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THE BOUNDARIES OF UNIFORM ALGEBRAS OF OPERATOR FIELDS

0. Introduction. The paper is devoted to the investigation of the geometrical structure of the spectrum of operator-valued functional algebras — a noncommutative analog of usual uniform algebras. The objects of such kind were first introduced by Fell in [1] (so called maximal C^*-algebras of operator fields; detailed presentation of the theory see in [2]). Non-involutive closed subalgebras of such C^*-algebras present certain interest: a number of papers (see e. g. [3]—[5]) are devoted to generalizations of some results known for classical (commutative) uniform algebras.

In [6] the systematic investigation of uniform algebras of operator fields of a specific form was started, illustrating many difficulties inherent in the general case.

Let T be a compactum (for simplicity everywhere supposed metrizable), A be a C^*-algebra with identity (called a fibre) and let $C(T, A)$ be the C^*-algebra of all continuous A-valued functions on T (from algebraic point of view $C(T, A)$ is the tensor product $C(T)\otimes A$). A uniform algebra $\mathfrak{M}$ is, by definition, a closed subalgebra of $C(T, A)$ separating points and containing the constants. The notion of relative spectrum $Sp_A\mathfrak{M}$ (generalizing the maximal ideal space) is defined as the set of all homomorphisms from $\mathfrak{M}$ into A which may be extended to a conditional expectation of $C(T, A)$ onto its subalgebra isomorphic to A. In the important case of A-algebras (uniform algebras which are A-submodules of $C(T, A)$) with simple fibre, the relative spectrum coincides with the space of all A-linear homomorphisms. The presence of a spectrum in the noncommutative case permits to consider also the other objects connected with it. Corresponding results for various classes of algebras are presented in [6].

In the paper we continue the investugation of noncommutative algebras in two directions. On the one hand, the new objects connected with the geometrical structure of the spectrum are introduced and investigated; on the other hand, the class of fibre C^*-algebras which permits a detailed analysis of corresponding uniform algebras is extended essentially.

In the first two sections we investigate the compactum $P_0(\mathfrak{M})$ (playing the role of "states" on the algebra $\mathfrak{M}$), the extreme points of which are characterized in terms of representing measures. In Section 3 we introduce the notion of reduced spectrum $Sp_A^r\mathfrak{M}$, adapted for uniform algebras with arbitrary fibre and coincided with $Sp_A \mathfrak{M}$ for an algebra A with trivial center. Instead of algebras with simple fibre we suggest for consideration the weakly*-transitive algebras, which permit to revise some results. The important notion of peak point replacing the notion of pure state for involutive algebras does not allow a literal generalization on the noncommutative case; the discussions of some possibilities are placed in Section 5. The main result of the paper consists of "function-theoretic" description of the set of extreme points of $P_0(\mathfrak{M})$; providing the existence of weak peak points of an A-algebra with weakly*-transitive fibre (Section 6). Note, that in [3] Taylor had given a description of peak points in terms of orthogonal operator-valued measures, and in [4], [5] the Hoffman-Wermer theorem was proved in the factual assumption of the existence of peak points. In Section 7 the notion of Shilov boundary is introduced and the maximum principle is proved. The present paper may be considered as a continuation of [6], the results of which are used constantly. A certain (very small) part of facts were announced in [7].

All the information, pertinent concepts and results, used without mentioning the source, can be found in the monographs [2], [8], [9].

1. **The compactum $P_0(T, A)$.** In this Section we introduce and investigate the space $P_0(T, A)$ — a subset of the space of all conditional expectations of the algebra $C(T, A)$ onto its subalgebra A[1]), coinciding with it for the algebras with trivial center. This space is compact for every fibre, which permits essentially extend the class of algebras under consideration. In the next Section the obtained results are generalized to arbitrary uniform subalgebras of $C(T, A)$.

Let T be a metrizable compactum (non single-point), A be a unital C^*-algebra, and $C(T, A)$ be the C^*-algebra of all continuous mappings from T into the algebra A. Let $P(T, A)$ be the set of all conditional expectations from $C(T, A)$ onto the subalgebra A equipped with the topology of pointwise convergence (the details see in [6], Section 2). This set is a closed convex subset of the space of all continuous A-linear mappings of $C(T, A)$ into A. Denote by $P_0(T, A)$ the subset of those conditional expectations, the restrictions of which on $C(T)$ are linear functionals (i. e. $p \in P_0(T, A)$ iff $p(C(T)) \subseteq Ce$, e being the identity of A).

Proposition 1.1. *The sets $P_0(T, A)$ and $P(T, A)$ are coinciding iff algebra A is with trivial center.*

Proof. Let $Z(N)$ be the center of an algebra N. Note at first, that $C(T) \subseteq Z(C(T, A))$ (see [6], Proposition 1.4), therefore for each $p \in P(T, A)$ we have $p(C(T)) \subseteq Z(A)$ (cf. [6], the beginning of the proof of Theorem 3.3). Hence, if the algebra A is with trivial center then $P_0(T, A) = P(T, A)$. Conversely, let $Z(A) = C(Y)$ for a non single-point compact Y, and h be some Uryson function on Y, where $0 < h(y) \leqslant 1$. For $x \in C(T, A)$ put $p(x) = x(t_1)h + x(t_2)(1 - h)$, $t_1 \neq t_2$. Evidently,

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[1] In contrast with [6], in this paper we make no difference between the algebras A, $C(T)$, and their isomorphic images in $C(T, A)$.

p is a conditional expectation and $p \in P_0(T, A)$ since its valus on $C(T)$ are not constants (by the properties of h).

Thus, for the algebras with non-trivial center, $P_0(T, A)$ is a proper subset of $P(T, A)$ which is, obviously, convex and closed in the topology of pointwise convergence; in addition T is imbedded in $P_0(T, A)$ (identifying the compact T with the set of atomic conditional expectations, $p_t(x) = x(t)$, $t \in T$, $x \in C(T, A)$).

Proposition 1.2. *The set $P_0(T, A)$ is a convex subcompactum of the space of all continuous A-linear mappings of $C(T, A)$ into the algebra A.*

Proof. Show, that $P_0(T, A)$ is homeomorphic to the state space of the algebra $C(T)$, from which the result is following at once.

Clearly, every $p \in P_0(T, A)$ determines some state on $C(T)$. namely the restriction of p on $C(T)$. Here the different p_1, p_2 from $P_0(T, A)$ determine the different states (see the beginning of the proof of Theorem 3.3, [6]).

Conversely, each extreme point of the state space of $C(T)$ is determined by some t from T and in turn determines a conditional expectation $p_t \in P_0(T, A)$. The finite convex combinations of such states are corresponding to some conditional expectations from $P_0(T, A)$ (as this set is convex). The restriction mapping $p \to p | C(T)$ is continuous hence t is extendable to the whole of the state space of $C(T)$ (remind that $C(T, A)$ is generated by its subalgebras A and $C(T)$, see [6], Proposition 1.4).

