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INVARIANT ALGEBRAS OF OPERATOR FIELDS ON COMPACT ABELIAN GROUPS

0. Introduction. Let T be a compact Hausdorff space (everywhere assumed metrizable), let A be a unital C^*-algebra, and let $C(T, A)$ be the algebra of all continuous A-valied functions on T. By a *uniform algebra* we mean a closed subalgebra M of $C(T, A)$ which separates the points of T and contains all constants (definition in a more general case see in [5], [6], and [13]—[15]). Some properties of such noncommutative algebras were investigated in [2]—[4]. The present paper, which is an immediate continuation of the just quoted papers, is devoted to the iniform algebras on compact Abelian groups invariant with respect to all group shifts. The rich structure of such algebras permits to revise some early obtained results. The main restrictions as usually are the following: the algebra M is an *A-algebra* (i. e. supposed that M is a submodule of the A-module $C(T, A)$; the algebra A (the fibre) is *weakly*-transitive* (i. e. it obtains a pure state which is weakly total in the state space of A, see [4]).

Preliminary results of general nature are collected in Sections 1 and 2: we consider the Gelfand—Naimark—Segal (GNS) construction for conditional expectations which uses the notion of Hilbert C^*-module, a realization for the case of Abelian groups, and a property of the ideal set of weakly*-transitive algebras. The structure and properties of invariant uniform algabras on Abelian groups are investigated in Sections 3—5 (in the centre of attention are orthogonal decompositions, the notion of spectrum, peak points, and maximality).

A small part of the results were announced in [2]. All general concepts and facts, used without mentioning the source can be found in the monographs [5], [7], [12].

Notations (for details see [3], [4].

The algebras A and $C(T)$ we identify with their isomorphic images in $C(T, A)$: thus, an element $a \in A$ denotes also the function $a(t) = a$ for all $t \in T$, and analogously, an element $f \in C(T)$ denotes also the function $f(t) = f(t)1$, where $t \in T$, and 1 is the identity of A (and of $C(T, A)$ too).

The set of all conditional expectations of the aigebra $C(T, A)$ onto the subalgebra A we denote by $P(T, A)$.

Let S be the dual group of a compact Abelian group T and let γ^s be a natural representation of S in $C(T)$, then obviously $\gamma^{s+t} = \gamma^s \gamma^t$ (we use on T multiplicative, and on S additive notations of group operations).

Denote by $Sp_A\mathrm{M}$ the compact set of all continuous A-linear homomorphisms from an A-algebra M to the weakly*-transitive fibre A (see Definition 3.1 of [3] and Corollary 4.4 of [4]).

Denote by $S(A)$ and $P(A)$ the state space and the set of pure states of a C^*-algebra A.

1. Preliminaries. In this Section we consider general questions connected with the notion of Hilbert C^*-module; the definition (see e. g. [10] [11]) we adduce in a convenient form.

Let A be a C^*-algebra with identity 1, and let L be a linear space which is a right A-module and on which one determine a map $L \times L \to A$ (A-product) with the following properties:

1. $(x+y, z) = (x, z) + (y, z)$
2. $(xa, y) = (x, y)a$
3. $(x, y) = (y, x)$
4. $(x, x) \geqslant 0$
5. If $(x, x) = 0$, then $x = 0$.

Here x, y, z are from L, $a \in A$.

The space L with the norm

$$\|x\| = \|(x, x)\|^{1/2}$$

is a normed space, which is called *Hilbert A-module* if L is complet in this norm.

A standard example of Hilbert A-module is a $l^2_S(A)$-completion of $\sum_{s\in S}^{\oplus} A$ in the norm determined by the product

$$(x, y) = \sum_{s\in S} y_s^* x_s,$$

where S is a set, $x, y \in L$, $x = (x_s)$, $x_s \in A$.

Another important example of Hilbert A-module, which is a generalization of GNS-construction arises in the following way.

Let A, B be C^*-algebras with common identity, $A \subset B$.

Lemma 1.1. *Let p be a conditional expectation, $p: B \to A$, then the formula*

$$(x, y) = p(y^* x) \tag{1.1}$$

determines an A-product on B satisfying the conditions 1—4, and

6. $(1, 1) = 1$
7. $(xz, y) = (z, x^* y)$.

