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ON THE EQUIVALENCE OF CLASSES OF INFINITELY DIFFERENTIABLE FUNCTIONS

1. Let I be an arbitrary subinterval of R and let $M=(M_n)$ be a given sequence of positive numbers. We denote by $C_M(I)$ the class of functions $f\in C^\infty(I)$ such that

$$|f^{(n)}(x)|\leqslant Ak^n M_n \quad \forall n\geqslant 0, \quad \forall x\in I \quad k=k(f)$$

and by $\overset{\bullet}{C}_M(I)$ the class of functions $f\in C^\infty(I)$ such that for each $x_0\in I$ there is a compact subinterval $J_{x_0}\subset I$ with the property that

$$|f^{(n)}(x)|\leqslant Ak^n M_n \quad \forall n>0, \quad \forall x\in J_{x_0} \quad k=k(f, J_{x_0}),$$

If I is a compact interval, then clearly $\overset{\bullet}{C}_M(I)\equiv C_M(I)$.

The problem of the equivalence of classes $C_M(I)$ for a fixed interval consists in finding necessary and sufficient conditions on sequences L and M in order that the classes $C_L(I)$ and $C_M(I)$ (resp. $\overset{\bullet}{C}_L(I)$ and $\overset{\bullet}{C}_M(I)$) be identical. A solution of this problem follows immediately from the analogous problem for inclusion viz that $C_L(I)\subseteq C_M(I)$ (resp. $\overset{\bullet}{C}_L(I)\subseteq\overset{\bullet}{C}_M(I)$). This inclusion problem was first posed by T. Carleman [3] (p. 76) and conditions which vary according to the nature of interval I, were found successively in cases when I is the whole line (for classes $C_M(I)$), an open or closed finite interval (for classes $\overset{\bullet}{C}_M(I)$), a closed halfline (for classes $C_M(I)$) by A. Gorny [6] (cf. also S. Mandelbrojt [7], [8]), H. Cartan and S. Mandelbrojt [4] and S. Agmon [1] respectively. J. Boman and L. Hörmander [2] gave simple proof of the theorem on equivalence of classes when $I=$ R using Baire's category theorem.

The problem of inclusion for clesses $C_M(\overset{\bullet}{R}_+)$, $\overset{\bullet}{C}_M(R_+)$ and $\overset{\bullet}{C}_M(\overset{\bullet}{R}_+)$, where $R_+=[0,\infty[$ and $\overset{\bullet}{R}_+=]0,\infty[$, however, remains open. In this paper, we settle this by using the Bair's category theorem. In order to give a unified approach to the treatment of the problem by this method for classes $C_M(I)$ and $\overset{\bullet}{C}_M(I)$, where I is any type of linear interval, we also give simple alternative proofs of the results of the above authors.

2. In order to formulate the theorems which we prove in the sequel, we need a few definitions and results about the regularizations

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of sequences and of classes (cf. S. Mandelbrojt [9]). We recall that a sequence $M=\{M_n\}$ of positive numbers is called log-convex if the sequence $\{\log M_n\}$ is a convex sequence. A sequence $M^c=\{M_n^c\}$ is called a log-convex regularization of M if $\{\log M_n^c\}$ is the largest convex minorant of $\{\log M_n\}$, it is defined by the relations

$$T_M(r)=\sup_{n>0}\frac{r^n}{M_n},\quad M_n^c=\sup_{r>0}\frac{r^n}{T_M(r)}.$$

It is easily seen that (cf. S. Mandelbrojt [9]):

$$T_{M^c}(r)=\sup_{n>0}\frac{r^n}{M_n^c},\quad M_n^c=\sup_{r>0}\frac{r^n}{T_{M^c}(r)}. \tag{1}$$

We define the sequence M^d by the following relations

$$N_n=n^n M_n,\quad N_n^c=n^n M_n^d \qquad (n\geqslant 1).$$

It follows readily from (1) that

$$\tilde{T}_{M^d}(r)=\sup_{n>0}\frac{r^n}{n^n M_n^d},\quad n^n M_n^d=\sup_{r>0}\frac{r^n}{\tilde{T}_{M^d}(r)}. \tag{2}$$

Let M be a sequence of positive numbers. We recall that the exponential regularization M^0 of M is defined by the relations

$$S_M(r)=\max_{n<r}\frac{r^n}{M_n}\ (r>1),\quad M_n^0=\sup_{r>n}\frac{r^n}{S_M(r)}.$$

It is easily seen that (cf. S. Mandelbrojt [9])

$$S_{M^0}(r)=\max_{n<r}\frac{r^n}{M_n^0},\quad M_n^0=\sup_{r>n}\frac{r^n}{S_{M^0}(r)}. \tag{3}$$

We define the sequence M^f by the following relations

$$N_n=(n^n M_n)^{\frac{1}{2}},\quad N_n^0=(n^n M_n^f)^{\frac{1}{2}}. \qquad (n>1).$$

It follows from (3) that

$$U_{M^f}(r)=\max_{n<r}\frac{r^{2n}}{n^n M_n^f},\quad n^n M_n^f=\sup_{r>n}\frac{r^{2n}}{U_{M^f}(r)}. \tag{4}$$

The sequences M^c, M^d, M^0, M^f are called the regularized sequences of M and possess the property of being the smallest sequences representing the classes $C_M(\mathbf{R})$, $C_M(\mathbf{R}^d)$, $\overset{*}{C}_M(I)$ (I a finite or infinite open interval), $C_M^*(I)$ (I a finite closed or a finite or infinite semi-closed interval) respectively. The processus of defining the classes by regularized sequences is called the regularisation of classes and is realized by using the Gorny—Cartan inequalities between the bounds of the moduli of the successive derivatives of a functions which vary according to the type of the interval considered. The following theorems due to the above cited authors describe these classes.