Remark 1.3. Thus, $P_0(T, A)$ may be identified with the state space of the algebra $C(T)$, hence with the space $M(T)$ of probability Borel measures on T. That permits to define an A-valued integral on T for each $x \in C(T, A)$ with respect to the measure μ by the formula

$$\int x(t)\, d\mu(t) = p_\mu(x) \tag{1.1}$$

where p_μ is the conditional expectation from $P_0(T, A)$, corresponding to the measure μ In turn, every measure corresponding to some $p \in P_0(T, A)$ will be called *representing* for p

Proposition 1.4. *The space $P(T, A)$ is homeomorphic to the set of all continuous mappings of the maximal ideal space of $Z(A)$ (the center of the algebra A) into $P_0(T, A)$, equipped with the topology of pointwise convergence.*

Proof. Let $p \in P(T, A)$, and p_0 be the restriction of p on $C(T)$. Then (see the proof of Proposition 1.1) p_0 maps $C(T)$ into $Z(A)$, hence it uniquely determines a continuous mapping of the compactum Y into the state space of $C(T)$ (here we identify $Z(A)$ with $C(Y)$ for some compactum Y). It follows from Remark 1.3 that this mapping is coinciding with the mapping of Y into $P_0(T, A)$.

Now let $h: Y \to P_0(T, A)$; show that this continuous mapping generates some conditional expectation from $P(T, A)$. Two elementary facts from the theory of C^*-algebras are necessary for this purpose. The proofs we adduce for the completness (and on account of the absence of references).

L e m m a 1.4.1. *The restriction of any pure state of an arbitrary unital C^*-algebra A on its center is also the pure state.*

P r o o f. Let φ be a pure state on A, $\varphi \in P(A)$, and φ_0 be the restriction of φ on $Z(A) = C(Y)$. If φ_0 is not a pure state then it is determined by some non-atomic measure μ on Y. Let F be the support of this measure and F_0 be a closed subset of F such that $0 < \mu(F_0) < 1$ and let f be a continuous function on Y equal to 0 outside of F_0 and such that $\int fd\mu \neq 0$. Thus $0 < \varphi_0(f) < 1$. Then the functional φ_f on A defined by the formula $\varphi_f(a) = \varphi(af)$ is positive and is majorized by the state φ (the last statement is verified by follows: $(\varphi - \varphi_f)(a) = = \varphi(a - af) = \varphi((e-f)a) \geqslant 0$). Therefore, $\varphi_f = \lambda\varphi$, which is impossible since φ_{0f} and φ_0 have different supports.

L e m m a 1.4.2. *For each pure state φ of a unital C^*-algebra A, $\varphi(af) = \varphi(a)\varphi(f)$, where $a \in A$, $f \in Z(A)$.*

P r o o f. Let $\varphi(a) = 1$ and let φ_a be a functional on $Z(A)$, defining by the formula $\varphi_a(f) = \varphi(af)$. Show that $\varphi_a = \varphi | Z(A)$. Let $y_0 \in Y$ be such that $\varphi(f) = f(y_0)$ for each $f \in C(Y)$ (see Lemma 1.4.1), then, if there exists $f \in C(Y)$ for which $|\varphi_a(f) \neq f(y_0)$, then the function $g = f - f(y_0)$ has the property: $\varphi_a(g) \neq 0$. On the other hand $|\varphi_a(g)|^2 = |\varphi(ag)|^2 \leqslant \leqslant \varphi(|g|^2)\varphi(aa^*) = 0$, since $\varphi(|g|^2) = |g|^2(y_0) = 0$. Thus, for every $f \in Z(A)$, $\varphi(af) = \varphi(f)$. Hence for each $a \in A$, $a \neq 0$, $\varphi(af) = \varphi(a)\varphi(f)$. If $\varphi(a) = 0$, then $\varphi((1-a)f) = \varphi(1-a)\varphi(f)$, from which we obtain $\varphi(af) = 0$.

Return to the proof. So, let $h: Y \to P_0(T, A)$ then, for every $\in C(T)$ one can define the mapping $\alpha_h: C(T) \to C(Y)$ by the formula $\alpha_h(f)(y) = h(y)(f)$ (note, that we identify everywhere $C(Y)$ with $Z(A)$ and $P_0(T, A)$ with the state space of $C(T)$). Thus, α_h is a linear mapping from $C(T)$ into A. Show that it can be extended to conditional expectation on $C(T, A)$.

Let $x = \sum a_i f_i$, $x \in C(T, A)$, $a_i \in A$, $f_i \in C(T)$. Let us verify the inequality

$$\| \sum a_i \alpha_h(f_i) \| \leqslant \| \sum a_i f_i \|. \tag{1.2}$$

In fact, if $P(A)$ is the pure state space of A, then by Lemma 1.4.2

$$\| \sum a_i \alpha_h(f_i) \|^2 = \sup_{\varphi \in P(A)} \Big| \sum_{i,j} \varphi(a_i^* a_j)\varphi(\alpha_h(f_i)^*)\varphi(\alpha_h(f_j)) \Big|.$$

Let $\varphi(f) = f(y)$ for each $f \in C(T)$ (see Lemma 1.4.1), then

$$\| \sum a_i \alpha_h(f_i) \|^2 = \sup_{\varphi \in P(A)} \| \sum_{i,j} \varphi(a_i^* \overline{\alpha_h(f_i)(y)} a_j \alpha_h(f_j)(y)) \| \leqslant$$

$$\leqslant \sup_{y \in Y} \| \sum_{i,j} a_i^* \overline{h(y)(f_i)} a_j h(y)(f_j) \| =$$

$$= \sup_{y \in Y} \| \sum_i h(y)(a_i f_i) \|^2 \leqslant \| \sum_i a_i f_i \|^2.$$

It follows from (1.2) first the possibility to define correctly an operator p_h on $C_0(T, A)$ (see Proposition 1.4, [6]), $p_h(\sum a_i f_i) = \sum a_i a_h(f_i)$ and, secondly, its continuity on this set, which permits to extend it to an operator $\hat{p}_h: C(T, A) \to A$. This operator is an A-linear projector and it follows from (1.2) that its norm is equal to 1. Then by the Tomiyama theorem (see [9]), $\hat{p}_h \in P(T, A)$.

It is easy to show that the mapping $h \to p_h$ is inverse to the mapping which was constructed in the first part of the proof, hence we obtain the bijection of both the sets appearing in the statement. The continuity of all these mappings is obvious.

2. The compactum $P_0(\mathfrak{M})$. For a noncommutative uniform algebra $\mathfrak{M} \subset C(T, A)$ we introduce the space $P_0(\mathfrak{M})$, the object playing the key role in the further considerations. In a sense it replaces the state space concept for noninvolutive algebra $\mathfrak{M}$. The main result (Proposition 2.2) characterizes the extreme points of $P_0(\mathfrak{M})$ (Choquet boundary) in terms of representing measures.

Let $\mathfrak{M}$ be a uniform algebra, $\mathfrak{M} \subset C(T, A)$, and let $P_0(\mathfrak{M})$ be the restriction of $P_0(T, A)$ on $\mathfrak{M}$. It follows from Proposition 1.2 that this set is convex.

Remark 2.1. It is obvious by Remark 1.3, that every $p \in P_0(\mathfrak{M})$ is corresponding to some measure $\mu \in M(T)$, howevere to not a unique one. Each such a measure will also be called representing for p.

The following statement describes the extreme point set $\operatorname{ext} P_0(\mathfrak{M})$ of the compact $P_0(\mathfrak{M})$.

Proposition 2.2. *Let $\mathfrak{M}$ be a uniform algebra, $\mathfrak{M} \subset C(T, A)$. Then $\operatorname{ext} P_0(\mathfrak{M})$ is a subset of T and is coinciding with the set of those $t \in T$, for which p_t has a unique representing measure.*

Proof. Show at first that each extreme point of $P_0(\mathfrak{M})$ is corresponding to an atomic measure on T. Let μ be a representing measure for $p \in \operatorname{ext} P_0(\mathfrak{M})$ which is not atomic. Then overcoming some technical difficulties one can show that there exist measures μ_1 and μ_2, such that $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$, where $0 < \lambda < 1$ and $\int x d\mu_1 \neq \int x d\mu_2$, $x \in \mathfrak{M}$, hence p is a convex combination of elements p_1, p_2 from $P_0(\mathfrak{M})$ (which are corresponding to μ_1 and μ_2).

