To Annette with affection and admiration

TANGENTIAL APPROXIMATION AND UNIVERSALITY

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Many approximation theorems on unbounded sets in vastly different contexts have essentially the same proofs, which can be considerably shortened by invoking an axiomatization of this method due to Sinclair.

INTRODUCTION

Uniform approximation, $|f-g|<\varepsilon$, is concerned with approximation of a function f by a (usually nicer) function g, which depends on ε, where ε is a (arbitrarily small) positive **constant**. Tangential approximation is more ambitious; it allows ε to be an arbitrary positive **function**. Of course, if the set on which we approximate is compact, the two notions are equivalent, but on unbounded sets, tangential approximation is (infinitely ?) stronger, since the error function, $\varepsilon(z)$ is allowed to decay to zero with arbitrary speed, as $z \to \infty$.

Annette Sinclair [18] formulated an axiomatic theorem on tangential approximation. The theorem of Sinclair is not a theorem in axiomatic potential theory. Rather, the reason her theorem is called axiomatic is that she has attempted to formulate her result in the most general formal context in which the proof holds. This result has not received the attention it deserves. In fact, since Sinclair published her theorem, several papers have been written (some very recently), which could have been considerably shortened by referring to [18]. To illustrate the wide applicability of Sinclair's theorem, we shall deduce tangential type approximation theorems in three different contexts : several complex variables, elliptic partial differential equations and

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axiomatic potential theory.

Since most readers are familiar with at most one of these three fields, in the next section we shall state the basic theorem on **uniform** approximation in each of these areas. In Section 3 we obtain tangential approximation theorems by coupling the uniform approximation theorems of Section 2 with the axiomatic theorem of Sinclair. In Section 4 we point out that many universality theorems are easy consequences of these tangential theorems.

In the remainder of the present section we formulate the theorem of Sinclair. We shall resist the temptation, in stating her theorem, to introduce terminology of direct or indirect limits and sheaf theory, in order to emphasize our thesis that Sinclair's theorem in its original formulation continues to have striking applications.

Let Ω be a topological space. A sequence $\{\Omega_i\}$ of subsets which satisfy the following conditions will be called an **exhaustion** of Ω :

(1) $\overline{\Omega}_i$ is interior to Ω_{i+1};

(2) $\bigcup_{i=1}^{\infty} \Omega_i = \Omega$.

A sequence $\{K_i\}$ such that $K_k \cap K_m = \emptyset$ for $k \neq m$ is said to be a **decomposition** of a set F, if $F = K_1 \cup K_2 \cup \cdots$. An exhaustion $\{\Omega_i\}$ and a decomposition $\{K_i\}$ are said to be **compatible**, if for every n, $K_n \subset \Omega_n$, but $\overline{\Omega}_n \cap K_{n+1} = \emptyset$.

For a given set $F \subset \Omega$ suppose, that an exhaustion $\{\Omega_i\}$ of Ω and a compatible decomposition $\{K_i\}$ are given. Let there be defined classes $\mathcal{K}_n$ and $\mathcal{W}_n$ of functions transforming K_n and Ω_n respectively into the complex plane, $n = 1, 2, \cdots$. Suppose that each function of $\mathcal{W}_n$ defines a function in $\mathcal{W}_{n-1}$, $n = 2, 3, \cdots$.

Theorem 1 [Sinclair]. **Let $F, \Omega, \Omega_n, K_n, \mathcal{W}_n$ and $\mathcal{K}_n$, $n = 1, 2, \cdots$ be defined as above with $\{\Omega_n\}$ and $\{K_n\}$ compatible. Suppose that**

(1) If $\{g_i\}$ is a sequence of functions of $\mathcal{W}_{n+1}$, which converges uniformly on closed subsets of Ω_{n+1}, then $\lim_{i\to\infty} g_i$ defines a function of $\mathcal{W}_n$.; and

(2) Any function defined on Ω_n by an arbitrary function of the class $\mathcal{W}_n$ and on K_{n+1} by a function of $\mathcal{K}_{n+1}$ can be uniformly approximated arbitrarily closely on $\Omega_n \cup K_{n+1}$ by a function of $\mathcal{W}_{n+1}$, $n = 0, 1, 2, \cdots$ (where Ω_0 is the null set).