Theorem A. (i) *If* $\lim\inf M_n^{1/n}=0$, C_M (R)$\equiv$\{*constants*\}; *if* $0<\lim\inf M_n^{1/n}<\infty$, $C_M(R)=C_1(R)$; *if* $\lim M_n^{1/n}=\infty$, $C_M(R)\equiv C_{M^c}(R)$.

(ii) *If* $\lim\inf n\, M_n^{1/n}<\infty$, $C_M(dR)\subseteq Cn^{-n}(dR)$; *if* $\lim nM_n^{1/n}=\infty$, $C_M(dR)\equiv C_{M^d}(dR)$, *where* dR *denotes a half-line.*

(iii) $C_M^*(I)\equiv C_{M^e}^*(I)$, *where I is a finite or infinite open interval.*

(iv) $C_M^*(I)=C_{M^f}^*(I)$, *where I is a compact interval or a finite or infinite semi-closed interval.*

3. We begin by showing that a category argument can be used to give a short and simple proof of the following basic theorem of S. Mandelbrojt (cf. [7], pp. 91—93) which gives a necessary condition for the inclusion of classes.

Theorem B. *If* $C_L(I)\subseteq C_M(I)$, *then* $(L_n^c)^{1/n}=O\,[(M_n^c)^{1/n}]$ *for any arbitrary interval* $I\subseteq R$.

Proof: Since $L_n^c\leqslant L_n$ for $n\geqslant 0$, it follows that $C_{L^c}(I)\subseteq C_M(I)$. Let

$$B=\{f\in C^\infty(I)\colon \|f^{(n)}\|_\infty\leqslant AL_n^c\ \forall n>0,\ A=A(f)\}.$$

B is a Banach space if we introduce the norm $f\to\|f\|_B=\sup\limits_{n>0}\{\|f^{(n)}\|_\infty/L_n^c\}$. For $j=1,2,\cdots$, let

$$V_j=\{f\in B\colon \|f^{(n)}\|_\infty\leqslant j^{n+1}M_n,\ \forall n\geqslant 0\}.$$

Since $C_{L^c}(I)\subseteq C_M(I)$, $B=\bigcup\limits_{j=1}^{\infty}V_j$. Since V_j's are closed in B, by Baire's category theorem, there exist a V_j and a $\delta>0$ such that $\|f\|_B\leqslant\delta$ implies that $f\in V_j$.

Put

$$f(x)=\frac{\delta e^{ixr}}{T_{L^c}(r)}\quad (r>0)$$

Clearly $\|f\|_B\leqslant\delta$ and hence $f\in V_j$ i. e.

$$\frac{\delta r^n}{T_{L^c}(r)}\leqslant j^{n+1}M_n\ (r>0)$$

and hence, by (1),

$$\delta L_n^c=\delta\sup_{r>0}\frac{r^n}{T_{L^c}(r)}<j^{n+1}M_n$$

or

$$(L_n^c)^{1/n}=O\,[(M_n)^{1/n}].$$

Since M^c is the largest convex minorant of M, it follows that we also have

$$(L_n^c)^{1/n}=O\,[(M_n^c)^{1/n}].$$

This theorem can be extended to classes $C_M^*(I)$ as follows.

Theorem B′. *If* $C_L^*(I)\subseteq C_M^*(I)$, *then* $(L_n^c)^{1/n}=O\,[(M_n^c)^{1/n}]$ *for any arbitrary interval* $I\subseteq R$.

Poof: We proceed as abowe and define the Banach space B. Let $I=\bigcup_{1}^{\infty} J_k$, a countable union of non-overlapping compact intervals. For each $j=1, 2, \cdots$ and $k=1, 2, \cdots$ let

$$V_{j,k}=\{f\in B: |f^{(n)}(x)|\leqslant j^{n+1} M_n,\ \forall x\in J_k \text{ and } \forall n\geqslant 0\}.$$

Since $C^*_{L^c}(I)\subseteq C^*_M(I)$, $B=\bigcup_{j,k=1}^{\infty} V_{j,k}$. The rest of the proof goes through as above.