The rest of the statement is an immediate consequence of the fact that $\mathfrak{M}$ is point separating.

Corollary 2.3. *For a uniform algebra $\mathfrak{M} \subset C(T, A)$, the space $P_0(\mathfrak{M})$ is the restriction of $P(T, A)$ on $\mathfrak{M}$ iff the algebra is with trivial center.*

Proof. If $Z(A) = Ce$ then by Proposition 1.1, the statement is evident.

Conversely, let $Z(A) = C(Y)$, where Y is not single-point. Show that $P(T, A)|\mathfrak{M} \neq P_0(\mathfrak{M})$. Let us suppose at first that the maximal ideal space of $Z(A)$ is connected and let h be a Uryson function on Y, $0 \leqslant h(y) \leqslant 1$, $h \in C(Y)$ and t_1, t_2 be two different points of T. Put $p_n(x) = x(t_1) h^n + x(t_2)(1-h^n)$, where $x \in C(T, A)$, $n = 1, 2, \cdots$. It is obvious that all p_n are distinct on $\mathfrak{M}$ and belong to $P(T, A)$. If for each n,

$p_n|\mathfrak{M} \in P_0(\mathfrak{M})$, then in view of the compactness of this set, there exists a subsequence $p_{n_k}|\mathfrak{M}$ which converges to some p_0 from $P_0(\mathfrak{M})$. Le $x_0 \in \mathfrak{M}$ be such that $x_0(t_1) = e$, $x_0(t_2) = 0$, then $p_{n_k}(x_0) \to p_0(x_0)$. Ont the other hand $p_{n_k}(x_0) = h^{n_k}$; but this sequence may has not a limit.

Suppose now that Y is not connected, and let q_1, q_2 be two orthogonal projectors from $Z(A)$, $q_1 + q_2 = e$, $q_1 q_2 = 0$. Let $\mathfrak{M}_i = q_i \mathfrak{M}$, $i = 1, 2$, then it is easy to check that $\mathfrak{M}_i$ are uniform algebras, $\mathfrak{M}_i \subset C(T, q_i A)$. Let $p_{t_i} \in \operatorname{ext} P_0(\mathfrak{M}_i)$, $t_1 \neq t_2$, then if we put $p(x) = x(t_1) q_1 + x(t_2) q_2$, for $x \in C(T, A)$ then p is a conditional expectation from $P(T, A)$. If $p|\mathfrak{M} \in P_0(\mathfrak{M})$, then $p(x) = \int x d\mu$ for some representing measure on T. Hence $p(x) = q_1 \int x d\mu + q_2 \int x d\mu = q_1 x(t_1) + q_2 x(t_2)$, from which $q_1 (\int x d\mu - x(t_1)) = q_2 (x(t_2) - \int x d\mu)$. It follows from the orthogonality of q_1 and q_2 that $\int q_i x d\mu = q_i x(t_i)$, $i = 1, 2$, that is $\int y d\mu = y(t_i)$ for each $y \in \mathfrak{M}_i$. Therefore μ is a representing measure for $p_{t_1} \in \operatorname{ext} P_0(\mathfrak{M}_1)$ and for $p_{t_2} \in \operatorname{ext} P_0(\mathfrak{M}_2)$. This is impossible since Proposition 2.2, as $t_1 \neq t_2$.

This fact generalizes Proposition 1 1.

3. **The reduced spectrum of a uniform algebra.** The objects which were introduced in the previuos sections permit to revise the concept of the s ectrum of a noncommutative uniform algebra, suitable for an essentially larger class of algebras. The notion of reduced spectrum is coinciding with $Sp_A \mathfrak{M}$ for algebras with trivial center, and in particular is homeomorphic to the compactum of maximal ideals for the usual uniform algebras (i. e. when $A = \mathbf{C}$). For a specific class of algebras one can give an "inner" definition of reduced spectrum (see Corollary 4.4, cf. Theorem 3.6, [6]).

Definition 3.1. *The reduced spectrum of a uniform algebra $\mathfrak{M}$ from $C(T, A)$ will be called the set $Sp_A^0 \mathfrak{M}$ of all homomorphisms of $\mathfrak{M}$ into the algebra A, extendable to conditional expectations from $P_0(T, A)$, equipped with the topology of pointwise convergence.*

Thus, $Sp_A^0 \mathfrak{M} = Sp_A \mathfrak{M} \cap P_0(\mathfrak{M})$.

Proposition 3.2. *The reduced spectrum of a uniform algebra $\mathfrak{M}$ is compact, moreover, $Sp_A^0 C(T, A)$ coincides with T. If the algebra A is with trivial center, then $Sp_A^0 \mathfrak{M} = Sp_A \mathfrak{M}$.*

Proof. The set $Sp_A^0 \mathfrak{M}$ is the image of a continuous mapping (the restriction) of some closed subset of $P_0(T, A)$ which is compact (by Proposition 1.2). If $Z(A)$ is trivial then it follows from Corollary 2.3 that $Sp_A^0 \mathfrak{M}$ coincides with $Sp_A \mathfrak{M}$. At last, Remark 1.3 permits evidently to conclude that $Sp_A^0 C(T, A) = T$.

Note that this result is a generalization of Theorem 3.3 and a reision of Corollary 3.5 from [6].

4. **Uniform algebras with weakly*-transitive fibre.** In the paper [6] for uniform A-algebras with simple fibre there was established the coincidence of the relative spectrum with the set all A-linear homomorphisms of $\mathfrak{M}$ into A (Theorem 3.6, [6]). This is not true for algebras with arbitrary fibre. Receding a little from the main subject, let us show (corollary 4.4) in more general case, that continuous homomorphisms can be extended to conditional expectations (in other words, that $P_0(\mathfrak{M})$ coincides with the set of all A-linear continuous homomorphisms of $\mathfrak{M}$ into A).

Let φ be a state of a unital C^*-algebra A, $\varphi \in S(A)$, and u be a unitary from A, $u \in U(A)$. Denote by φ_u the state on A defining by the formula $\varphi_u(a) = \varphi(u^*au)$, $a \in A$. It is easy to check that if φ is pure, then φ_u is also pure, $\varphi, \varphi_u \in P(A)$.

Definition 4.1. *A state φ on a C^*-algebra A is called total, if the set of finite convex combinations of the states φ_u, when u runs over $U(A)$, is weakly*-dense in $S(A)$, i. e. $\overline{co\{\varphi_u, u \in U(A)\}} = S(A)$. A C^*-algebra is called weakly*-transitive if it possess a pure total state.*

Proposition 4.2. (i) *Every weakly*-transitive algebra has trivial center.*

(ii) *A C^*-algebra is simple iff each its pure state is total.*

Proof. (i) Let A be a weakly*-transitive algebra, and φ be a total pure state. Suppose $Z(A)$ is not trivial. Then $Z(A) = C(Y)$ for some non single-point compact Y. We have $\varphi | Z(A) \in P(Z(A))$ (see Lemma 1.4.1) from which we obtain $\varphi(f) = f(y_0)$ for some y_0 from Y and al $f \in C(Y)$. It is evident, that for each $u \in U(A)$ we have $\varphi_u | Z(A) = \varphi | Z(A)$, therefore the state $\psi \in S(A)$, the restriction of which coincides with the state $\psi(f) = f(y_1)$ on $C(T)$, where $y_1 \neq y_0$, can not belongs to $\overline{co\{\varphi_u, u \in U(A)\}}$.