Let f be a function defined on F in such a way as to determine a function of $\mathcal{K}_i$ for each i. Then, given a sequence $\{\epsilon_i\}$ of positive numbers, there exists a function g defined on Ω, which determines a function of $\mathcal{W}_n$ for

each n, such that $|f(x) - g(x)| < \varepsilon_i$ when $x \in K_i$, $i = 1, 2, \cdots$.

§1. RUNGE APPROXIMATION

The classical approximation theorem of Carl Runge (1885) states that a necessary and sufficient condition, in order for polynomials to be uniformly dense in the holomorphic functions on a compact set $K \subset \mathbf{C}$, is that $\mathbf{C} \setminus K$ be connected. In this section we recall generalized versions of Runge's theorem in several complex variables, elliptic partial differential equations, and axiomatic potential theory. In all three contexts we shall attempt to give similar formulations, in order to spare the reader the task of deducing this similarity from the literature, where the terminology, notation and formulations vary considerably.

Let us say, that a compact set $K \subset \mathbf{C}^n$ is a **Runge** set, if the polynomials are uniformly dense in the holomorphic functions on K. Runge's theorem gives a topological characterization of Runge sets in $\mathbf{C}$. For $n > 1$ a topological characterization is impossible. For a continuous function f on K, denote $\|f\|_K = \max\{|f(z)| : z \in K\}$. The polynomial hull of K is the set

$$K^\wedge = \{z : |p(z)| \leq \|p\|_K, \text{ for all polynomials } p\}.$$

The set K is said to be **polynomially convex**, if $K^\wedge = K$. In $\mathbf{C}$, a compact set is polynomially convex, if and only if its complement is connected. Thus, Runge's theorem states, that a compact set $K \subset \mathbf{C}$ is Runge, if and only if it is polynomially convex. Formulated thus, the sufficiency also holds in $\mathbf{C}^n$, but the necessity fails, due to the Hartogs phenomenon. However, the necessity can be recuperated if we consider open rather than compact sets.

Let $X \subset \mathbf{C}^n$ be compact or open. X is said to be **Runge**, if the polynomials are dense in the holomorphic functions on X, in the topology of uniform convergence on compact subsets. An open set X is said to be **polynomially convex**, if $K^\wedge \subset X$ for each compact set $K \subset X$. The following extension of Runge's theorem to $\mathbf{C}^n$ is due to Kiyoshi Oka and André Weil (see [17], p. 221).

Theorem 2 [Oka-Weil]. Let X be an open or compact subset of $\mathbf{C}^n$. If X is polynomially convex, then X is Runge. If X is open and Stein, the converse is also true.

We now turn to partial differential equations. Let

$$L = \sum_{|\alpha| \leq m} a_\alpha \frac{\partial^\alpha}{\partial x^\alpha},$$

be a linear partial differential operator on $\mathbb{R}^n$ with constant complex coefficients a_α, where each $\alpha = (\alpha_1, \cdots, \alpha_n)$, is a multi-index whose entries α_j, $j = 1, \cdots$ are natural numbers and $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and

$$\frac{\partial^\alpha}{\partial x^\alpha} = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

The operator L is said to be **elliptic**, if

$$\sum_{|\alpha|=m} a_\alpha \xi^\alpha \neq 0, \quad \xi \in \mathbb{R}^n \setminus 0.$$

Let W be an open subset of $\mathbb{R}^n$. A function $f : W \longmapsto \mathbf{C}$ is said to be L-**harmonic**, if it is a solution of the homogeneous elliptic partial differential equation $Lf = 0$. A function is said to be L-**harmonic** on a subset X of $\mathbb{R}^n$, if it is L-harmonic in an open neighborhood of this set. A function which is L-harmonic on all of $\mathbb{R}^n$ is called an **entire** L-harmonic function.