Conversely, every A-product on B satisfying the conditions 1—4, 6, 7 is generated by a unique conditional expectation. The condition 5 is equivalent to the exactness of the corresponding conditional expectation.

Proof. The first part of Lemma is an immediate consequence of the properties of conditional expectations (see e. g. S. 2, [3]). Conversely, let there exists on B an A-product satisfying the conditions 1—4, 6, 7. It is easy to check that the map $p: B \to A$, $p(b) = (b, 1)$ is a conditional expectation (a unique one) which generates the given A-product by the formula (1.1). The last part of the statement is evident.

Thus, let p be a conditional expectation, $p: B \to A$; then the set $N_p = \{x \in B.\ p(x^*x) = 0\}$ is a left ideal in B, and simultaneously, an A-bimodule (a consequence of the inequality $\|p(y^*x)\|^2 \leqslant \|y\|^2 \|p(x^*x)\|$ see (iii), Proposition 2,2, [3]. The linear space B/N_p is a right A-module with an A-product determined by p (Lemma 1.1); its completion $L^2(B, A, p)$ is, evidently a Hilbert A-module.

By analogy with usual GNS-construction one can indicate a standard (left) A-representation of the algebra B in $L^2(B, A, p)$ related with the starting conditional expectation and another facts which are, however, out of bounds of purposes of our paper.

Let T be a compact space, and let $C(T, A)$ be the A-bimodule of all continuous functions on T with values in a C^*-algebra A. For a conditional expectation $p \in P(T, A)$ the Hilbert A-module $L^2(C(T, A), A, p)$ will be denoted by $L^2(T, A, p)$.

Let T be a compact Abelian group, and let the map k_r, where $r \in T$, $k_r: T \to C(T, A)$, is defined by the formula

$$(k_r x)(t) = x(rt). \tag{1.2}$$

We call a conditional expectation $p_0 \in P(T, A)$ *T-invariant* iff[1] $p_0 \cdot k_r = p_0$ for every $r \in T$.

Proposition 1.2. *There exists and is unique on $C(T, A)$ an exac invariant conditional expectation (i. c. e.).*

Proof. Let σ be the probability Haar measure on T, and let φ_σ be the corresponding functional on $C(T)$. Then (see the proof of Proposition 1.4, [4]) there exists a conditional expectation p_0 on $C(T, A)$ such that the restriction of p_0 on $C(T)$ coincides with φ_σ. Obviously, p_0 is an i. c. e.

Let p be an i. c. e. from $P(T, A)$, and let s be a character on T. Then, for every $t \in T$ we have.

$$p(\gamma^s) = p(k_t \gamma^s) = \gamma^s(t)\, p(\gamma^s) \tag{1.3}$$

from which we obtain that if s is not a neutral character, then $p(\gamma^s) = 0$. Consequently, the restriction of p on $C(T)$ is a T-invariant state, hence, by the uniqueness of the Haar measure we obtain the uniqueness of p_0 (we use again Proposition 1.4, [4]). Verify now the exactness of p_0. Let $x \in C(T, A)$, $p_0(x^*x) = 0$, and let φ be any pure state on A. Then $\Phi = \varphi \circ p_0$ is a state on $C(T, A)$ the restriction of which on A coincides with φ. We have for some probability measure μ on T

$$\Phi(x) = \int \varphi(x(t))\, d\mu(t)$$

(by Theorem 1.5, [3]). It is evident that the functional Φ is invariant with respect to group shifts, hence μ coincides with the Haar measure σ. Further, as

$$\int \varphi(x^*x(t))\,d\sigma(t) - \Phi(x^*x) = \varphi(p_0(x^*x)) = 0$$

we obtain that $\varphi(x^*x(t)) = 0$ for each t. Since φ is arbitrary then, finally, $x = 0$.

Thus, $L^2(T, A, p_0)$ is a completion of $C(T, A)$ in the norm of the Hilbert A-module, corresponding to p_0. The i. c. e. just obtained can be defined, with necessary justifications, as $\int x d\sigma$ for $x \in C(T, A)$, see Remark 1.3, [4].

The following result clarifies the structure of the Hilbert A-module $L^2(T, A, p_0)$.

Proposition 1.3. *Let S be the group of characters of a group T let p_0 be an i. c. e. Then the Hilbert modules $L^2(T, A, p_0)$ and $l_S^2(A)$, are isomorphic.*

Proof. By an isomorphism we mean an A-linear bijection which preserves the A-product.