(ii) Let A be a simple unital C^*-algebra, $\varphi \in P(A)$. Then every pure state ψ is approximable (in the weakly*-topology) by the states of the form of φ_b, $\varphi_b(a) = \varphi(b^*ab)$ (see 3.4.3 of [2]) and by the transitivity theorem (see 2.8.2, [2]) it is sufficient b being unitary.

Conversely, let the algebra A is such that every φ from $P(A)$ i total and let I be an ideal of A, $I \neq A$. Then there exists a pure state ψ, which annuls I. For each $u \in U(A)$, ψ_u annuls I too, therefore all states from $\overline{co\{\varphi_u, u \in U(A)\}}$ are zero on I, so $I = \{0\}$.

Note, that the algebra $B(H)$ of all bounded linear operators on an infinite dimensional Hilbert space H is an example of weakly*-transitive, but not simple algebra.

Let $\mathfrak{M}$ be a uniform A-algebra, $R(\mathfrak{M})$ be the set of all A-linear normalized mappings of $\mathfrak{M}$ into A which are identity preserving.

In contrast with the case of commutative uniform algebras, the set $R(\mathfrak{M})$ is not coinciding with the set of all restrictions of the space $P(T, A)$ on the algebra $\mathfrak{M}$. We shall show that it is true when the fibre A is weakly*-transitive.

Theorem 4.3. *Let $\mathfrak{M}$ be a uniform A-algebra, $\mathfrak{M}\subset C(T, A)$, where A is weakly*-transitive, then each mapping from $R(\mathfrak{M})$ can be extended to a conditional expectation on $C(T, A)$.*

Proof. Let $p\in R(\mathfrak{M})$, φ be a pure total state on A, then $\varphi\circ p$ is a functional on $\mathfrak{M}$ with norm 1, and $\varphi\circ p(e)=1$. Therefore such a functional can be extended to some state Φ on $C(T, A)$ such that its restriction on A is pure. We have then, by Theorem 1.5 of [6]

$$\Phi(x)=\int \varphi(x(t))\,d\mu(t) \tag{4.1}$$

for a probability measure μ, $x\in C(T, A)$. Denote by $\hat{p}$ the conditional expectation, corresponding to this measure (note that it follows from Propositions 1.1 and 4.2(i) that $P_0(T, A)=P(T, A)$). Then $\Phi=\varphi\circ\hat{p}$. This equality is easily checked for the elements of A and $C(T)$ taking into account (1.1) and (4.1) then it is fulfilled on $C_0(T, A)$, and therefore on $C(T, A)$ (see Proposition 1.4 [6]).

Show, that $\hat{p}|\mathfrak{M}=p$. It is sufficient to verify that $Ker\, p\subset Ker\, \hat{p}|\mathfrak{M}$. Really, by virtue of A-linearity of p, we have $\mathfrak{M}=Ker\, p+A$, therefore if $x\in\mathfrak{M}$, then $x=y+a$, where $y\in Ker\, p$, $a\in A$ and, consequently $\hat{p}(x)=\hat{p}(y)+\hat{p}(a)=a=p(x)$.

Thus, let $x\in$ Ker p, then for each $u\in U(A)$ we have $\varphi_u(\hat{p}(x))=$ $=\varphi(u^{*}\hat{p}(x)\ u)=\varphi(\hat{p}(u^{*}xu))=\Phi(u^{*}xu)=\varphi_u(p(x))=0$.

Hence as A is weakly*-transitive (see the beginning of the proof) we may conclude, that $\psi(\hat{p}(x))=0$ for every $\psi\in S(A)$ and, consequently $\hat{p}(x)=0$.

Corollary 4.4. *The reduced spectrum of an A-algebra with weakly*-transitive fibre coincides with the set of all nontrivial continuous A-linear homomorphisms of $\mathfrak{M}$ into A.*

Proof. Let α be a continuous A-linear homomorphism of the algebra $\mathfrak{M}$ into A. Show, that $\|\alpha\|=1$. In just the same way as in the proof of Theorem 3.6, [6], suppose that $\|\alpha\|>1$ and $x\in\mathfrak{M}$ is such that $\|x\|<1$, but $\|\alpha(x)\|=1$. For $y=\alpha(x)^{*}x$ we have: $\|y\|<1$ and $\|\alpha(y)\|=1$, therefore $e-\alpha(y)$ is not invertible in A, which is contradictory to the invertibility of $e-y$.

Hence $\alpha\in R(\mathfrak{M})$ and, consequently, by the previous theorem, α can be extended to a conditional expectation on $C(T, A)$.

5. **Peak points.** In this Section we introduce and investigate some concepts of peak points. In contrast with the classical case, there exist several possibilities which are coinciding for algebras with trivial fibre. We choose as a basic the definition of a weak peak point which describes extreme points of $P_0(\mathfrak{M})$ (see Theorem 6.1) and which corresponds thus to the notion of a pure state (for involutive algebras).

Definition 5.1. *Let* $\mathfrak{M}$ *be a uniform algebra,* $\mathfrak{M} \subset C(T, A)$. *A point* t_0 *from* T *is called*

(i) *weak peak (for the algebra* $\mathfrak{M}$*) if for every* $d > 1$ *and each neighborhood* U *of* t_0*, there exists* $x \in \mathfrak{M}$ *such that* $\|x\| < d$, $x(t_0) > 0$, $\|x(t_0)\| = 1$ and also $\|x(t)\| < (2d)^{-1}$ when $t \in \bar{U}$.

(ii) *norm peak, if there exists* $x \in \mathfrak{M}$ *such that* $x(t_0) > 0$, $\|x(t_0)\| = 1$, *and* $\|x(t)\| < 1$ *when* $t \neq t_0$.

(iii) *peak, if there exists* $x \in \mathfrak{M}$, *such that* $x(t_0) = e$, $\|x(t)\| < 1$, *when* $t \neq t_0$.

The set of peak points in the sense of (j) denote by $\Pi_j(\mathfrak{M})$, $j = 1, 2, 3$. It is easy to check that these sets are connected by the following relations: $\Pi_3(\mathfrak{M}) \subseteq \Pi_2(\mathfrak{M}) \subseteq \Pi_1(\mathfrak{M})$.

These notions are essentially different; their characterizations and the relations between them are given in Proposition 5.2 (see also Corollaries 6.2 and 6.3). In [7] the question of existence of peak point has been formulated (in the sense of Definition 5.1 (iii)) which in general has a negative answer. A description of peak point in terms of operator-valued orthogonal measures (without discussions on the existence question) has been given in [3] (in the case of general uniform algebras of operator fields).

At the same time, Definition 5.1 (i) satisfies several natural requirements: for example it is concordant with the geometrical structure of the compactum $P_0(\mathfrak{M})$ (Theorem 6.1) and it satisfies the norm maximum principle in the Gelfand representation (Theorem 7.2). Note here that besides the trivial examples of usual uniform algebras, these three definitions are coincide for A-algebras (with matrix fibre (it follows from Propositions 1.8 and 3.7 of [6]) and also for invariant algebras on compact Abelian groups (see Section 7 of [7] and the extensive paper which will appear in the same journal).