We shall say, that X is a L-**Runge set**, if each function, which is L-harmonic on X is the limit (uniformly on compact subsets of X) of L-entire functions. A **hole** of a set X is a bounded component of the complement $\mathbb{R}^n \setminus K$. We define the **topological hull** of X as the union of X and all of its holes. A subset X of $\mathbb{R}^n$ is said to be **topologically convex**, if it coincides with its topological hull, that is, if it has no holes.

The following Runge theorem for L-harmonic functions is due to Peter Lax [13] and Bernard Malgrange [15].

Theorem 3 [Lax-Malgrange]. Let L be an elliptic linear partial differential operator on $\mathbb{R}^n$, and let X be an open or compact subset of $\mathbb{R}^n$. If X is topologically convex, then X is L-Runge. If X is open, the converse is also true.

Proof: In [10] the sufficiency is proved under the hypothesis that L has a fundamental solution which is analytic in $\mathbb{R}^n \setminus 0$. First of all, Ehrenpreis and Malgrange independently showed (see [10], p. 189), that every linear partial differential operator with constant coefficients has a fundamental solution E (a distribution $E \in \mathcal{D}'(\mathbb{R}^n)$ with $LE = \delta_0$). Moreover, if L is elliptic, then by Weyl's lemma, E is analytic on $\mathbb{R}^n \setminus 0$. The necessity is proved in the Remark on p. 113 in [10].

Note that for holomorphic functions of one complex variable, this theorem is equivalent to the classical Runge theorem, since holomorphic functions of one complex variable can be described as solutions of the homogeneous Cauchy-Riemann equation $\bar{\partial} f = 0$. Moreover, in one complex dimension, polynomial convexity and topological convexity coincide. For functions of more than one complex variable, on the other hand, Theorem 2 is not a special case of Theorem 3.

Indeed, let L be a linear differential operator with analytic coefficients. It follows from the Cauchy-Kovalevskaya theorem that there exists a point p and a real hyperplane H through p, such that Cauchy data can be prescribed locally near p along H for a solution of $Lu = 0$. But holomorphic functions are already determined on **complex** hyperplanes and so there values cannot be prescribed on a **real** hyperplane. Hence, for $n > 1$, there is no linear differential operator L (elliptic or not) with analytic coefficients on $\mathbf{C}^n$, for which the solutions of $Lu = 0$ are holomorphic.

The Lax-Malgrange theorem states that topological convexity is necessary and sufficient in order for an **open** set to be L-Runge and that it is sufficient in order for a **compact** set to be L-Runge. However, as mentioned in the previous paragraph, necessity **does** hold for compact sets in the case where L is the Cauchy-Riemann operator $\bar{\partial}$ in $\mathbb{R}^2 \equiv \mathbf{C}$. For general operators L, however, necessity fails. Stephen Gardiner [7] has given a complete characterization of compact Runge sets for the Laplacian $L = \Delta$.

The third generalization of Runge's theorem will be in the framework of axiomatic potential theory (see [16]). Let $\mathcal{H}$ denote a sheaf of functions on $\mathbb{R}^n$, such that $\{\mathcal{H}, \mathbb{R}^n\}$ is a harmonic space in the sense of Brelot. The sections of the sheaf $\mathcal{H}$ are called $\mathcal{H}$-**harmonic** functions. A function is said to be $\mathcal{H}$-**harmonic** on a subset of $\mathbb{R}^n$, if it is $\mathcal{H}$-harmonic in an open neighborhood of this set. A function which is $\mathcal{H}$-harmonic on all of $\mathbb{R}^n$ is called an **entire** $\mathcal{H}$-harmonic function.