Evidently, it sufficient to indicate an "A-basis" in $L^2(T, A, p_0)$ corresponding to the standard basis in $l_S^2(A)$. The set $\gamma^s = \{\gamma^s, s \in S\}$ is the required one. More precisely, every element x from $L^2(T, A, p_0)$ have a unique orthogonal expansion of the form

$$x = \sum a_s \gamma^s, \tag{1.4}$$

where $s \in S$, $a_s \in A$, and the convergence of the series is understood in the norm of $L^2(T, A, p_0)$.

Remark 1.4. Let $x \in C(T, A)$, then the "coeficients" a_s in the expansion (1.4) are determined by the formula

$$a_s = p_0(x\gamma^{-s}). \tag{1.5}$$

Remark 1.5. Roughly speaking, $L^2(T, A, p_0)$ is a "tensor product" of $L^2(T, \sigma)$ and A, just as $l_S^2(A) = l^2 \otimes A$, therefore Proposition 2.2 is just a reformulation of the classical result concerning an isomorphism of Hilbert spaces of the same dimension.

2. Ideals in weakly*-transitive algebras. We represent here an important property of the ideal set in weakly*-transitive algebras, necessary in the sequel.

Lemma 2.1. *Every two closed two-sided non-zero ideals in a weakly*-transitive algebra have a non-zero intersection.*

Proof. Note, at first, that if I_1 and I_2 are closed two-sided non-zero ideals with zero intersection in a C^*-algebra A, then each pure state on A annuls one of these ideals. Indeed, let H_φ be a Hilbert space corresponding to φ by GNS-construction, and let H_1, H_2 be invariant and, obviously, mutually orthogonal subspaces of H_φ generated by the ideals I_1, I_2. Since the GNS-representation is irreducible, one of the subspaces H_1 or H_2 must be zero, which means that φ annuls the corresponding ideal.

Now, let A be weakly*-transitive, and let φ be a total state. Then φ does not annul any non-zero ideal: otherwise, a state which is non-zero on this ideal can not be approximated by convex combinations of the unitary shifts of φ, see [4].

3. Structure of invariant uniform algebras. Let T be a compact Abelian group, let M be an invariant (with respect to group shifts) A-algebra, $M \subset C(T, A)$, and let p_0 be the i. c. e., defined by Proposition 1.2.

Lemma 3.1. *Let A be a C^*-algebra with trivial centre, and let $x \in C(T, A)$. Then $x \in M$ iff $a_s\gamma^s \in M$ for all $s \in S$ in the expansion (1.4)*

Proof. We use the following easily checking general fact ("Fubini theorem").

F) Let T_1, T_2 be compact spaces, let ω be a state on $C(T, A)$ and let $p \in P(T_2, A)$. Define on $C(T_1 \times T_2, A)$ the operators

$$\hat{\omega} : C(T_1 \times T_2, A) \to C(T_2) \subset C(T_2, A),$$

$$\hat{p} : C(T_1 \times T_2, A) \to C(T_1, A)$$

by the formulas:

$$(\hat{\omega} y)(t_2) = \omega(y(\cdot, t_2)), \tag{2.1}$$

$$(\hat{p} y)(t_1) = p(y(t_1, \cdot)), \tag{2.2}$$

where $y \in C(T_1 \times T_2, A)$, Then

$$p \cdot \hat{\omega} = \omega \cdot \hat{p} \tag{2.3}$$

(i. e. $p \circ \hat{\omega}(y) = (\omega \circ p(y))1$).

Now, let $x \in M$, $x \sim \sum a_s \gamma^s$, where the convergence of the series is understood in the norm of $L^2(T, A, p_0)$. Check that $a_s\gamma^s$ belongs to M for each $s \in S$. It is sufficient to show that for every functional ψ on $C(T, A)$ which annuls M, one have $\psi(a_s\gamma^s) = 0$. Evidently, for each $s \in S$, the function y, defined by the formula.

$$y(t, r) = \gamma^{-s}(r)\, x(tr) \tag{2.4}$$

belongs to $C(T \times T, A)$, moreover

$$p_0(y) = a^s \gamma^s.$$

Indeed, by the formulas (2.2), (1.1) and (1.5) we have

$$p_0(y)(t) = p_0(\gamma^{-s} k_t x) = \gamma^s(t)\, p_0(\gamma^{-s} x) = a_s \gamma^s(t).$$

Therefore, by (2.3) and (2.4),

$$\psi(a_s\gamma^s) = \psi \cdot p_0(y) = p_0 \cdot \hat{\psi}(y) = 0$$

since $\hat{\psi}(y)(r) = \gamma^{-s}(r)\psi(k_r x) = 0$ (note, that by invariantness of M, we have $k_r x \in M$ for all r.