4. Using the category argument as in the proof Theorem A, we give a simple proof of the following theorem which solves the inclusion problem for different classes:

Theorem C. *Let L and M be two sequences of positive numbers.*

(i) *If* $\liminf_{n\to\infty} L_n^{1/n}=\liminf_{n\to\infty} M_n^{1/n}=0$, $C_L(\mathrm{R})=C_M(\mathrm{R})=\{constants\}$, *if* $0<\liminf_{n\to\infty} L_n^{1/n}<\infty$ *and* $0<\liminf_{n\to\infty} M_n^{1/n}<\infty$, $C_L(\mathrm{R})=C_M(\mathrm{R})=C_1(\mathrm{R})$, *if* $\lim_{n\to\infty} L_n^{1/n}=\infty$, *then* $C_L(\mathrm{R})\subseteq C_M(\mathrm{R})$ *if and only if* $\lim_{n\to\infty} M_n^{1/n}=\infty$ *and* $(L_n^c)^{1/n}=O(M_n^{1/n})$ *or* $(L_n^c)^{1/n}=O[(M_n^c)^{1/n}]$.

(ii) *If* $\liminf_{n\to\infty} L_n^{1/n}>0$ *and* $\liminf_{n\to\infty} M_n^{1/n}>0$, *then* $C_L(d\mathrm{R})\subseteq C_M(d\mathrm{R})$ *if and only if* $(L_n^d)^{1/n}=O(M_n^{1/n})$ *or* $(L_n^d)^{1/n}=O(M_n^d)^{1/n}$, *where* $d\mathrm{R}$ *represents any open or closed halfline.*

(iii) $C_L^*(I)\subseteq C_M^*(I)$ *if and only if* $(L_n^0)^{1/n}=O(M_n^{1/n})$ *or* $(L_n^0)^{1/n}=O[(M_n^0)^{1/n}]$ *for any finite or infinite open interval.*

(iv) $C_L^*(I)\subseteq C_M^*(I)$ *if and only if* $(L_n^f)^{1/n}=O(M_n^{1/n})$ *or* $(L_n^f)^{1/n}=O[(M_n^f)^{1/n}]$ *for any closed finite interval or any semi-closed finite or infinite interval.*

Proof: Since the conditions given in (i)—(iv) are clearly sufficient, we need only prove that they are necessary.

(i) If $\lim L_n^{1/n}=\infty$, it follows from Theorem A (i) that $C_L(\mathrm{R})=C_{L^c}(\mathrm{R})$. Applying Theorem B, we get the result.

(ii) By Theorem A (ii), $C_L(d\mathrm{R})=C_{L^d}(d\mathrm{R})$ and $C_M(d\mathrm{R})=C_{M^d}(d\mathrm{R})$. Suppose that $C_{L^d}(d\mathrm{R})\subseteq C_{M^d}(d\mathrm{R})$. Without loss of generality, we can suppose that $d\mathrm{R}=\mathrm{R}_+=[0,\infty)$. Put $\mathrm{B}=\{f\in C^\infty(\mathrm{R}): \|f^{(n)}\|_\infty\leqslant AL_n^d,\ \forall n>0\}$ and define the norm $f\to\|f\|_B$ by setting $\|f\|_B=\sup_{n>0}\{\|f^{(n)}\|_\infty/L_n^d\}$. B is a Banach space and for every integer $j\geqslant 1$, $V_j=\{f\in B: \|f^{(n)}\|_\infty\leqslant j^{n+1}M_n^d,\ \forall n>0\}$ is a closed subspace of B. Since $C_{L^d}(d\mathrm{R})\subseteq C_{M^d}(d\mathrm{R})$, $B=\bigcup_{j=1}^{\infty} V_j$. By Baire's category theorem, there exist a $\delta>0$ and a V_j such that $\|f\|_B\leqslant\delta$ implies that $f\in V_j$. Consider the function g defined by

$$g(x)=\delta e^{-x}\Lambda_{[r]}(x)/2a\,\widetilde{T}_{L^d}(r),\quad (r>1),$$

where Λ_n is a Laguerre polynomial of degree n defined by setting

$$\Lambda_n(x) = \sum_{k=0}^{n} C_{n,\,n-k} \frac{(-x)^k}{k!} \quad (n \geq 1)$$

and where $a = \max[1, L_1^d, L_1^d (L_0^d)^{-1}]$. It is known (cf. [9], p. 208) that for all $x \in R_+$

$$e^{-x} |\Lambda_n(x)| \leqslant 1,$$

$$|[e^{-x} \Lambda_n(x)]^{(k)}| \leqslant 2\,(4e^2 n k^{-1})^k, \; 0 < k < n,$$

$$|[e^{-x} \Lambda_n(x)]^{(k)}| \leqslant 4^k, \; k \geq n.$$

It follows that for $k \geqslant [r]$

$$\|g^{(k)}\|_\infty \leqslant \frac{\delta 4^k}{2a\,\tilde{T}_{L^d}(r)} \leqslant \frac{\delta 4^k}{a} L_1^d.$$