We shall say, that a set $X \subset \mathbb{R}^n$ is an $\mathcal{H}$-**Runge** set, if each function which is $\mathcal{H}$-harmonic on X is the limit (uniformly on compact subsets of X) of $\mathcal{H}$-entire functions. Under certain assumptions on the sheaf $\mathcal{H}$, Arnaud de la Pradelle [16] proved a Runge type theorem for approximation by such harmonic functions. Let us call a harmonic space satisfying the hypotheses of de la Pradelle's theorem a Runge harmonic space. In this case we also say that $\mathcal{H}$ is a **Runge sheaf** of harmonic functions on $\mathbb{R}^n$. Rather than state these hypotheses here, we refer the reader to Theorem 10 in [16], for a decompression of the following Runge-type theorem for harmonic spaces.

Theorem 4 [de la Pradelle]. Let $\mathcal{H}$ be a Runge sheaf of harmonic functions

on $\mathbb{R}^n$, and let X be an open or compact subset of $\mathbb{R}^n$. If X is topologically convex, then X is $\mathcal{H}$-Runge. If X is open, the converse is also true.

An important example of a Runge sheaf is furnished by real solutions of the partial differential equation $Lf = 0$, where

$$L = \sum_{i,j=1}^{n} a_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + c \tag{1}$$

is an operator with real coefficients $a_{i,j} = a_{j,i} \in C^{2,1}$, $b_i \in C^{1,1}$, $c \leq 0$ and $c \in C^{0,1}$, where $C^{k,1}$ denotes the class of functions, which are k times continuously differentiable and whose k-th order partial derivatives are Lipschitz. We also suppose, that the quadratic form associated with L is positive definite, so that the operator L is elliptic. The ground breaking papers of Lax [13] and Malgrange [15] considered Runge type approximation for solutions of elliptic partial differential equations with infinitely differentiable coefficients. The assumptions on the smoothness of the coefficients were considerably relaxed in the investigations of Felix Browder [5]. However, even these stronger versions of Theorem 3 due to Browder do not seem to include the operator (1) and so do not imply Theorem 4. Nor does the axiomatic Theorem 4 imply the Lax-Malgrange Theorem 3.

Indeed, the $\mathcal{H}$ – *harmonic* functions are real-valued, whereas L-harmonic functions may be complex-valued. But, even the real version of the Lax-Malgrange Theorem 3 (trivially deduced from the complex version) does not follow from the axiomatic Runge Theorem 4, for the class of elliptic operators whose solutions satisfy the axioms of a harmonic space is quite restricted.

§2. TANGENTIAL APPROXIMATION

For a compact subset $K \subset \mathbf{C}^n$ we denote

$$m(K) = \min\{|z| : z \in K\} \quad \text{and} \quad M(K) = \max\{|z| : z \in K\}.$$

In order to state a theorem simultaneously for $\mathbf{C}^n$, as well as for the unit ball B^n therein, we shall denote the ball of radius r by $B(r) = \{z \in \mathbf{C}^n : |z| \leq r\}$, where $0 < r \leq +\infty$. Thus, $B(+\infty) = \mathbf{C}^n$. Let us say, that a sequence of compact subsets K_j of $B(r)$ tends to the boundary if $m(K_j) \to r$, and in this case we write $K_j \to \partial_r$. If moreover, $M(K_j) < m(K_{j+1})$, $j = 1, 2, \cdots$, we say that $K_j \to \partial_r$ **strictly.**

Theorem 5. Let $B(r)$ denote the ball of radius r in $\mathbf{C}^n$, where $0 < r \leq +\infty$. If $\{K_j\}$ is a sequence of convex compact sets in $B(r)$ and $K_j \to \partial_r$ strictly,

then for each sequence $\{f_j\}$ of functions holomorphic respectively on K_j and each sequence $\{\varepsilon_j\}$ of positive numbers, there is a g holomorphic on $B(r)$, such that $|f_j(z) - g(z)| < \varepsilon_j$ for $z \in K_j$, $j = 1, 2, \cdots$.

Proof : Since $K_j \to \partial_r$ strictly, we may construct an exhaustion of $B(r)$ by balls $\{B_j\}$, which is compatible with $\{K_j\}$. Since the union of two disjoint closed convex sets $\overline{B}_n \cup K_{n+1}$ is polynomially convex [12], the conclusion follows from the Oka-Weil theorem and Sinclair's theorem.

