UNIVERSAL TRIGONOMETRIC AND POWER SERIES

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An expository paper on the recent works of Nestoridis, Melas and J.-P. Kahane.

The notion of a universal trigonometric series was introduced and developed by D. Menshov in 1945 and 1947 [1], [18], [19]. As usual, a trigonometric series and its partial sums are written either

$$S = (\text{formal}) \sum_{-\infty}^{\infty}(a_n \cos nt + b_n \sin nt), \quad S_N(t) = \sum_{0}^{N}(a_n \cos nt + b_n \sin nt),$$

or

$$S = (\text{formal}) \sum_{-\infty}^{\infty} c_n e^{int}, \quad S_N(t) = \sum_{-N}^{N} c_n e^{int}.$$

A trigonometric series S is called **universal** in the sense of Menshov if, given f, any Lebesgue measurable function on the circle $T = \mathbb{R}/2\pi\mathbb{Z}$, there exists an increasing sequence of integers n_j, such that

$$f(t) = \lim_{j\to\infty} S_{n_j}(t) \quad \text{a.e.} \quad (\text{almost everywhere}).$$

At a first look, it can seem paradoxical, that **one trigonometric series is able to represent all** functions. However Menshov proved the following :

$M1$. There exist Menshov universal trigonometric series.

$M2$. Given any trigonometric series, it can be written as the sum of two Menshov universal trigonometric series.

$M3$. There exist universal trigonometric series whose coefficients tend to zero.

$M4$. Given any trigonometric series whose coefficients tend to zero, it can be written as the sum of two Menshov universal trigonometric series.

The paper is divided in two parts. The first is related to $M1$ and $M2$, the second much shorter to $M3$ and $M4$. $M1$ and $M2$ can be proved and easily extended using the Baire category theorem. This theorem (easy to prove, but quite powerful) says that, given a complete metric space X any countable intersection of dense open subsets is dense (therefore, a dense G_δ set). It is convenient to say that quasi all x in X enjoy a given property if this property holds on dense G_δ subset of X.

A trigonometric series S can be identified with the sequence on the coefficients $(c_n)_{n\in\mathbb{Z}}$. The set of all (complex) trigonometric series will be identified with $C^{\mathbb{Z}}$. Let us choose $X = C^{\mathbb{Z}}$, equipped with a metric compatible with the product topology. We shall prove very easily that quasi all trigonometric series are universal in the sense of Menshov, and a little more.

Quasi all trigonometric series S have the following properties :

M' : given $f \in C(T)$, there exists a sequence n_j, such that for each $\alpha > 0$,

$$f(t) = \lim_{j\to\infty} S_{n_j}(t) \quad \textbf{uniformly on } [0, 2\pi - \alpha]$$

(therefore everywhere),

M'' : given f on T in the first class of Baire (pointwise limit of a sequence of continuous functions), there exists a sequence n_j, such that

$$f(t) = \lim_{j\to\infty} S_{n_j}(t) \quad \textbf{everywhere.}$$

M''' : given f Lebesgue measurable on T, there exists a sequence n_j, such that

$$f(t) = \lim_{j\to\infty} S_{n_j}(t) \quad \textbf{almost everywhere,}$$

the Menshov universality property.

Clearly $M' \Rightarrow M'' \Rightarrow M'''$. In order to prove M', let us consider

$$G(P, N, \varepsilon, K) = \{S \in X : \exists n > N,\ ||S_n - P||_{C(K)} < \varepsilon\},$$

where P is a trigonometric polynomial, N an integer, $\varepsilon > 0$ and K a proper compact subset of T. It is an open subset of X. Now, given any ball (Q, δ) in X, whose center Q is a trigonometric polynomial, there exists a trigonometric polynomial S, which approximates Q in X and P in $C(K)$, in such a way that S belongs both to (Q, δ) and $G(P, N, \varepsilon, K)$ (we simply chose $S_n = S$). Therefore, $G(P, N, \varepsilon, K)$ is dense in X. Choosing now for P_j a sequence of trigonometric polynomials dense in $C(T)$, a sequence $N_j \to \infty$, $\varepsilon_j \to 0$, and $K_j = [0, 2\pi - \frac{1}{j}]$. M' holds on $\bigcap_j G(P_j, N_j, \varepsilon_j, K_j)$, a dense G_δ subset of X. The proof is achieved.

Here are some variations around the statement and the proof. First, if X is a complete metric group (say, Abelian for simplicity) and if Y is a subset of X, invariant under translation, which contains a dense G_δ *– set* A, each x in X can be written as $x = y_1 - y_2$, y_1 and y_2 belonging to Y. We simply choose y_2 in $A \cap (A + x)$, hence $y_1 \in A$. Therefore, $M2$ derives from the above statement.