Conversely, let $x \in C(T, A)$ and let $a_s\gamma^s \in M$ for all s in the expansion $x \sim \sum a_s \gamma^s$. Show, that $x \in M$. Let a functional ψ be annuling M , but $\psi(x) \neq 0$. Then, the non-zero function $f(t) = \psi(k_t x)$ belongs to $C(T)$, therefore there exists a character $s \in S$ such that $p_0(\gamma^{-s} f) \neq 0$.

If y is a function from $C(T \times T, A)$, defined by the formula (2.4), then $\psi \cdot \hat{p}_0(y) = \psi(a_s\gamma^s) = 0$, i. e. $a_s\gamma^s \in M$. On the other hand,

$p_0 \cdot \hat{\psi}(y) = p_0(\gamma^{-s} f) \neq 0$, which is a contradiction with (2.3), and thus the proof is complete.

Denote for all $s \in S$

$$A_s(\mathbf{M}) = A_s = \{a \in A,\ a_s \gamma^s \in \mathbf{M}\}.$$

Obviously, A_s is a closed two-sided (consequently selfadjoint) ideal in A. Let S_M be the set of those $s \in S$, for which $A_s \neq 0$. By Lemma 3.1, $s \in S_M$ iff there exists $x \in \mathbf{M}$, such that $(x, \gamma^s) \neq 0$. In general, $\gamma^{S_M} = \{\gamma^s, s \in S_M\}$ is not a subset of $\mathbf{M}$. However, if $\gamma^{S_M} \subset \mathbf{M}$, then $\mathbf{M}$ is decomposable, i. e. it is generated by the algebra A and by a uniform subalgebra of $C(T)$ (see Propositions 1.8 and 3.7 of [3]). Chow, that the same is true for the case of an algebra with simple fibre.

Lemma 3.2. *Let A be simple C^*-algebra, and let $\mathbf{M}$ be a uniform invariant A-algebra on a compact Abelian group T. Then $\mathbf{M}$ is decomposable.*

Proof. By simlicity of A, for every $s \in S_M$ the ideal A_s coincides with A, therefore γ^s belongs to $\mathbf{M}$. Let M be a minimal closed subalgebra of $C(T)$ generated by γ^{S_M}. By Lemma 3.1 and by the expansion (1.4), this algebra is separating, and consequently is uniform. Evidently, the algebras $\mathbf{M}$ and A together generate the algebra $\mathbf{M}$.

Corollary 3.3. *In conditions of the Lemma, the algebra $\mathbf{M}$ coincides with $C(T, A)$ iff $Sp_A\mathbf{M}$ coincides with T.*

Note, that in contradistinction to the commutative theory, S_M is not a subsemigroup of S; however, this property is satisfied for a large class of algebras. The following statement contains in particular, an extension of one of the results of Arens and Singer, [1].

Theorem 3.4. *Let A be a weakly*-transitive algebra with identity, and let $\mathbf{M}$ be a uniform T-invariant A-algebra or a compact Abelian group T with the dual group S. Then*

(i) S_M is a subsemigroup of S;

(ii) there exists a natural homeomorphism of $Sp_A\mathbf{M}$ with the compact space (in the standard topology) of all homomorphisms of S_M to the unit circle.

Proof. (i) Let $s, u \in S_M$ then $A_s \gamma^s, A_u \gamma^u \subset \mathbf{M}$, therefore $A_s A_u \gamma^{s+u} \subset \mathbf{M}$. Evidently, $A_s A_u \subset A_{s+u}$. On the other hand, $A_s A_u = A_s \cap A_u$ (see [5], 1.9 12) is a non-zero ideal, by Lemma 2.1. Consequently, $s + u \in S_M$.