The following tangential type theorem is for solutions of differential equations. The proof is analogous to that of Theorem 5, invoking Sinclair's Theorem and, in place of the Oka-Weil Theorem, the Lax-Malgrange Theorem.

Theorem 6. Let $B(r)$ denote the ball of radius r in $\mathbb{R}^n$, where $0 < r \leq +\infty$. Let L be an elliptic linear partial differential operator on $\mathbb{R}^n$ with constant coefficients. If $\{K_j\}$ is a sequence of topologically convex compact sets in $B(r)$ and $K_j \to \partial_r$ strictly, then for each sequence $\{f_j\}$ of functions L-harmonic respectively on K_j and each sequence $\{\varepsilon_j\}$ of positive numbers, there is a L-harmonic function g on $B(r)$, such that

$$|f_j(z) - g(z)| < \varepsilon_j, \quad \text{for } z \in K_j, \quad j = 1, 2, \cdots.$$

As an application of Theorem 6, one can easily show the existence of solutions to such differential equations, which approach every value on every ray. This is a special case of a recent result [4].

The following tangential type theorem in axiomatic potential theory is a special case of a result in [2] and is proved in the same way as the two preceding ones, invoking Sinclair's Theorem and, now, Theorem 4 of de la Pradelle.

Theorem 7. Let $B(r)$ denote the ball of radius r in $\mathbb{R}^n$, where $0 < r \leq +\infty$. Let $\mathcal{H}$ be a Runge sheaf of harmonic functions on $\mathbb{R}^n$. If $\{K_j\}$ is a sequence of topologically convex compact sets in $B(r)$ and $K_j \to \partial_r$ strictly, then for each sequence $\{f_j\}$ of functions $\mathcal{H}$-harmonic respectively on K_j and each sequence $\{\varepsilon_j\}$ of positive numbers, there is a entire $\mathcal{H}$-harmonic function g on $B(r)$, such that $|f_j(z) - g(z)| < \varepsilon_j$ for $z \in K_j$, $j = 1, 2, \cdots$. For simplicity, we have stated the Runge theorems of Section 2 and consequently the tangential theorems of the present section on very restricted domains (balls), but they, of course, hold on more general domains. In the next section we give applications, which are in fact based on more general versions of these theorems.

§3. UNIVERSAL FUNCTIONS

In 1929 George D. Birkhoff [3] showed the existence of an entire function with the striking property that its translates approximate **every** entire function. Such a function is called a **universal entire** function. We now explain in more detail.

Let Ω be a complex manifold. We denote by $\mathcal{O}(\Omega)$ the space of holomorphic functions on Ω with the topology of uniform convergence on compact subsets of Ω. A compact set $K \subset \Omega$ is said to be **holomorphically convex** ($\mathcal{O}(\Omega)$-convex), if for each $x \in \Omega \setminus K$ there is an $F \in \mathcal{O}(\Omega)$, such that $|F(x)| > \|F\|_K$. Let $Aut(\Omega)$ denote the group of automorphisms of Ω. Let $\mathcal{F} \subset \mathcal{O}(\Omega)$ and $\Phi \subset Aut(\Omega)$. A function $g \in \mathcal{F}$ is called a **universal function** of $\mathcal{F}$ relative to Φ, if the family $g \circ \Phi = \{g \circ \varphi : \varphi \in \Phi\}$ is dense in $\mathcal{F}$, in the topology of uniform convergence on compact subsets of Ω.

Birkhoff [3] showed the existence of a universal entire function on $\mathbf{C}$ relative to the group of translations. Wladimir Seidel and Joseph Walsh [19] showed an analogous result for holomorphic functions in the unit disc relative to the group of non-Euclidian "translations". In 1999, Fernando Leon-Saavedra extended these results as follows.