Secondly, the proof relies on the fact that the system $\{e^{int}\}_{|n|>N}$ is total (dense span) in $C(K)$. The same is true, if we restrict ourselves to trigonometric series S of the form

$$S = \text{(formal)} \sum_{n \in \Lambda} C_n e^{int},$$

whenever $\{e^{int}\}_{n\in\Lambda}$ is total in all spaces $C(I)$ with $|I| < 2\pi$: in other words, when there is no non-zero entire function of exponential type $< \pi$, bounded on the real line, and vanishing on Λ. An explicit description of such Λ_1 is provided by the theorem of Beurling and Malliavin. Here $X = \mathbf{C}^\Lambda$.

In particular, the statement holds for trigonometric series of the Taylor type with $X = \mathbf{C}^{\mathbb{N}}$, where $\mathbb{N}$ is the set of all natural numbers :

$$S = \text{(formal)} \sum_{n \in \mathbb{N}} C_n e^{int},$$

therefore, with a slight change of notation, for power series

$$S = \text{(formal)} \sum_0^\infty c_n z^n.$$

But in this case we are free to choose for K not only a subset of the unit circle, but any compact subset of $\mathbf{C}$, which does not contain 0 and does not divide the plane (K^c connected). The main point is to check, that there is a countable family of such compact sets K_j, such that every K is contained in a K_j. Then the above proof applies and gives a generic Seleznev theorem.

Quasi all power series are universal in the sense of Seleznev.

S is universal in the sense of Seleznev if, given any K as above and any function $f(z)$ continuous on K and analytic in the interior of K, there exists a sequence n_j, such that

$$f(z) = \lim_{j\to\infty} S_{n_j}(z) \quad \text{uniformly on } K.$$

Mergelian's theorem says that $f(z)$ is a uniform limit of polynomials on K, and that is enough to show that the open sets $\{S : \forall N \exists n > N : \|S_n - P\| < \varepsilon\}$ are dense in X. The theorem of Seleznev (existence theorem) goes back to 1951 [22].

In the 1970's Luh, and independently Chui and Parnes, had the idea of considering Taylor series of analytic functions instead of formal power series. In the simplest case we choose now $X = H(D)$, the space of holomorphic functions in the open unit disc $D = \{z : |z| < 1\}$. Shortly afterwards Grosse-Erdmann gave a generic version of Luh's existence theorem, and universality as a generic phenomenon is explained by Grosse-Erdmann in his review paper [7] and in my expository article [8]. In the meantime, Nestoridis introduced a new notion of universality, that we shall describe now [21].

A power series S, convergent in the open unit disc D, is said to be universal (in the sense of Nestoridis) if, given any compact subset K of C, disjoint form D (that is, contained in $\{z : |z| \geq 1\}$) and not dividing the plane (K^c connected), and a function $f(z)$ continuous on K and analytic in K°, there exists a sequence n_j such that $S_{n_j}(z)$ converges to $f(z)$ uniformly on K.

The only difference with Luh, and Chui and Parnes, is that these authors considered K in the exterior of $D(\{z : |z| > 1\})$. There are more universal series in the sense of Luh (=Chui-Parnes), than in the sense of Nestoridis, but obviously universal series in the sense of Nestoridis have a closer relation with trigonometric series (because K in the unit circle is allowed).

Taking $X = H(D)$, Nestoridis proved : **Quasi all Taylor series of functions in $H(D)$ are universal (in the sense of Nestoridis).**

Here again a construction of a countable family of compact sets K_j is needed, and the Mergelian theorem is used to approximate both a polynomial in $C(K)$ and another polynomial in $H(D)$.

There are many other generic (=quasi sure) properties of $H(D)$. For example, it has been known for a long time, that quasi all Taylor series of functions in $H(D)$ are non-continuable across the circle $|z| = 1$. Consequently there are non continuable Taylor series that are N-universal (in the sense of Nestoridis). It is exactly what Nestoridis needed, in order to solve a problem of Pichorides, when he started this research.

Actually **Every N-universal Taylor series in non-continuable.**

This is a theorem of Melas and Nestoridis, that relies on a recent work of Gehlen, Luh and Müller on power series with Ostrowski gaps [5], [17]. It can be extended by introducing a new notion of universality for functions analytic in an open domain Ω (here $X = H(\Omega)$), provided Ω is simply connected and contained in the exterior of an angle. On the other hand, non-analytic continuation of universal functions is not a general fact, if non simply connected domains Ω are considered (Melas, to appear in Annales de l'Institut Fourier).