Since $\liminf L_n^{1/n} > 0$, there exists a positive constant $\lambda < 1$ such that $n^n L_n > \lambda^n n^n$. Since $\lambda^n n^n$ is log-convex and $n^n L_n^d$ is the largest log-convex minorant of $n^n L^n$ we have $n^n L_n^d \geqslant n^n \lambda^n$ or $L_n^d \geqslant \lambda^n$ for all $n \geqslant 1$. Thus for $k \geq [r]$

$$\|g^{(k)}\|_\infty \leqslant \delta 4^k \frac{L_1^d}{a\lambda^k} L_k^d \leqslant \delta 4^k \lambda^{-k} L_k^d$$

since $a = \max[1, L_1^d, (L_0^d)^{-1}]$. For $0 < k < [r]$,

$$\|g^{(k)}\|_\infty \leqslant \delta\,(4e^2)^k k^{-k} \frac{r^k}{\tilde{T}_{L^d}(r)} \leqslant \delta\,(4e^2)^k k^{-k} \sup_{r>0} \frac{r^k}{\tilde{T}_{L^d}(r)} \leqslant \delta\,(4e^2)^k L_k^d$$

by (2). Clearly for $k=0$, $\|g^{(k)}\|_\infty \leqslant \delta / a\tilde{T}_{L^d}(r) \leqslant \delta L_0^d$. If we set $f(t) = g(bt)$ where $0 < b < \frac{1}{4e^2}$ then for all $k \geqslant 0$ $\|f^{(k)}\|_\infty \leqslant \delta L_k^d$ so that $f \in B$ and $\|f\|_B \leqslant \delta$. Hence $f \in V_j$ so that

$$|f^{(k)}(0)| = |b^k g^{(k)}(0)| = \delta b^k |[e^{-x} \Lambda_{[r]}(x)]^{(k)}_{x=0}| / 2a\,\tilde{T}_{L^d}(r) < j^{k+1} M_k^d.$$

But we know (cf. [9], p. 210) that

$$|[e^{-x} \Lambda_{[r]}(x)]^{(k)}_{x=0}| \geqslant \left(\frac{[r]}{ak}\right)^k \geqslant \left(\frac{r}{2ak}\right)^k.$$

Hence for every $r \geq 1$

$$\frac{\delta b^k}{2a\,(2ak)^k} \cdot \frac{r^k}{\tilde{T}_{L^d}(r)} \leqslant j^{k+1} M_k^d \quad k \geqslant 1$$

so for large k

$$L_k^d = k^{-k} \sup_{r \geq 1} \frac{r^k}{\tilde{T}_{L^d}(r)} \leqslant \frac{2a}{\delta} (2ab^{-1})^k j^{k+1} M_k^d$$

i. e. $(L_k^d)^{1/k} = O[(M_k^d)^{1/k}]$.

(III) For any finite or infinite open interval I, we have by Theorem A (iii) $C^*_{L^0}(I) = C^*_L(I)$ and $C^*_{M^0}(I) = C^*_M(I)$. Suppose that $C^*_{L^0}(I) \subseteq \subseteq C^*_{M^0}(I)$. First suppose that I is a finite open interval. Without loss of generality, we may suppose that $I =]-1, 1[$. We can write $I = \bigcup_{l=1}^{\infty} I_l$, where $I_l = [-a_l, a_l]$ and (a_l) is an increasing sequence tending to 1. Let $F = \{f \in C^\infty(I): \forall I_l, \ \exists A_l: |f^{(n)}(x)| \leqslant A_l L^0_n, \ \forall n > 0, \forall x \in I_l, \ A_l = A_l(f)\}$ and let

$$p_l(f) = \sup_{n>0} \max_{x \in I_l} |f^{(n)}(x)| / L^0_n.$$

p_l is a semi-norm on F and F with this family of semi-norms is a Frèchet space. Let $V_{j,l} = \{f \in F: |f^{(n)}(x)| < j^{n+1} M^0_n, \ \forall n > 0, \ \forall x \in I_l\}$. If $f \in F$, $f \in C^*_{M^0}(I)$. Hence for every $x_0 \in I_l$, there exists an open interval $J_{x_0} \subset I$ such that

$$|f^{(n)}(x)| \leqslant A_{x_0} \lambda^n_{x_0} M^0_n, \ \forall x \in J_{x_0} \text{ and } \forall n \geqslant 0.$$

Applying Borel—Lebesgue theorem, we conclude that there exist constants A_l and λ_l such that

$$|f^{(n)}(x)| \leqslant A_l \lambda^n_l M^0_n, \ \forall x \in I_l \text{ and } \forall n \geqslant 0.$$

This shows that for each fixed l, there exists a $j = j(f, l)$ such that $f \in V_{j(f,l),l}$. Thus $F = \bigcup_{j,l=1}^{\infty} V_{j,l}$. Clearly $V_{j,l}$'s are closed in F. Hence, by Baire's category theorem, there exist a semi-norm p_{l_1}, a $\delta > 0$ and a V_{j_0, l_0} such that $p_{l_1}(f) \leqslant \delta$ implies that $f \in V_{j_0, l_0}$.