(ii) Let M be a uniform algebra in $C(T)$, generated by γ^{S_M}, and let $\mathbf{M}$ be a uniform A-algebra in $C(T, A)$, generated by M and A (decomposable, by definition). Then, by Proposition 3.7, [3] (which was proved for a simple fibre; however, it is true in weakly*-transitive case by Corollary 4.4. [4]), the maximal ideal space of M coincides with $Sp_A\mathbf{M}'$, and, by Arens—Singer theorem, it is homeomorphic to the compact set of all homomorphisms of the semigroup S_M to the unit circle. It remains to check only that $Sp_A\mathbf{M}'$ coincides with $Sp_A\mathbf{M}$. Clearly, that each $p \in Sp_A\mathbf{M}$ determines some $p' \in Sp_A\mathbf{M}$ (the restriction: $p' = p \mid M$). Show that the map $'$ is a monomorphism. Let

$\in Sp^{\prime\prime} M'$, $p_1 \neq p_2$, then there exists $s \in S_M$ for which $p_1(\gamma^s) = p_2(\gamma^s)$, therefore, for some $a \in A$, we obtain

$$p_1'(a\gamma^s) = p_1(a\gamma^s) = ap_1(\gamma^s) = ap_2(\gamma^s) = p_2'(a\gamma^s).$$

Finally, show that every homomorphism $q \in Sp_A$ M has an extension on M′. Let q' be a conditional expectation from $P(T, A)$, such that $q' | M = q$ (see Corollary 4.4, [4]). For every $s \in S_M$ and $a \in A_s$ we have

$$q(a\gamma^s) = q'(a\gamma^s) = aq'(\gamma^s),$$

and moreover, it is clear, that the value $q'(\gamma^s)$, which is scalar, does not depend on the choice of extension (see Propositions 1.1, and 4.2 (i) [4]). Thus, q' is well defined on γ^{S_M}, and consequently on M′ too. It remains to verify that the restriction of q' on γ^{S_M} is a character. Let, $u, s \in S_{M'} = S_M$, $a \in A_u \cap A_s$, $a > 0$, then $a\gamma^u a\gamma^s = a^2\gamma^{s+u} \neq 0$ and so $a^2 q'(\gamma^u) q'(\gamma^s) = a^2 q'(\gamma^{u+s})$ therefore, as above, $q'(\gamma^{u+s}) = q'(\gamma^s)$.

4. Maximality. Let T be a compact space, let M be a uniform algebra from $C(T, A)$, where A is a unital C^*-algebra. A uniform algebra M is called *maximal*, if $C(T, A)$ is the only uniform algebra containing M.

Let M be an invariant uniform algebra on a compact Abelian group T. Denote by $S_M(I)$ for each closed two-sided ideal $I \subset A$ the following set

$$S_M(I) = \{s \in S_M, A_s = I\}.$$

Even in the case when S_M is a semigroup (see e. g. Theorem 3.4) $S_M(I)$ could not be a subsemigroup of S_M. However, $S_M(A)$ is always a semigroup.

Lemma 4.1. *Let I be a two-sided closed ideal in weakly*-transitive algebra A, and let M_1 be a uniform T-invariant A-algebra, generated by $(A_s(M) + I)\gamma^s$, where $s \in S_M$. Then*

$$A_s(M_1) = A_s + I.$$

Proof. Note first, that $A_s(M) + I$ is also a two-sided closed ideal (see [5]). Further, we have $M_1 = \overline{M + M'}$, where M′ denotes a closed subalgebra generated by $I\gamma^s$, when s runs over all S_M. Indeed, M + M′ forms a dense subalgebra in M_1, since M′ is an ideal in M. Introduce for every $s \in S_{M_1} = S_M$ a linear operator $p_s : M \to A$ by the formula: $p_s(x) = p_0(x\gamma^{-s})$, where p_0 is the i. c. e. (see Proposition 1.2). Using Lemma 3.1 and Remark 1.4 we obtain at last

$$A_s(M_1) = p_s(M_1) = A_s(M) + I.$$

The following results are evident.

Corollary 4.2. *Let I be a two-sided closed ideal in A, such that for some $s \in S_M$, $A_s \subset I$, and let M_1 be a uniform T-invariant A-algebra generated by M and $I\gamma^s$. Then $A_s(M_1) = I$.*

Corollary 4.3. *Let $s_1, s_2 \in S_M$, s_2, and $A_{s_1} \neq A_{s_2} \neq A$. Then, if M it a uniform T-invariant A-algebra generated by M and $(A_{s_1} + A_{s_2})\gamma^{s_1}$, one have $A_{s_1}(M_1) = A_{s_2}(M)$.*

The following result presents a description of maximal invariant uniform algebras and, in particular, it extends one of the results of the Hoffman and Singer, [9].