There are subtle questions about universal Taylor series in $H(\Omega)$. The first step is to define a universality property for the Taylor expansion of a function $f \in H(\Omega)$ around a point $\zeta \in \Omega$; this is done in a natural way, considering compacts sets K disjoint from Ω and not dividing the plane ; when the universality holds, we can write $f \in \mathcal{U}(\Omega, \zeta)$. How does the class $\mathcal{U}(\Omega, \zeta)$ depend on ζ ? Melas and Nestoridis exhibited a large class of sets Ω (those that we just described), such that $\mathcal{U}(\Omega, \zeta)$ does not depend on Ω and is not empty. Melas exhibited sets Ω for which $\mathcal{U}(\Omega, \zeta)$ is empty for all $\zeta \in \Omega$ except one [16], [17]. There is still much to do in that direction.

From now on $\Omega = D$. Then every N-universal Taylor series, considered on the unit circle ($z = e^{it}$), is a M (Menshov) and even an M' (see the definition above) universal trigonometric series. Is it true, that a Taylor series that is universal in the sense of Menshov (M) or in the restricted sense (M'), when considered on the unit circle is necessarily N-universal ? The answer is negative in both cases and it can be seen in many ways. The clearest way is to enlarge the universality class $\mathcal{U}(D) = \mathcal{U}(D, O)$ by restricting the class of compact sets under consideration, and to see that the new class is different from $\mathcal{U}(D)$. Given $r \geq 1$, if we restrict ourselves to compact K contained in the closed annulus $1 \leq |z| \leq r$, we obtain a class $\mathcal{U}_r(D)$. Here are two statements showing that all these classes are different, and different from $\mathcal{U}(D)$:

1. $\mathcal{U}_r(D)$ is strictly decreasing as r increases. Given $r > 1$, there exists $S \in \mathcal{U}_r(D)$, such that $\lim_{n\to\infty} |S_n(z)| = \infty$ for almost all z in $\{z : |z| > r\}$.

2. There exists $S \in \mathcal{U}_1(D)$, such that the sequence of partial sums S_n is not dense in any $C(K)$, such that $K^\circ \neq \emptyset$.

Let us turn to Menshov's statement $M3$ and $M4$, about universal trigonometric series, whose coefficients c_n tend to 0, that is, $(c_n)_{n\in\mathbb{Z}} \in c_0(\mathbb{Z})$. There is no hope to extend our first statement about M' being a quasi sure property to such series for M' excludes that the coefficients tend to 0.

However, replacing $X = \mathbb{C}^{\mathbb{Z}}$ by $X = c_o(\mathbb{Z})$, we can apply Baire's theorem : **Quasi all trigonometric series, whose coefficients tend to zero are universal in the sense of Menshov.** It needs some work to replace $c_o\mathbb{Z}$ by $c_o\mathbb{N}$. However the result still holds. **Quasi all Taylor series whose coefficients tend to zero, considered on the unit circle ($z = e^{it}$), are universal in the sense of Menshov.**

There are two extensions : one consists in replacing c_o by a smaller space, for example l^p with $p > 2$ (certainly l^2 does not work!), and the other is to replace $\mathbb{N}$ by a (lacunary) subset of $\mathbb{N}$. We state only the existence theorem (to apply Baire's theorem needs a regularity condition) for the first extension.

Let $\gamma(x) = o(x^2)$ $(x \to 0)$, $\gamma(x) \geq 0$, and let $l_\gamma(\mathbb{N})$ be the set of $(c_n)_{n\in\mathbb{N}}$, such that $\sum \gamma(|c_n|) < \infty$. There exists $(c_n) \in l_\gamma(\mathbb{N})$, such that the series $\sum\limits_0^\infty c_n e^{int}$ is universal in the sense of Menshov. The statement on quasi all Taylor series, whose coefficients tend to zero holds true, if we restrict ourselves to series $\sum\limits_{n\in\Lambda} c_n e^{int}$, when $\Lambda \subset \mathbb{N}$ has the following property : for each positive integer a, there exists an integer $\lambda > 2a$, such that Λ contains all points $m + \lambda n$ with $m \in \{-a, -a+1, ..., a-1, a\}$ and $n \in \{1, 2, ..., a\}$.

Here the principal tool is a lemma already used by Kahane and Katznelson in order to obtain another kind of universality of Taylor series, related to the radial behavior. Here is the result [9] : **Given γ as above and a Lebesgue-mensurable function f on the unit circle ($|z| = 1$) with values in $\{\mathbb{C}\} \cup \{\infty\}$, there exists a Taylor series $\sum\limits_0^\infty c_n z^n$ with $(c_n) \in l_\gamma(\mathbb{N})$, such that**

$$\lim_{r\uparrow 1} \sum_0^\infty c_n r^n e^{int} = f(e^{it}) \quad a.e.$$

АБСТРАКТ. Объяснительная статья на недавние работы Несторидиса, Меласа и Дж.-П. Кахана.

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