Let

$$z_n(t) = \frac{1}{2}[T_{n-1}(t) + T_n(t)],$$

where T_n denotes the Cebyšev polynomial of degree n. It is known (cf. [9], p. 206) that

$$\frac{e^{-k}}{2} n^k \leqslant |z^{(k)}_n(0)| < n^k \ (n > 1) \tag{4'}$$

and

$$|T^{(k)}_n(x)| \leqslant \frac{3^k n^k}{(1-x^2)^k} \ (0 \leqslant k \leqslant n), \ -1 < x < 1. \tag{5}$$

Put for $r \geqslant 1$,

$$f(x) = \frac{\delta}{S_{L^0}(r)} z_{[r]}(bx),$$

where $0 < b < \min\left(\frac{1}{2a_{l_1}}, \frac{1}{4}\right)$ and

$$S_{L^0}(r) = \max_{n<r} \frac{r^n}{L^0_n} \qquad (r > 0).$$

For $n \leqslant [r]$ and $|x| \leqslant a_{l_1}$

$$|f^{(n)}(x)| \leqslant \frac{\delta}{S_{L^\circ}(r)} \cdot b^n \, |z^{(n)}_{[r]}(bx)| \leqslant \frac{\delta}{S_{L^\circ}(r)} \cdot \frac{3^n b^n r^n}{(1-a^2_{I_1} b^2)^n} \leqslant \frac{\delta}{S_{L_\circ}(r)} \cdot r^n$$

the last inequality being true since $3b < (1-a^2_{I_1} b^2)$. Hence for $n \leqslant r$

$$\max_{x \in I_{l_1}} |f^{(n)}(x)| \leqslant \sup_{n<r} \frac{r^n}{S_{L^\circ}(r)} = \delta L^0_n.$$

But for $n > [r]$, $f^{(n)} \equiv 0$. Hence $p_{l_1}(f) \leqslant \delta$ so that $f \in V_{j_0, l_0}$ i. e.

$$\frac{\delta b^n}{S_{L_\circ}(r)} \cdot |z^{(n)}_{[r]}(bx)| \leqslant j_0^{n+1} M^0_n, \ \forall n \geqslant 0 \text{ and } \forall x \in I_{l_0}$$

so that

$$\frac{\delta b^n}{S_{L_\circ}(r)} \frac{e^{-n}}{2} [r]^n \leqslant \frac{\delta}{S_{L_\circ}(r)} |z^{(n)}_{[r]}(0)| \leqslant j_0^{n+1} M^0_n$$

or

$$\frac{r^n}{S_{L_\circ}(r)} \leqslant \delta^{-1} 2^n e^n b^{-n} j_0^{n+1} M^0_n.$$

Since this holds for r such $[r] > n$, taking the supremum with respect to all $r > n$, we get

$$L^0_n = \sup_{n<r} \frac{r^n}{S_{L^\circ}(r)} \leqslant \lambda^n M^0_n,$$

i. e. $(L^0_n)^{1/n} = O\,[(M^0_n)^{1/n}]$.

Next suppose that $I = \mathrm{R}$. We write $\mathrm{R} = \bigcup_{1}^{\infty} I_l$, where $I_l = [-l, l]$.

We repeat the steps of the above proof and conclude that there exist a semi-norm p_{l_1}, a $\delta > 0$ and a V_{j_0, l_0} such that if an $f \in F$ is such that $p_{l_1}(f) \leqslant \delta$, then $f \in V_{j_0, l_0}$.

Let

$$f(x) = \frac{\delta x^{[r]}}{l_1^r S_{L_\circ}(r)}.$$

Then

$$\max_{x \in I_l} |f^{(n)}(x)| \leqslant \frac{\delta r^n l^r}{S_{L^\circ}(r) \cdot l_1^r}$$

if $n \leqslant [r]$ and $\equiv 0$ otherwise. Hence

$$p_{l_1}(f) = \sup_{n>0} \frac{\max_{x \in I_{l_1}} |f^{(n)}(x)|}{L^0_n} \leqslant \delta \left(\sup_{n<r} \frac{r^n}{L^0_n} \right) \frac{1}{S_{L^\circ}(r)} \leqslant \delta.$$

Hence, for all $[r] \geqslant n$ and for $n \geqslant 1$

$$\frac{\delta}{l_1^r S_{L^\circ}(r)} [r]\,([r]-1) \cdots ([r]-n+1)\, l_1^{[r]-n} \leqslant j^{n+1} M^0_n.$$

or

$$\frac{\delta}{S_{L^\circ}(r)} 2^{-n} r^n e^{-n} l_1^{[r]-r-n} \leqslant j^{n+1} M^0_n.$$

Thus

$$L^0_n = \sup_{r>n} \frac{r^n}{S_{L^\circ}(r)} \leqslant \lambda^{n+1} M^0_n.$$

Let us suppose that I is an open half-line. Without loss of generality, we suppose that $I =]0, \infty[$. We write $I_l = \left[\frac{1}{l}, l\right]$ so that $I = \bigcup I_l$. If we repeat the arguments for $I = R$, we get the result.

(iv) For any closed finite interval or any semi-closed finite or infinite interval I, we have by Theorem A (iv), $C_L^*(I) = C_{L^f}^*(I)$ and $C_M^*(I) = C_{M^f}^*(I)$.