Theorem 4.4. *Let* **M** *be a uniform T-invariant maximal A-algebra with weakly*-transitive fibre. Then* **M** *is a Dirichlet algebra (i. e* $\overline{Re\,\mathbf{M}} = Re\,C(T, A)$*), and the semigroup* $S_{\mathbf{M}}(A)$ *defines on S an Archimedian order. In addition:*

i) If $S_{\mathbf{M}} \neq S$*, then* $S_{\mathbf{M}} = S_{\mathbf{M}}(A)$ *(hence* **M** *is decomposable). In this case the algebra A is simple and*

$$Sp_A(\mathbf{M}) = T \times [0, 1]/T \times \{0\}.$$

(ii) If $S_{\mathbf{M}} = S$*, then there exists a maximal two-sided ideal I of A, such that* $S_{\mathbf{M}}(I) \cup \{0\} = -S_{\mathbf{M}}(A)$*. In this case* $Sp_A(\mathbf{M}) = T$*.*

Proof. (i) If $S_{\mathbf{M}} \neq S_{\mathbf{M}}(A)$, then using Lemma 4.1 and the fact that $S_{\mathbf{M}}$ is a semigroup we can construct an algebra $\mathbf{M}_1$, for which $\mathbf{M} \underset{\neq}{\subset} \mathbf{M}_1$ and so we obtain a contradiction with the maximality of **M**. Thus, $S_{\mathbf{M}} = S_{\mathbf{M}}(A)$. The algebra A has an identity, hence the algebra **M** is generated by the algebras A and M, where M is a uniform subalgebra of $C(T)$ generated by $\gamma^{S_{\mathbf{M}}}$ (i. e. **M** is decomposable). Since **M** is maximal in $C(T, A)$, M is maximal in $C(T)$. By the Hoffman—Singer theorem (see [9]), $S_{\mathbf{M}}$ defines on S an Archimedian order.

It remains to prove the simplicity of A. Let I be a non-trivial two-sided closed ideal of A. Put $A_s(\mathbf{M}_2) = I$, for $s \in S - S_{\mathbf{M}}$ and $A_s(\mathbf{M}_2) = A$, for $s \in S_{\mathbf{M}}$; using Lemma 4.1, construct a new algebra $\mathbf{M}_2$, for which $\mathbf{M} \underset{\neq}{\subset} \mathbf{M}_2$ and obtain a contradiction with the maximality of **M**.

The last statement of (i) can be obtained from Proposition 3.7 of [3] ($Sp_A\mathbf{M} = Sp\,M$) and the corresponding result for commutative uniform algebras, [7], [9].

(ii) Let $S_{\mathbf{M}} = S$. Note, that $S_{\mathbf{M}}(A) \neq S$, because otherwise we obtain $\mathbf{M} = C(T, A)$. By Corollaries 4.2 and 4.3 and by the maximality of the algebra **M**, we have an existence of a two-sided closed ideal I, such that for every $s \in S_{\mathbf{M}} - S_{\mathbf{M}}(A)$, $A_s = I$.

Note, that $S_{\mathbf{M}}(A)$ is a maximal subsemigroup of S (if S' is such that $S_{\mathbf{M}}(A) \underset{\neq}{\subset} S'$; then putting $A_s(\mathbf{M}') = A$ for all $s \in S'$, we can obtain, by Lemma 4.1, a new algebra $\mathbf{M}'$, $\mathbf{M} \underset{\neq}{\subset} \mathbf{M}'$; this contradicts with the maximality of M).