Theorem 8. (Universality [14]). Let Ω be a Stein manifold and $\Phi \subset Aut(\Omega)$, such that there exist an exhaustion $\{Q_j\}$ of Ω by holomorphically convex compacta and a sequence $\{\varphi_j\}$ in Φ, such that $Q_j \cap \varphi_j(Q_j) = \emptyset$ and $Q_j \cup \varphi_j(Q_j)$ is holomorphically convex for each j. Then there exists a function in $\mathcal{O}(\Omega)$, which is universal relative to the family Φ.

Proof : Set $K_j = \varphi_j(Q_j)$ and $F = K_1 \cup K_2 \cup \cdots$. By choosing a subsequence, for which we retain the same notation, we may assume that the decomposition $\{K_j\}$ of F is compatible with the exhaustion $\{Q_j\}$ of Ω. The Oka-Weil Theorem 2 also holds on Stein manifolds (see [11]) and so, by the Sinclair Theorem 1, we obtain a tangential Theorem on Ω. For each sequence $f_j \in \mathcal{O}(K_j)$ and each sequence $\varepsilon_j > 0$, there is a $g \in \mathcal{O}(\Omega)$, such that $|f_j - g| < \varepsilon_j$ on K_j.

Now $\mathcal{O}(\Omega)$ is separable and so there is a dense sequence $g_j \in \mathcal{O}(\Omega)$. If we set $f_j = g_j \circ \varphi_j^{-1}$ and apply the tangential theorem of the previous paragraph to this sequence $\{f_j\}$ and a sequence $\{\varepsilon_j\}$ which converges to zero, we obtain a universal function $g \in \mathcal{O}(\Omega)$. This completes the proof.

As emphasized in [14], this yields universal functions in the ball as well as the polydisc of $\mathbf{C}^n$, and of course in $\mathbf{C}^n$ itself. But these three cases follow even more directly from Theorem 5 and the polydisc version thereof, whose proof is the same, since the union of two disjoint closed polydiscs is always polynomially convex.

One can give other interesting examples of domains Ω, whose automorphisms satisfy the hypotheses of the previous theorem. However, for generic Ω, the group $Aut(\Omega)$ of automorphisms of Ω is trivial (consists of the identity mapping only). In order to state a universality theorem, which holds "universally", that is for all Stein manifolds, we relax the notion of universality (see [1]).

Let Ω be a Stein manifold of dimension n, and let B^n denote the unit ball in $\mathbf{C}^n$. We shall say that a holomorphic function $g \in \mathcal{O}(\Omega)$ is **universal with respect to balls**, if the family of functions $g \circ \varphi$, where φ ranges over all biholomorphic mappings of B^n into Ω is dense in $\mathcal{O}(B^n)$. If φ is such a mapping and $\overline{B}$ is a closed ball in B^n, we call $\varphi(\overline{B})$ a closed ball in Ω.

Let K_j be a sequence of closed balls in Ω, where $K_j = \varphi_j(\overline{B}_j)$. Given $f \in \mathcal{O}(B^n)$ and $g \in \mathcal{O}(\Omega)$, we shall say that f **can be approximated by restrictions of** g **to the balls** K_j, if $\{\overline{B}_j\}$ is an exhaustion of B^n and the maximum of $|g - f \circ \varphi_j^{-1}|$ on K_j converges to zero, as $j \to \infty$. It is easy to see, that $g \in \mathcal{O}(\Omega)$ is universal with respect to balls if and only if each $f \in \mathcal{O}(B^n)$ can be approximated by the restriction of g to some sequence of closed balls in Ω.

Theorem 9 (Universality). Let Ω be a Stein manifold. Then there is a function $g \in \mathcal{O}(\Omega)$, which is universal with respect to balls.