Suppose that $C_{L^f}^*(I) \subseteq C_{M^f}^*(I)$. First let I be a compact interval which we may suppose, without loss of generality, to be $[-a, a]$ with $a > \frac{e}{2}$. In this case we have $C_{L^f}(I) \subseteq C_{M^f}(I)$. Set $B = \{f \in C^\infty(I) : \|f^{(n)}\|_\infty \leqslant \leqslant A L_n^f \ \forall n \geqslant 0, A = A(f)\}$. For every $f \in B$, define $\|f\|_B = \sup_{n \geqslant 0} \|f^{(n)}\|_\infty / L_n^f$. B is a Banach space with this norm. Eor every integer $j \geqslant 1$, define V_j by setting $V_j = \{f \in B : \|f^{(n)}\|_\infty \leqslant j^{n+1} M_n^f \ \forall n \geqslant 0\}$. V_j's are closed in B and $B = \bigcup_{j=1}^{\infty} V_j$. By Baire's theorem, there exist a $\delta > 0$ and a j such that $\|f\|_B \leqslant \delta$ implies that $f \in V_j$.

Put

$$f(x) = \delta \frac{T_{[r]}(a^{-1}x)}{U_{L^f}(r)},$$

where $r \geqslant 1$. Clearly

$$\max_{-a<x<a} |f^{(n)}(x)| = \frac{\delta a^{-n}}{U_{L^f}(r)} \cdot |T_{[r]}^{(n)}(1)| \text{ for } n \leqslant [r]$$

$$= 0 \qquad \text{for } n > [r].$$

But

$$\left(\frac{2}{en}\right)^n [r]^{2n} \leqslant |T_{[r]}^{(n)}(1)| \leqslant \left(\frac{e}{2n}\right)^n [r]^{2n}. \tag{6}$$

Hence

$$\|f\|_B \leqslant \frac{\delta a^{-n}}{U_{L^f}(r)} \left(\frac{e}{2}\right)^n \sup_{n<r} \frac{r^{2n}}{n^n L_n^f} \leqslant \delta$$

since $a > \frac{e}{2}$. But then $f \in V_j$ i. e. for all $r \geqslant n$ and for all $n \geqslant 1$,

$$\left(\frac{2}{en}\right)^n [r]^{2n} \frac{\delta a^{-n}}{U_{L^f}(r)} \leqslant j^{n+1} M_n^f$$

or for some A

$$\frac{r^{2n}}{U_{L^f}(r)} \leqslant A^{n+1} n^n M_n^f.$$

Thus

$$n^n L_n^f = \sup_{r>n} \frac{r^{2n}}{U_{L^f}(r)} \leqslant A^{n+1} n^n M_n^f$$

i. e.

$$(L_n^f)^{1/n} = O[(M_n^f)^{1/n}].$$

Let I be a finite semi-closed interval. We can suppose, without loss of generality that $I=]-a,\ a]$, where $a>\frac{e}{2}$. Set $B=\{f\in C^{\infty}(I)$: $\|f^{(n)}\|_{\infty}\leqslant AL_n^f\ \forall n\geqslant 0,\ A=A(f)\}$. For each $f\in B$ define $\|f\|_B=\sup\limits_{n\geqslant 0}\frac{\|f^{(n)}\|_{\infty}}{L_n^f}$. B is a Banach space with this norm. We can write $I=\bigcup\limits_{l}^{\infty} I_l$, where $I_l=\left]-a+\frac{1}{l},\ a\right]$. For any integer $j\geqslant 1$ define $V_{j,\,l}$ by setting $V_{j,\,l}=\{f\in B\colon \max\limits_{x\in I_l}|f^{(n)}(x)|\leqslant j^{n+1}M_n^f\ (n\geqslant 0)\}$. $V_{j,\,l}$'s are closed in B and $B=\bigcup\limits_{j,\,l=1}^{\infty} V_{j,\,l}$. By Baire's category theorem, there exist a $\delta>0$, a „j" and an „l" such that $\|f\|_B\leqslant\delta$ implies that $f\in V_{j,\,l}$.

Put

$$f(x)=\frac{\delta T_{[r]}(a^{-1}x)}{U_{Lf}(r)},$$

where $r\geqslant 1$. We have

$$\max_{x\in I}|f^{(n)}(x)|=\frac{\delta a^{-n}|T_{[r]}^{(n)}(1)|}{U_{Lf}(r)}\quad\text{for } n\leqslant[r]$$

$$=0\quad\text{for } n>[r].$$

By (6) we conclude that $f\in B$ and that

$$\|f\|_B=\frac{\delta}{U_{Lf}(r)}\sup_{n<r}\frac{a^{-n}|T_{[r]}^{(n)}(1)|}{L_n^f}\leqslant\frac{\delta}{U_{Lf}(r)}\sup_{n<r}\frac{a^{-n}r^{2n}}{n^n L_n^f}\left(\frac{e}{2}\right)^n\leqslant$$