Proposition 5.2. *Let* $\mathfrak{M}$ *be a uniform algebra,* $\mathfrak{M} \subset C(T, A)$ *Then* (i) the point t_0 from T *is a weak peak point iff for every* $d > 1$, $\varepsilon > 0$. *and each neighborhood* U *of* t_0*, there exists* $x \in \mathfrak{M}$*, such that* $\|x\| < d$, $x(t_0) \geqslant 0$, $\|x(t_0)\| = 1$ and $\|x(t)\| < \varepsilon$ when $t \in \bar{U}$;

(ii) *the point* t_0 *from* T *is a norm peak point iff there exist a bounded sequence* $\{x_n\} \subset \mathfrak{M}$, *a positive number* $\varepsilon < 1$, *a state* $\varphi \in S(A)$ *and a decreasing sequence of open neighborhoods* U_n *of the point* t_0, $\cap U_n = \{t_0\}$, *such that* $x_n(t_0) \geqslant 0$, $\|x_n(t_0)\| = 1$, $\varphi(x_n(t_0)) = 1$ *and* $\max \|x_n(t)\| < \varepsilon$, *if* $t \in T - U_n$, $n = 1, 2, \cdots$.

(iii) *the point* t_0 *from* T *is a peak point iff the conditions of* (ii) *are satisfied for each* $\varphi \in S(A)$, *i. e. if there exist a bounded sequence* $\{x_n\} \subset \mathfrak{M}$, *a positive* $\varepsilon < 1$ *and a decreasing sequence of open neighborhoods* U_n *of the point* t_0, $\cap U_n = \{t_0\}$ *such that* $x_n(t_0) = e$, $\max \|x_n(t)\| < \varepsilon$, *where* $t \in T - U_n$, $n = 1, 2, \cdots$.

Proof. (i) Sufficiency is clear. Let $t_0 \in \Pi_1(\mathfrak{M})$ and d, ε, U are as in the conditions of (i). Let n be such that $\varepsilon > 2^{-n}$, then, there exists, by Definition 5.1 (i), some $y \in \mathfrak{M}$ for which $\|y\| < d^{1/n}$

$y(t_0)\geqslant 0$, $\|y(t_0)\|=1$, $\|y(t)\|<(2d^{1/n})^{-1}$ when $t\overline{\in} U$. It is easy to verify that the function $x=y^n$ has needed properties.

(ii) Let $t_0\in \Pi_2(\mathfrak{M})$, and a function y from $\mathfrak{M}$ satisfies the conditions of Definition 5.1 (ii) and let $\{U_n\}$ be any decreasing sequence of open sets, $\cap U_n=\{t_0\}$. Let also ε be a positive number and φ be a state, $\varphi\in S(A)$, such that $\varphi(y(t_0))=1$. By virtue of compactness of $T-U_n$ there exists some m_n such that $\|y(t)\|^{m_n}<\varepsilon$ when $t\overline{\in} U_n$. The sequence $x_n=y^{m_n}$ is desired.

The proof of sufficiency is a modification of the proof of Theorem 11.1 of [8]. Assume the conditions of (ii) are satisfied. Let $M=\sup_n\|x_n\|$, $\varepsilon_n=s^{n+1}(1-\varepsilon)(4(1-s^{n+1}))^{-1}$, where $s=(2M-1-\varepsilon)(2(M-\varepsilon))^{-1}$, $n=1,2,\cdots$. It is evident that $s\in]0,1[$ and ε_n is a sequence decreasing to 0. Let $F_n=T-U_n$ and choose a subsequence h_n, $n=0,1,\cdots$ of the sequence x_n as follows. Let $h_0=e$, $h_1=x_1$ and assume $h_0, h_1, h_2,\cdots,h_n$ are already choosen. Put

$$W_n=\{t\in T,\ \max_{1\leqslant j\leqslant n}\|h_j(t)\|\geqslant 1+\varepsilon_n\},\ n=1,2\cdots.$$

Then, since $t_0\overline{\in} W_n$ we have $W_n\subset F_{m_n}$ for some m_n. Denote by h_{n+1} he function x_{m_n}. It is obvious that this function satisfies the condition $\max\|h_{n+1}(t)\|<\varepsilon_n$, where $t\in F_{m_n}$.

Put now $x=(1-s)\sum_{n=0}^{\infty}s^n h_n$. Clearly, $x\in\mathfrak{M}$; show that this function realizes a norm peak at the point t_0. We have $x(t_0)\geqslant 0$, $\varphi(x(t_0))=1$ and also $\|x(t_0)\|\leqslant(1-s)\sum_{n=0}^{\infty}s^n\|h_n(t_0)\|=1$, from which $\|x(t_0)\|=1$. Let $t\neq t_0$, then we have $\|x(t)\|<1$. Really, if $t\overline{\in}\bigcup_{n=1}^{\infty}W_n$, then $\|h_n(t)\|\leqslant 1$ for all $n=0,1,\cdots$, however $t\in F_m$ for some m and therefore $\|h_m(t)\|<\varepsilon<1$, so $\|x(t)\|<1$.

If $t\in\bigcup_{n=1}^{\infty}W_n$, then as the sets W_n form an increasing sequence of closed sets, there exists m, such that $t\in W_{m+1}-W_m$, hence $\|h_n(t)\|\leqslant 1+\varepsilon_m$, when $n=0,1,\cdots,m$, $\|h_{m+1}(t)\|\leqslant M$, and $\|h_n(t)\|\leqslant\varepsilon$ when $n>m+1$. Therefore

$$\|x(t)\|\leqslant(1-s)\left\{(1+\varepsilon_m)\sum_{n=0}^{m}s^n+Ms^{m+1}+\varepsilon\sum_{n=m+2}^{\infty}s^n\right\}<1$$

by virtue of the choosing of s and ε_m, and finally we obtain that $\|x(t)\|<1$ when $t\neq t_0$, i. e. $t_0\in\Pi_2(\mathfrak{M})$.

(iii) The proof is the same (it is necessary only to check $x(t_0)=e$).

6. Characterization of weak peak points. Here we prove the main result of the paper: for the important class of uniform algebras (including A-algebras with simple fibre) the weak peak points are extreme points of $P_0(\mathfrak{M})$. This fact provides the existence of weak peak points. Using that the weak peak points are described in other terms, which complements Proposition 5.2.

Theorem 6.1. *Let $\mathfrak{M}$ be a uniform algebra, $\mathfrak{M} \subset C(T, A)$. Then each weak peak point is an extreme point of the compactum $P_0(\mathfrak{M})$. If $\mathfrak{M}$ is an A-algebra with weakly*-transitive fibre then the converse statement is also true.*

Proof. Let $t_0 \in \Pi_1(\mathfrak{M})$ and assume that $t_0 \overline{\in}$ ext $P_0(\mathfrak{M})$, then, by virtue of Proposition 2.2, p_{t_0} has a nonatomic representing measure μ. Let F be a closed subset of T not containing t_0, for which $0 < \mu(F) < 1$ and let d be a number satisfying the inequality $1 < d < 1 + \mu(F)$. By Proposition 5.2 (i), there exists $x \in \mathfrak{M}$, such that $\|x\| < d$, $\|x(t_0)\| = 1$, $\|x(t)\| < \mu(F)$, when $t \in F$. Then

$$1 = \|p_{t_0}(x)\| \leqslant \int_F \|x(t)\| d\mu(t) + \int_{T-F} \|x(t)\| d\mu(t) < \mu(F)^2 + d(1 - \mu(F)) < 1.$$

It follows from this that $\Pi_1(\mathfrak{M}) \subset$ ext $P_0(\mathfrak{M})$.

To prove the second part of the theorem we need of the following anxiliary facts, the first of which is well known.