Proof : Since Ω is a Stein manifold, we may consider Ω to be embedded in some ambient space $\mathbf{C}^N$. We claim that a compact subset $E \subset \Omega$ is $\mathcal{O}(\Omega)$-convex, if and only if it is $\mathcal{O}(\mathbf{C}^N)$-convex. First of all, if E is $\mathcal{O}(\mathbf{C}^N)$-convex, it is clear that E is $\mathcal{O}(\Omega)$-convex, since $\mathcal{O}(\mathbf{C}^N) \subset \mathcal{O}(\Omega)$. Conversely, suppose E is $\mathcal{O}(\Omega)$-convex and choose $z \in \mathbf{C}^N \setminus E$. If $z \notin \Omega$, then one can find $F \in \mathcal{O}(\mathbf{C}^N)$, such that $F(z) \neq 0$, but $F = 0$ on Ω (see [11]). If, on the other hand, $z \in \Omega$, then one can find $f \in \mathcal{O}(\Omega)$, such that $f(z) = 2$, but $|f| < 1$ on E.

Now extend (see [9], 4.1.8) f to a function $F \in \mathcal{O}(\mathbf{C}^N)$. Then $|F(z)| > \|F\|_E$. Thus, E is $\mathcal{O}(\mathbf{C}^N)$-convex. We have established the claim that a compact set $E \subset \Omega$ is $\mathcal{O}(\Omega)$-convex, if and only if it is $\mathcal{O}(\mathbf{C}^N)$-convex. Of course, this in turn is equivalent to E being polynomially convex in $\mathbf{C}^N$.

Let Q be a $\mathcal{O}(\Omega)$-convex compact set in Ω, and let $x \in \Omega \setminus Q$. We claim there is a closed ball K in Ω containing x and disjoint from Q, such that $Q \cup K$ is $\mathcal{O}(\Omega)$-convex. Let K be the intersection with Ω of a closed ball X in B^N containing x. Since X

is $\mathcal{O}(\mathbf{C}^N)$-convex, $K = X \cap \Omega$ is trivially $\mathcal{O}(\Omega)$-convex. Moreover, if X is sufficiently small, by the implicit function theorem. K is a closed ball in the manifold Ω. Since Q is $\mathcal{O}(\Omega)$-convex, we have seen that it is polynomially convex and so there is a polynomial p, such that p is bounded by 1 on Q and $p(x) = 2$. By continuity, if X is small enough, then $[p(Q)]^\wedge \cap [p(K)]^\wedge = \emptyset$. Since both Q and K are polynomially convex, it follows from the separation lemma of Eva Kallin [12], that their union $Q \cup K$ is also polynomially convex, hence $\mathcal{O}(\Omega)$-convex.

Since Ω is Stein, we have an exhaustion Q_j of Ω by $\mathcal{O}(\Omega)$-convex compact sets. From the previous paragraph, we can construct a compatible sequence K_j of closed balls (in Ω), which tends to infinity (in $\mathbf{C}^N$). Just as in the proof of Theorem 5, we may now conclude from Sinclair's theorem, that tangential approximation is possible on the union of the closed balls K_j. In particular, for each $f \in \mathcal{O}(B^N)$ and sequence $\varepsilon_j > 0$, there exists $g \in \mathcal{O}(\Omega)$ such that $|g - f \circ \varphi_j^{-1}| < \varepsilon_j$ on K_j, where φ_j is the biholomorphic mapping of B^n into Ω, which defines K_j. We may assume that the $\overline{B}_j = \varphi_j^{-1}(K_J)$ exhaust B^n. As a consequence, we obtain the existence of a universal function with respect to balls. This concludes the proof.

Remark 1. Universal functions for the class $\mathcal{H}^\infty$ of bounded holomorphic functions and for the unit ball $B \subset \mathcal{H}^\infty$ were considered by Maurice Heins [8] and Chee Pak Soong [6] respectively. Those results are not covered by the preceding theorems.

Remark 2. The tangential Theorems 6 and 7 can also be used to obtain universal functions as we did using Theorem 5, providing, again, there are sufficiently many automorphisms. For example, Gardiner [7] shows the existence of universal harmonic functions.

Remark 3. In all situations, in which we have asserted the existence of a universal function, one can in fact show (with the same technique) that, in the sense of Baire category, most functions are universal!

АБСТРАКТ. Много теорем приближения на неограниченных множествах в различных контекстах имеют по существу одинаковые доказательства, которые могут быть значительно сокращены использованием аксиоматизации Синклера для этого метода.

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