$$\leqslant\frac{\delta}{U_{Lf}(r)}\sup_{n<r}\frac{r^{2n}}{n^n L_n^f}=\delta.$$

But then $f\in V_{j,\,l}$ i. e. for all $r\geqslant n$ and for all $n\geqslant 1$

$$\left(\frac{2}{en}\right)^n[r]^{2n}\frac{\delta a^{-n}}{U_{Lf}(r)}\leqslant j^{n+1}M_n^f$$

or

$$\frac{r^{2n}}{U_{Lf}(r)}\leqslant\delta^{-1}a^n\left(\frac{e}{2}\right)^n 2^{2n}j^{n+1}n^n M_n^f$$

or

$$n^n L_n^f=\sup_{r>n}\frac{r^{2n}}{U_{Lf}(r)}\leqslant\lambda^{n+1}n^n M_n^f.$$

Thus

$$(L_n^f)^{1/n}=O\left[(M_n^f)^{1/n}\right].$$

Finally, let I be a semi-closed half-line which we may suppose to be $[0,\ \infty[$. Let $I_l=[0,\ l]$. Then $I=\bigcup\limits_{l=1}^{\infty}L_l$. Let

$$F=\{f\in C^{\infty}(I)\colon\forall l\geqslant 1,\ \max_{x\in I_l}|f^{(n)}(x)|\leqslant A_l\,L_n^f,\ A_l=A_l(f)\}$$

and let

$$p_l(f) = \sup_{n>0} \max_{x \in I_l} |f^{(n)}(x)|/L_n^f.$$

p_l is a semi-norm on F and F with (p_l) as a family of semi-norm is a Fréchet space. Let

$$V_{j,l} = \{f \in F: |f^{(n)}(x)| \leqslant j^{n+1} M_n^f \ (n \geqslant 0),\ x \in I_l\}.$$

If $f \in F$, $f \in C^*_{M^f}(I)$. Hence, for every $x_0 \in I_l$, there exists a relatively open interval $J_{x_0} \subset I$ such that

$$|f^{(n)}(x)| \leqslant A_{x_0} \lambda_{x_0}^n M_n^f \ \forall x \in I_{x_0} \text{ and } \forall n \geqslant 0.$$

Applying Borel—Lebesgue theorem, we conclude that there exist constants A_l and λ_l such that

$$|f^{(n)}(x)| \leqslant A_l \lambda_l^n M_n^f \ \forall x \in I_l \text{ and } \forall n > 0.$$

This shows that for each fixed l, there exists a $j = j(f, l)$ such that $j \in V_{j(f,l),l}$. Thus $F = \bigcup_{j,l=1}^{\infty} V_{j,l}$. Clearly $V_{j,l}$'s are closed in F. Hence, by Baire's category theorem, there is a semi-norm p_{l_0}, a $\delta > 0$ and a V_{j_0, l_0} such that $p_{l_0}(f) \leqslant \delta$ implies that $f \in V_{j_0, l_0}$.

Let

$$f(x) = \frac{\delta e^{-arx} \Lambda_{[r]}(arx)}{2U_{Lf}(r)},$$

where $0 < a < 4^{-1} e^{-2}$. We have

$$\|f^{(n)}\|_\infty \leqslant \frac{\delta a^n r^n 4^n}{U_{Lf}(r)} \leqslant \frac{\delta r^n}{U_{Lf}(r)}, \text{ if } n \geqslant [r]$$

and

$$\|f^{(n)}\|_\infty \leqslant 2\left(\frac{4e^2}{n}\right)^n \delta \frac{[r]^n a^n r^n}{2U_{Lf}(r)} \leqslant$$

$$\leqslant \delta \frac{r^{2n}}{n^n U_{Lf}(r)}, \text{ if } n < [r].$$

Hence

$$p_{l_0}(f) = \sup_{n>0} \max_{x \in I_{l_0}} |f^{(n)}(x)|/L_n^f <$$

$$\leqslant \delta \sup_{n<r} \frac{r^{2n}}{n^n U_{Lf}(r)} / L_n^f =$$

$$= \frac{\delta}{U_{Lf}(r)} \sup_{n<r} \frac{r^{2n}}{n^n L_n^f} = \delta.$$

Therefore, for every $n \geqslant 1$,

$$|f^{(n)}(0)| \leqslant j_0^{n+1} M_n^f$$

or

$$\delta a^n r^n |[e^{-x}\Lambda_{[r]}(x)]^{(n)}_{x=0}| \frac{1}{2U_{Lf}(r)} \leqslant j_0^{n+1} M_n^f$$

or

$$\delta a^n r^n \left(\frac{r}{2\alpha n}\right)^n \frac{1}{2U_{Lf}(r)} \leqslant j_0^{n+1} M_n^f$$

or

$$\frac{r^{2n}}{U_{Lf}(r)} \leqslant 2 j_0^{n+1} n^n M_n^f \delta^{-1} a^{-n} (2\alpha)^n .$$

or

$$n^n L_n^f = \sup_{n<r} \frac{r^{2n}}{U_{Lf}(r)} \leqslant n^n \lambda^{n+1} M_n^f.$$

Thus

$$(L_n^f)^{1/n} = O\left[(M_n^f)^{1/n}\right].$$

5. We make the folllowing concluding remarks:

(i) If I is a finite closed interval then $C_M(I) = C_M^*(I) = C_{Mf}^*(I)$ and hence the problem of inclusion of classes is solved for classes $C_M(I)$.