Lemma 6.1.1. *Let A be a C^*-algebra with identity, $a \in A$ $a = u + iv$, where u and v are Hermitian and $u \leqslant 0$. Then $\|\exp a\| \leqslant 1$.*

Lemma 6.1.2. *Let $\mathfrak{M}$ be a uniform A-algebra, $\mathfrak{M} \subset C(T, A)$ and let $t_0 \in$ ext $P_0(\mathfrak{M})$, $f \in C(T)$, $f(t_0) = 1$, $0 < f(t) < 1$, when $t \neq t_0$. Let $a \in A$, $a > 0$ and let φ be a total state on A, such that $\varphi(a) = \|a\|$. Put*

$$\widehat{\mathfrak{M}}_a = \{u \in \mathrm{Re}\, \mathfrak{M},\ u \leqslant af,\ \varphi(u(t_0)) > 0\}.$$

Then $\sup \varphi(u(t_0)) = \|a\|$, when $u \in \widehat{\mathfrak{M}}_a$.

Proof. Let $\sup\limits_{u \in \widehat{\mathfrak{M}}_a} \varphi(u(t_0)) = c < \|a\|$. Put $\varphi_{t_0} = \varphi \circ p_{t_0}$ on $\mathfrak{M}$. It is clear, that φ_{t_0} is a linear continuous normalized functional and $\varphi_{t_0}(e) = 1$.

If $af \in \mathrm{Re}\mathfrak{M}$, then the statement is trivial. Let now $af \overline{\in} \mathrm{Re}\, \mathfrak{M}$ and let us extend φ_{t_0} on $\mathfrak{M}' = \mathfrak{M} \oplus \mathbf{C}af$ (which is a closed subspace of $C(T, A)$) by the formula

$$\psi_{t_0}(x + \alpha af) = \varphi_{t_0}(x) + \alpha c.$$

Verify that ψ_{t_0} is a normalized linear functional. Suppose the opposite, $\|\psi_{t_0}\| \neq 1$, then there exists $x + \alpha af \in \mathfrak{M}'$, $\alpha \neq 0$, such that $\|x + \alpha af\| = 1$, but $\|\psi_{t_0}(x + \alpha af)\| > 1$. Let $\psi_{t_0}(x + \alpha af) = re^{i\theta}$, then $\|e^{-i\theta}(x + \alpha af)\| = 1$ however $\psi_{t_0}(e^{-i\theta}(x + \alpha af) = r > 1$. Thus, we may assume that $\psi_{t_0}(x + \alpha af) = r > 1$, where $\|x + \alpha af\| = 1$. As ψ_{t_0} is Hermitian (i. e. it takes real values on Hermitian elements of $\mathfrak{M}'$), therefore the following linear functional on Re $\mathfrak{M}'$ is uniquely defined: if $v \in \mathrm{Re}\, \mathfrak{M}'$, then for some $\hat{v} \in \mathfrak{M}'$, $v = \mathrm{Re}\, \hat{v}$, put $\psi_{t_0}(v) = \mathrm{Re}\, \psi_{t_0}(\hat{v})$. Let $\mathrm{Re}\,(x + \alpha af) = w + \beta af$, and $\beta = \mathrm{Re}\, \alpha$, then $\psi_{t_0}(w + \beta af) = r$. In addition $\|w + \beta af\| \leqslant 1$, and since it is Hermitian, we obtain

$$e - (w + \beta af) \geqslant 0. \tag{6.1}$$

Granting this we have

$$\psi_{t_0}(e-(w+\beta af))=1-r<0. \tag{6.2}$$

On the other hand:

(i) If $\beta>0$, then it follows from (6.1) that $e-w(t_0)-\beta a\geqslant 0$, whence

$$\psi_{t_0}(e-w-\beta af)=1-\varphi(w(t_0))-\beta c>1-\varphi(w(t_0))-\beta\|a\|=$$
$$=\varphi(e-w(t_0)-\beta a)\geqslant\varphi(0)=0,$$

which contradicts to (6.2).

(ii) If $\beta<0$, then it follows from (6.1) that

$$\beta^{-1}(e-w)\leqslant af. \tag{6.3}$$

Show that

$$\varphi(\beta^{-1}(e-w(t_0))>0. \tag{6.4}$$

Indeed, $\psi_{t_0}(e-w)=1-r+\beta c<\beta c<0$, whence (6.4).

It means together with (6.3) that $\beta^{-1}(e-w)\in\widehat{\mathfrak{M}}_a$, hence $\varphi(\beta^{-1}\times\times(e-w(t_0))\leqslant c$. On the other hand,

$$\varphi(\beta^{-1}(e-w(t_0))=\psi_{t_0}(\beta^{-1}(e-w)=$$
$$=\beta^{-1}\psi_{t_0}(e-w)=\beta^{-1}(1-r+\beta c)>\beta^{-1}\beta c=c.$$

This contradiction proves finally that $\|\psi_{t_0}\|=1$.

This functional can be extended by Hahn—Banach theorem to a normalized functional Φ_{t_0} on $C(T, A)$ which is, evidently, a state. Recall, that the restriction $\Phi_{t_0}|A=\varphi$ is a pure state, and $\Phi_{t_0}(x)=\varphi(x(t_0))$ for $x\in\mathfrak{M}$. Hence, by Theorem 1.5 [6], for some probability measure μ on T and for each $x\in\mathfrak{M}$ we have:

$$\varphi(x(t_0))=\int\varphi(x(t))\,d\mu(t).$$

The algebra $\mathfrak{M}$ is an A-bimodule, whence, for each $u\in U(A)$,

$$\varphi_u(x(t_0))=\int\varphi_u(x(t))\,d\mu(t).$$

At last, since φ is total on A, then, for each state ψ on A we have $\psi(x(t_0))=\int\psi(x(t))\,d\mu(t)$. Therefore μ is a representing measure for p_{t_0}, and so it is atomic (Proposition 2.2). Again, by Theorem 1.5, [6], for each $x\in C(T, A)$ we obtain $\Phi_{t_0}(x)=\varphi(x(t_0))$. Therefore $\Phi_{t_0}(af)=$ $=\varphi(af(t_0))=f(t_0)\varphi(a)=\|a\|$.

On the other hand, $\Phi_{t_0}(af)=\psi_{t_0}(af)=c$, which completes the proof of the Lemma.

Return to the proof of the theorem. Let $t_0\in\operatorname{ext}P_0(\mathfrak{M})$, $d>1$ and let Ube, a neighborhood of the point t_0. Assume at first that $d<(e/2)^{1/2}$, and let $f\in C(T)$ such that $f(t_0)=1$, $0<f(t)<1$ for $t\neq t_0$, moreover, $f(t)<$ $<1-\ln 2d^2$ for $t\overline{\in}U$. By virtue of Lemma 6.1.2, for the total state φ, putting $a=e$, we obtain $\sup\limits_{u\in\widehat{\mathfrak{M}}_e}\varphi(u(t_0))=1$, hence, for each $\varepsilon>0$, there exists $u\in\operatorname{Re}\mathfrak{M}$, such that $\varphi(u\ t_0))>1-\varepsilon$ and $u\leqslant f$.