Further, $S_{\mathbf{M}}(A)$ does not contain any subgroup of S. Indeed, let S' be a maximal subgroup of S, which is contained in $S_{\mathbf{M}}(A)$. Consider a subgroup T_0 of T, $T_0 = \{t \in T,\ \gamma^s(t) = 1,\ s \in S'\}$. Let M denotes a uniform subalgebra of $C(T)$ generated by $S_{\mathbf{M}}(A)$. Then, as it follows from Theorem 2.2 of [7], the restriction of M on T_0 does not coincide with $C(T_0)$. Therefore, there exists a charcter s_0 of the group T_0, which impossible to approximate on T_0 by linear combinations of characters from $S_M(A)$. Show, that the restrictions of M on T_0 is not dense in $C(T_0, A)$. It is sufficient to verify, that γ^{s_0} can not be approximated on T_0 by the finite lihear combinations of the elements of M of t e form $\sum a_s\gamma^s$. Let φ be a state on A, which annuls I, then

$$\|\gamma^{s_0} - \sum a_s\gamma^s\|_{(T_0, A)} \geqslant$$

$$\max_{t\in T_0} |\varphi(\gamma^{s_0}(t)1 - \varphi(\sum_{s\in S_M(A)} a_s\gamma^s(t) + \sum_{s\in S-S_M(A)} a_s\gamma^s(t))| =$$

$$= \max_{t\in T_0} |\gamma^{s_0}(t) - \sum_{s\in S_M(A)} \varphi(a_s)\gamma^s(t)| =$$

$$= \|\gamma^{s_0} - \sum_{s\in S_M(A)} \varphi(a_s)\gamma^s\|_{C(T_0)} > 1.$$

Therefore $\overline{M|T_0} \neq C(T_0, A)$, and consequently, for each $t\in T$, $M|T_0t \neq C(T_0t, A)$.

Let t be such that $T_0t \neq T_0$, then there exists a function $x\in C(T, A)$ for which $x|T_0 = \gamma^{s_0}$ and $x|T_0t = 0$. Denote by M′ a uniform algebra generated by M and x. It is easy to check, that $M \underset{\neq}{\subset} M' \underset{\neq}{\subset} C(T, A)$ which is a contradiction with the maximality of M.

Thus, $S_M(A)$ does not contain any subgroup of S. Then by maximality of M we obtain that $S_M(A)$ defines on S an Archimedian order (see e. g. [8]). By the way, we have $\{0\} \cup S_M(I) = -S_M(A)$.

The statement concerning the spectrum of the algebra M is a trivial consequence of Theorem 3.4 (ii).

Since $S_M(A)$ defines an Archimedian order on S, in both cases (i) and (ii), the algebra M_1 generated by $S_M(A)$ and A is, obviously, a Dirichlet algebra ($M_1 + M_1^*$ is dense in $C(T, A)$, however $Re(M_1 + M_1^*) = Re\,M_1 \subset Re\,M$).

5. Peak points. In [4] we consider different concepts of peak point. They coincide in the case of invariant uniform algebras. This fact is an immediate consequence of the following result.

Theorem 5.1. *Let* M *be an invariant A-algebra on a compact Abelian group t. Then every point of T is a peak point for* M.

Proof. By invariantness of M, it is sufficient to verify that there exists $x\in M$, such that $x(1) = 1$, $\|x(t)\| < 1$, when $t \neq 1$, 1 being a neutral element of T.

By Lemma 3.1, the linear combinations of elements of the form $a\gamma^s$, where $a\in A_s$, $s\in S_M$ are dense in M. Since A_s is a two-sided closed ideal of A (and, consequently a C^*-algebra), then, clearly, the linear combinations of the same form with $a\in A_+$ are dense in M too.

Let $t_0\in T$, $t\neq 1$. Show, that for each state φ on the algebra A, there exist $a\in A_+$, and $s\in S_M$, such that $\varphi(a)\neq 0$, $\gamma^s(t_0)\neq 1$. Indeed, et q_φ be a corresponding conditional expectation of the algebra $C(T, A)$, onto $C(T)$, $q_\varphi\in Q(T, A)$, see Proposition 2.6, [3],

$$(q_\varphi x)(t) = \varphi(x(t)).$$

By Corollary 2.7 of the 'same paper, there exists $z \in M$, such that $(q_\varphi z)(1) = 1$, and $(q_\varphi z)(t_0) = 0$. Therefore, there exists $a\gamma^s$ in M, for which $\varphi(a)\gamma^s(t_0) \neq \varphi(a)\gamma^s(1) = \varphi(a)$, and so $\varphi(a) \neq 0$, $\gamma^s(t_0) \neq 1$.