(ii) If I is a finite open or semi-closed interval then $C_M(I) = C_M(\bar{I}) = C_{Mf}^*(\bar{I})$ by an obvious extension of the C^∞-functions belonging to $C_M(I)$. Thus if $I = [a, b[$ and $f \in C_M(I)$, $f^{(n)} \in BV[0, 1]$ for every n. Hence $f^{(n)}(b-0)$ exsist for every $n > 0$ so that we define $f^{(n)}(b) = f^{(n)}(b-0)$ for each $n \geqslant 0$. A similar extension can be made if $I =]a, b[$. Thus the problem of inclusion for classes $C_M(I)$, where I is a finite open or semi-closed interval is also solved.

(iii) In view of the above remarks, Theorem C gives us the solutions of the problems of inclusion for classes $C_M(I)$ and $C_M^*(I)$ for *every* type of linear interval.

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Ջ. Ա. ՍԻԴԻՔԻ. Անվերջ դիֆերենցելի ֆունկցիաների դասերի համարժեքության մասին

(ամփոփում)

Դիցուք J-ն իրական առանցքի վրա գտնվող կամայական միջակայք է, իսկ $M = \{M_n\}$*-ը դրական թվերի հաջորդականություն է։ Նշանակենք* $C_M(J)$ *այն* $f \in C^\infty(J)$ *ֆունկցիաների դասը, որոնց համար*

$$|f^{(n)}(x)| \leqslant A k^n M_n \quad \forall n \geqslant 0 \quad \forall x \in J, \quad k = k(f)։$$

Համանմանորեն սահմանվում է պայմաններին լոկալ բավարարող ֆունկցիաների $C_M^*(J)$ *դասը։*

Տ. Կառլեմանի [3] կողմից դրվել էր հետևյալ խնդիրը. սևեռած միջակայքի դեպքի գտնել անհրաժեշտ և բավարար պայմաններ և M հաջորդականությունների վրա, որոնց դեպքում տեղի կունենա $C_L(J) \subset C_M(J)$ *(համապատասխանորեն* $C_L^*(J) \subset C_M^*(J)$*) ընդգրկումը։ Միջակայքերի որոշ տարբեր տիպերի համար դրա լուծումը տրվել է Ա. Գորնու [6], Հ. Կարտանի ու Ս. Մանդելբրոյտի [4] և Ս. Ագմոնի [1] կողմից։* $J = (-\infty, +\infty)$ *դեպքում Բոմանի և Լ. Հյորմանդերի [2] կողմից տրվել է այս խնդրի պարզ լուծումը՝ հիմնված կատեգորիաների վերաբերյալ Բէրի թեորեմի վրա։*

Տվյալ հոդվածը նվիրված է դրված խնդրի լուծմանը միջակայքերի կամայական տիպերի համար $C_M(J)$ *և* $C_M^*(J)$ *դասերի դեպքերում։ Առաջարկված մեթոդը, որը նույնպես հիմնված է կատեգորիաների մասին Բէրի թեորեմի վրա, թույլ է տալիս բերել ոչ միայն տվյալ աշխատանքում ստացված, այլև վերոհիշյալ հեղինակների կողմից ստացված արդյունքների պարզ ապացույցներ։*

Дж. А. СИДДИКИ. *Об эквивалентности классов бесконечно дифференцируемых функций (резюме)*

Пусть J — произвольный интервал на вещественной оси и $M = \{M_n\}$ — последовательность положительных чисел. Обозначим через $C_M(J)$ класс функций $f \in C^\infty(J)$ таких, что

$$|f^{(n)}(x)| \leqslant kA^n M_n \ \forall n \geqslant 0, \ \forall x \in J, \quad k = k(f).$$

Аналогично можно ввести классы $\overset{*}{C}_M(J)$, локально удовлетворяющих поставленным условиям.

Т. Карлеманом [3] была поставлена задача: при фиксированном интервале J найти необходимые и достаточные условия на последовательности L и M, при которых имеет место включение $C_L(J) \subset C_M(J)$ (соотв. $\overset{*}{C}_L(J) \subset \overset{*}{C}_M(J)$). Для различных типов интервалов ее решение было дано А. Горным [6], А. Картаном и С. Мандельбройтом [4] и С. Агмоном [1]. В случае $J = (-\infty, +\infty)$ простое решение этой задачи, основанное на применении теоремы Бэра о категориях, было дано Дж. Боманом и Л. Хёрмандером [2].

Данная статья посвящена решению поставленной задачи для произвольных типов интервалов J и в обеих случаях классов $C_M(J)$ и $\overset{*}{C}_M(J)$. Предложенный при этом метод, также основанный на применении теоремы Бэра о категориях, позволяет дать простые доказательства не только установленных в данной работе, но и известных ранее результатов вышеназванных авторов.

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