Show, that there exists also $v \in \widehat{\mathfrak{M}}_e$, $v \geqslant 0$, for which $\varphi(v(t_0)) > 1 - \varepsilon$. Indeed, if $u(t_0) = a_+ - a_-$, then $a_+ - a_- < e$ implies $a_+ < e$. In addition,

$$\varphi(a_+) > \varphi(u(t_0)) > 1 - \varepsilon.$$

Put $v = a_+^{1/2} u a_+^{1/2}$, $v \geqslant 0$, then, by virtue of $\mathfrak{M}$ being an A-module, $v \in \operatorname{Re} \mathfrak{M}$. In addition

$$v = a_+^{1/2} u a_+^{1/2} \leqslant a_+^{1/2} f a_+^{1/2} + a_+ f \leqslant f,$$

and so $v \in \widehat{\mathfrak{M}}_e$. Further

$$\varphi(v(t_0)) = \varphi(a_+^2) \geqslant \varphi(a_+)^2 > (1 - \varepsilon)^2 > 1 - 2\varepsilon. \tag{6.5}$$

Now, let $\hat{y} \in \mathfrak{M}$, $\operatorname{Re} \hat{y} = v$ and let $y = \hat{y} - i \operatorname{Im} \hat{y}(t_0)$. Put

$$x = \exp y \, (\|\exp v(t_0)\|)^{-1}$$

and check that this function realizes a weak peak at the point t_0.

It is obvious that $x \in \mathfrak{M}$, $x(t_0) \geqslant 0$, and $\|x(t_0)\| = 1$.

Further, since $v(t) < e$, then $\operatorname{Re}(y(t) - e) < 0$. By Lemma 6.1.1, we have $\|\exp(y(t) - e)\| \leqslant 1$, whence

$$\|x(t)\| = \frac{\|\exp y(t)\|}{\|\exp v(t_0)\|} \leqslant \frac{e}{\|\exp a_+^2\|} = e^{1 - \|a_+^2\|} < e^{2\varepsilon},$$

where $\|a_+^2\| > 1 - 2\varepsilon$ by (6.5). Choosing $\varepsilon = 1/2 \ln d$, we obtain $\|x\| < d$.

Let $t \in U$. Since $\operatorname{Re}(y(t) - f(t)) \leqslant 0$, using Lemma 6.1.2, we have

$$\|x(t)\| = \frac{\|\exp y(t)\|}{\|\exp v(t_0)\|} < \frac{\exp f(t)}{\exp \|a_+^2\|} < e^{1 - \ln 2d^2 - \|a_+^2\|}.$$

Again, since $\|a_+^2\| > 1 - \ln d$, we obtain $\|x(t)\| < (2d)^{-1}$. Hence, the function x realizes a weak peak at the point t_0.

Now, let d is arbitrary, $d > 1$ and let n be such, that $d^{1/n} < (e/2)^{1/2}$, and let y be a function for which $\|y\| < 1/n$, $y(t_0) \geqslant 0$, $\|y(t_0)\| = 1$, and $\|y(t)\| < (2d^{1/n})^{-1}$ when $t \in U$. Then evidently, the function $x = y^n$ is desired. The theorem is completely proved.

Corollary 6.2. *A point t_0 from T is a weak peak point of a uniform A-algebra with weakly*-transitive fibre iff for each $d > 1$ and every neighborhood U of the point t_0, there exist $x \in \mathfrak{M}$ and a state φ on the algebra A such that the function $g(t) = \varphi(x(t))$ has the properties: $g(t) < d$, $g(t_0) = 1$, $g(t) < (2d)^{-1}$ if $t \in U$.*

Proof. Prove only sufficiency. Let t_0 be a point satisfying the conditions of the corollary. Taking into consideration the previous theorem it is sufficient to verify that $p_{t_0} \in \operatorname{ext} P_0(\mathfrak{M})$. Let us assume the contrary and let μ be a corresponding nonatomic measure (see Proposition 2.2). It is obvious that there exists a closed subset F, not containing the point t_0, and such that $0 < \mu(F) < 1$. Let d be such that $1 < d < (2 - \mu(F))\,(2(1 - \mu(F)))^{-1}$ and let U be a neighborhood of the point t_0, $U \cap F = \varnothing$. According to conditions, there exist a function $x \in \mathfrak{M}$ and a state φ on A, such that the function $g(t) = \varphi(x(t))$ has the properties: $g(t) < d$, $g(t_0) = 1$, $g(t) < (2d)^{-}$ when $t \bar{\in} U$. Then

$$1 = g(t_0) = \int \varphi(x(t))\, d\mu(t) = \int_F \varphi(x(t))\, d\mu(t) + \int_{T-F} \varphi(x(t))\, d\mu(t) <$$

$$< (2d)^{-1}\mu(F) + d\mu(T - F) < 1.$$

This contradiction proves the corollary.

Corollary 6.3. *A point $t_0 \in T$ is a weak peak point for a uniform A-algebra with weakly*-transitive fibre iff there exist a bounded sequence $\{x_n\} \subset \mathfrak{M}$, an $\varepsilon \in]0, 1[$, and a decreasing sequence of open neighborhoods $\{U_n\}$, $\cap U_n = \{t_0\}$, such that $x_n(t_0) \geqslant 0$, $\|x_n(t_0)\| = 1$, and $\|x_n \mid T - U_n\| < \varepsilon$ for $n = 1, 2, \cdots$.*

Proof. Let $t_0 \in \Pi_1(\mathfrak{M})$, U_n be a sequence of neighborhoods, $\cap U_n = \{t_0\}$. Fix $d > 1$ and for any n let a function $x_n \in \mathfrak{M}$ satisfies the properties: $\|x_n\| < d$, $x_n(t_0) \geqslant 0$, $\|x_n(t_0)\| = 1$, and $\|x_n(t)\| < \varepsilon$ for $t \in T - U_n$ (see Proposition 5.2 (i)). The sequence $\{x_n\}$ is desired. Conversely, let $\{x_n\}$ be a sequence satisfying the conditions of the corollary. Assume, that $p_{t_0} \bar{\in}$ ext $P_0(\mathfrak{M})$ (we use again Theorem 6.1). Let μ be a nonatomic representing measure (by Proposition 2.2) of the conditional expectation p_{t_0} and let $M = \sup\limits_n \|x_n\|$. For each n, obviosly, we have

$$\varepsilon\mu(T - U_n) + \mu(\{t_0\}) < \alpha < 1.$$

Let n_0 be such that $\mu(U_{n_0} - \{t_0\}) < (1 - \alpha) M^{-1}$. Then, for x_{n_0} we obtain

$$1 = \|x_{n_0}(t_0)\| \leqslant \int_{U_n - \{t_0\}} \|x_{n_0}(t)\| d\mu(t) + \int_{T_n - U} \|x_{n_0}(t)\| d\mu(t) + \mu(\{t_0\}) <$$

$$< M\mu(U_n - \{t_0\}) + \varepsilon\mu(T - U_n) + \mu(\{t_0\}) < 1.$$

This contradiction proves the corollary.

This result is a supplement to Proposition 5.2. Thus, Corollary 6.3, Proposition 5.2 (ii) and Proposition 5.2 (iii) compose a sequence of gradually strengthening requirements.

7. **Maximum principle.** We introduce here the notion of Shilov boundary and for some classes of uniform algebras prove the maximum principle.

Definition 7.1. *Let $\mathfrak{M}$ be a uniform algebra $\mathfrak{M} \subset C(T, A)$. The closure of the set of weak peak points will be called the Shilov boundary and will be denoted by $\partial(\mathfrak{M})$.*

Thus, $\partial(\mathfrak{M}) = \overline{\Pi_1(\mathfrak{M})}$. It is clear, that for uniform A-algebras with weakly*-transitive fibre the Shilov boundary coincides with the closure of the Choquet boundary (see Section 2 and Theorem 6.1), $\partial(\mathfrak{M}) = \overline{\exp P_0(\mathfrak{M})}$.