Further, by the weak compactness of the state space $S(A)$, there exist $a_1, a_2, \cdots, a_n \in A$, and $s_1, s_2, \cdots, s_n \in S_M$, such that for each $\varphi \in S(A)$, we have $\varphi(a_k) \neq 0$, $\gamma^{s_k}(t_0) \neq 1$ for some couple (a_k, s_k).

We denote by y the element from M defined by the formula

$$y = 1 + (\sum \|a_k\|)^{-1} \sum a_k (\gamma^{s_k} - 1).$$

Obviously, $y(1) = 1$. Since y is a normal element of $C(T, A)$, and taking into account the relations

$$|\gamma^{s_k}(t) - 1|^2 = 2(1 - Re\,\gamma^{s_k}(t)) < 4, \ k = 1, 2, \cdots,$$

we have, for each $t \in T$,

$$\|y(t)\|^2 = \sup_{\varphi \in S(A)} |\varphi(y(t))|^2 <$$

$$1 + 2(\sum \|a_k\|)^{-1} \sum \varphi(a_k)\,(Re\,\gamma^{s_k}(t) - 1) +$$

$$(\sum \|a_k\|)^{-2} (\sum \varphi(a_k) |Re\,\gamma^{s_k}(t) - 1|)^2 \leqslant$$

$$1 + 2(\sum \|a_k\|)^{-1} \sum \varphi(a_k)(Re\,\gamma^{s_k}(t) - 1) +$$

$$(\sum \|a_k\|)^{-2} \sum \varphi(a_k) \sum \varphi(a_k) |Re\,\gamma^{s_k}(t) - 1|^2 =$$

$$1 + 2(\sum \|a_k\|)^{-1} \sum \varphi(a_k)(Re\,\gamma^{s_k}(t) - 1) +$$

$$2(\sum \|a_k\|)^{-1} \sum \varphi(a_k)(1 - Re\,\gamma^{s_k}(t)) = 1.$$

Besides, it is easy to check that when $t = t_0$, we have a strong inequality: $\|y(t_0)\| < 1$. Indeed, the equality is possible only when $\gamma^{s_k}(t_0) = \gamma^{s_m}(t_0)$ for those $k, m = 1, 2, \cdots, n$, for which $\varphi(a_k)$, and $\varphi(a_m) \neq 0$; then we have $\gamma^{s_k}(t_0) = 1$ for all k, what contradicts to the choice of $a_1, \cdots, a_n$; $s_1, \cdots, s_n$.

Now, let U_n be a countable system of neighbourhoods of the neutral element, $\cap U_n = \{1\}$. Then, by compactness of the sets $F_n = T - U_n$, there exists a system of elements x_n of M with the properties: $x_n(1) = 1$, $\|x_n\| = 1$, $\|x_n(t)\| < 1$, when $t \in F_n$. Therefore, the element

$$x = \sum_{1}^{\infty} 2^{-n} x_n$$

realizes the peak at identity.

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Վ. Ա. ԱՐԶՈՒՄԱՆՅԱՆ, Ս. Ա. ԳՐԻԳՈՐՅԱՆ. Օպերատորային դաշտերի ինվարիանտ հանրահաշիվներ կոմպակտ աբելյան խմբերի վրա (*ամփոփում*).

Աշխատանքը նվիրված է կոմպակտ աբելյան խմբերի վրա որոշված C*-*արժեքային անընդհատ ֆունկցիաների ոչ կոմուտատիվ հավասարաչափ հանրահաշիվներին, որոնք ինվարիանտ են տեղաշարժերի նկատմամբ։ Բնական պայմաններին բավարարող հանրահաշիվների համար ստացված են Արենսի, Զինգերի, Հոֆմանի հայտնի թեորեմների անալոգներ*

(մասնավորապես նվիրված հարաբերական սպեկտրին, մաքսիմալության հարցերին, պիկի կետերի գոյությանը)։

В. А. АРЗУМАНЯН, С. А. ГРИГОРЯН. *Инвариантные алгебры аопраторных полей на компактных абелевых группах (резюме).*

Работа посвящена некоммутативным равномерным алгебрам C^*-значных непрерывных функций на компактной абелевой группе, инвариантным относительно сдвигов. При естественных условиях на алгебру получены аналоги некоторых известных результатов Аренса, Зингера, Гоффмана из теории равномерных алгебр (в частности, описание относительного спектра, вопросы максимальности, существование точек пика).

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