted by $h(A)$. In this way, a finite measure h on $[\mathcal{H}, \mathcal{B}]$ is established, the so-called **directional measure** of Φ. The distribution of a SPHP Φ is completely determined by h. The intensity λ of Φ is equal to the total mass of $h : \lambda = h(\mathcal{H})$.

Every m-tuple of hyperplanes from Φ in general position has for the set of intersection points a $(d-m)$-dimensional affine subspace $(m = 1, ..., d)$. For fixed m, all these intersection $(d-m)$-flats form a stationary (non-Poisson) $(d-m)$-dimensional flat process Φ_m. Note that $\Phi_1 = \Phi$ and Φ_d is the point process of vertices of the tessellation formed by Φ. The $(d-m)$-content of Φ_m per unit volume is said to be **the m-intersection density** and denoted by $\rho_m(h)$. Note that $\rho_1 = \lambda$ and ρ_d equals the intensity of the point process of vertices.

To every finite measure h on $[\mathcal{H}, \mathcal{B}]$ different from the zero-measure there corresponds a convex body (more precisely a zonoid), the so-called Steiner compact $\mathcal{S}[h]$ [3], [4]. It is known, that

$$\rho_m(h) = V_m(\mathcal{S}[h]), \quad m = 1, ..., d, \tag{9}$$

if h is the directional measure of Φ [3], [4].

Let D be a random rotation about the origin. It transforms h in a random measure D_h and the Steiner compact $\mathcal{S}[h]$ in a random convex body $D\mathcal{S}[h] = \mathcal{S}[D_h]$. We define a measure $\mathbb{E}D_h$ on $[\mathcal{H}, \mathcal{B}]$ by $(\mathbb{E}D_h)(A) = \mathbb{E}(D_h)(A), \quad A \in \mathcal{B}$. It can be shown that

$$\mathcal{S}[\mathbb{E}D_h] = \mathbb{E}D\mathcal{S}[h]. \tag{10}$$

Corollary 1 and (10) lead to

$$V_m(\mathcal{S}[\mathbb{E}D_h]) \geq V_m(\mathcal{S}[h]), \quad m = 2, ..., d. \tag{11}$$

Combining (9) with (11), we get

$$\rho_m(\mathbb{E}D_h) \geq \rho_m(h), \quad m = 2, ..., d. \tag{12}$$

For $m = d$ the result can be found in [5], [6].

If the distribution of D is a Haar measure on the group of rotations about the origin, we say that D is **uniformly distributed**. In this case, $\mathbb{E}D_h$ is proportional to the uniform distribution γ on $[\mathcal{H}, \mathcal{B}]$: $\mathbb{E}D_h = \lambda\gamma$. A SPHP with directional measure $\lambda\gamma$ is called **isotropic**.

For an arbitrary random rotation D, it seems to be reasonable to say, that $\mathbb{E}D_h$ is "more isotropic" than h. We say also, that a SPHP $\tilde{\Phi}$ with directional measure $\mathbb{E}D_h$ is "more isotropic" than the SPHP Φ with directional measure h. Note that $\tilde{\Phi}$ and Φ have the same intensity λ.

In this context, formula (12) means that the intersection densities of a stationary Poisson hyperplane process Φ are not greater, than the corresponding intersection densities of a stationary Poisson hyperplane process $\tilde{\Phi}$, which is "more isotropic" than Φ, but has the same intensity.

As a special case, the result of Thomas [8], cf. [4] is reestablished : for fixed intensity λ, the intersection densities of stationary Poisson hyperplane processes take their maximal value in the isotropic case.

Резюме. Хорошо известное неравенство Брунна-Минковского, относящееся к выпуклому сложению измеримых множеств было обобщено Р. А. Витале для случая случайных множеств. В настоящей статье приводится его новое доказательство в частном случае случайных выпуклых тел, где вместо закона больших чисел для случайных множеств используется смешанная мера соответствующая площади. Этим путём получены неравенства для смешанных обёмов и внутренних обёмов случайных выпуклых тел. Обсуждаются следствия для стационарных случайных процессов гиперплоскостей.

REFERENCES

1. Z. Artstein, R. A. Vitale, "A strong law of large numbers for random compact sets", Ann. Probab., vol. 3, pp. 879 – 882, 1975.
2. K. Leichtweiss, Konvexe Mengen, Deutsche Verlag der Wissenschaften, Berlin, 1980.
3. G. Matheron, Random Sets and Integral Geometry, Wiley, New York, 1975.
4. J. Mecke, R. Schneider, D. Stoyan, W. Weil, Stochastische Geometrie, Birkhäuser, Basel, 1990.
5. J. Mecke, "Inequalities for the anisotropic Poisson polytope", Adv. Appl. Prob, vol. 27, pp. 56 – 62, 1995.
6. J. Mecke, "Inequalities for mixed stationary Poisson hyperplane tessellations", Adv. Appl. Prob., vol. 30, pp. 921 – 928, 1998.
7. R. Schneider, Convex Bodies : The Brunn–Minkowski Theory, Cambridge Univ. Press, Cambridge, 1993.
8. C. Thomas, "Extremum properties of the intersection densities of stationary Poisson hyperplane processes", Math. Operationsforsch. Statist., ser. Statist., vol. 15, pp. 443 – 449, 1984.
9. R. A. Vitale, "Expected convex hulls, order statistics and Banach space probabilities", Acta Appl. Math., vol. 9, pp. 97 – 102, 1987.
10. R. A. Vitale, "The Brunn–Minkowski inequality for random sets", J. Multivariate Anal., vol. 33, pp. 286 – 293, 1990.
11. W. Weil, J. A. Weacker, "Stochastic geometry" in Handbook of Convex Geometry, Eds. P. M. Gruber, J. M. Wills. Nort Holland, Amsterdam, pp. 1391 – 1438, 1993.

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