Theorem 7.2. *Let $\mathfrak{M}$ be a uniform A-algebra, $\mathfrak{M} \subset C(T, A)$, where the algebra A is weakly*-transitive, then, for every $x \in \mathfrak{M}$,*

$$\|x\| = \max_{t \in \partial(\mathfrak{M})} \|x(t)\|.$$

Proof. This fact expressing the maximum principle is an evident consequence of the following statement (which is in turn a generalization of a well known result of the theory of commutative uniform algebras, see, for example, Proposition 6.3, [10]): if M is a subspace of $C(Y)$, containing identity, where Y is compact, then for every function f of M, there exists $y \in$ ext $P_0(M)$ (the Choquet boundary of the space M), such that $\|f(y)\| = \|f\|$.

Proposition 7.3. *Assume that the conditions of Theorem 7.2 are satisfied and denote* $F_x = \{t \in T, \|x(t)\| = \|x\|\}$, *then* $F_x \cap \Pi_1(\mathfrak{M}) \neq \varnothing$.

Proof. Let φ be an arbitrary pure state on A and q_φ be a mapping of $\mathfrak{M}$ into $C(t)$ defined by the formula: $(q_\varphi x)(t) = \varphi(x(t))$. Obviously q_φ is a linear continuous operator. If $M_\varphi = \overline{q_\varphi(\mathfrak{M})}$ then M_φ is a closed linear subspace of $C(T)$. Show, that ext $P_0(M_\varphi)$ is contained in ext $P_0(\mathfrak{M})$. Really, let $t_0 \in$ ext $P_0(M_\varphi)$ and let μ be some representing measure for p_{t_0}, i. e. for every $x \in \mathfrak{M}$,

$$x(t_0) = \int x(t)\, d\mu(t), \tag{7.1}$$

whence

$$\varphi(x(t_0)) = \int \varphi(x(t))\, d\mu(t). \tag{7.2}$$

This means that μ is a representing measure for the functional f_t on M_φ given by the formula: $f_{t_0}(g) = g(t_0)$ for $g \in M_\varphi$, therefore μ is nesessarly atomic (since t_0 is from ext $P_0(M_\varphi)$), hence, by Proposition 2.2, we have $t_0 \in$ ext $P_0(\mathfrak{M})$.

Note, that the previous arguments are true for every uniform algebra. If $\mathfrak{M}$ is an A-algebra with weakly*-transitive fibre, then the same reasons as in the proof of Lemma 6.1 (ii) permit for any total state to deduce (7.1) from (7.2), i. e. in this case M_φ separates points of T and ext $P_0(M_\varphi) =$ ext $P_0(\mathfrak{M})$. This fact will not be used in the future.

Return to the proof. Let $t_0 \in F_x$ and let $y = x(t_0)^* x$, then $y \in \mathfrak{M}$, $y(t_0) \geqslant 0$, $\|y(t_0)\| = \|y\|$ and $\|y(t)\| < \|y(t_0)\|$ when $t \overline{\in} F_x$, i. e. $F_y = F_x$. Le φ be a pure state on A for which $\varphi(x(t_0)^* x(t_0)) = \|x(t_0)\|^2$, then $\varphi \circ y \in$ $\in M_\varphi$ and satisfies the conditions; $\varphi \circ y(t_0) = \|\varphi \circ y\|$, $|\varphi \circ y(t)| < |\varphi \circ y(t_0)|$ when $t \overline{\in} F_x$, therefore $F_x = F_{\varphi \circ y}$. From the fact which was formulated before Proposition 7.3, it follows that $F_x \cap$ ext $P_0(M_\varphi) \neq \varnothing$. It remains to note that ext $P_0(M_\varphi) \subset$ ext $P_0(\mathfrak{M})$ and ext $P_0(\mathfrak{M}) = \Pi_1(\mathfrak{M})$ (Theorem 6.1).

The maximum principle is true also in the orher, in some sense the opposite (see Proposition 4.2) class of uniform algebras with commutative fibre.

Proposition 7.4. *Let $\mathfrak{M}$ be a uniform algebra, $\mathfrak{M} \subset C(T, A)$ where A is a commutative C^*-algebra. Then for each $x \in \mathfrak{M}$,*

$$\|x\| = \max_{t \in \partial(\mathfrak{M})} \|x(t)\|.$$

Proof. Let $A = C(Y)$ for some compactum Y, then $C(T, A)$ can be identified with $C(T \times Y)$. Denote by $\hat{\mathfrak{M}}$ the uniform subalgebra of

$C(T \times Y)$ isomorphing to $\mathfrak{M}$ (if $x \in \mathfrak{M}$, then $\hat{x}(t, y) = x(t)(y)$).

Let $t_0 \in \Pi_1(\mathfrak{M})$, then for every neighborhood of the point t_0 there exists $x \in \mathfrak{M}$, such that $\|x(t_0)\| = 1$, and $\|x(t)\| < 1/2$ for $t \in U$ (see Proposition 5.2 (i)). Hence the function $\hat{x}$ achieves its maximum on the set $\operatorname{ext} P_0(\hat{\mathfrak{M}}) \cap (U \times Y) \neq \varnothing$ (we use again the very important fact mentioned in the beginning of the proof of Theorem 7.2). Since U is arbitrary we may conclude that $\operatorname{ext} P_0(\hat{\mathfrak{M}}) \subset \partial(\mathfrak{M}) \times Y$, whence, for every $x \in \mathfrak{M}$ we obtain

$$\|x\| = \sup_{(t, y) \in \operatorname{ext} P_0(\hat{\mathfrak{M}})} |\hat{x}(t, y)| \leqslant \max_{(t, y) \in \partial(\mathfrak{M}) \times Y} |\hat{x}(t, y)| = \max_{t \in \partial(\mathfrak{M})} |x(t)|$$

which completes the proof.

Corollary 7.5. *Let $\mathfrak{M}$ be a uniform algebra (or A-algebra), $\mathfrak{M} \subset C(T, A)$, for which the maximum principle is fulfilled. Then the restriction $\mathfrak{M}_1$ of $\mathfrak{M}$ on the Shilov boundary $\partial(\mathfrak{M})$ is also uniform algebra (correspondingly A-algebra), $\mathfrak{M}_1 \subset C(\partial(\mathfrak{M}), A)$.*

Proof. The non-trivial part of the proof is checking the completeness of $\mathfrak{M}_1$, which (as in the commutative case) is the easy consequence of the maximum principle.

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Վ. ԱՐԶՈՒՄԱՆՅԱՆ, Ս. ԳՐԻԳՈՐՅԱՆ. Օպերատորային դաշտերի հավասարաչափ հանրահաշիվների եզրերը *(ամփոփում)*

Աշխատանքում ուսումնասիրվում է ոչ կոմուտատիվ հանրահաշիվների սպեկտրի երկրաչափական կառուցվածքը։ Որոշ դասի հավասարաչափ հանրահաշիվների «վիճակների» կոմպակտի Շոքեյի եզրը բնութագրվում է ներկայացնող չափերի և թույլ պիկերի տերմիններով։ Նույն դասերի հանրահաշիվների համար ապացուցված է մաքսիմումի սկզբունքը։

В. АРЗУМАНЯН. С. ГРИГОРЯН, *Границы равномерных алгебр операторных полей* (резюме)

В работе изучается геометрическая структура спектра некоммутативной равномерной алгебры. Граница Шоке компакта «состояний» алгебр определенного класса характеризуется в терминах представляющих мер и точек слабого пика. Для алгебр того же класса доказан принцип максимума. Библиографий 10.

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