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ON TANGENTIAL APPROXIMATION BY HOLOMORPHIC FUNCTIONS

I. Introduction and Statement of Main Results

Let D be a domain of the finite complex plane **C** and E a (relatively) closed proper subset of D. *A speed* on E is a positive, continuous, bounded function on E. If α is a speed on E, then E is called *a set of approximation* (in D) *with speed* α, or simply, *a set of α-approximation* provided that for each function f continuous on E, holomorphic on its interior E^0, and each constant $\varepsilon>0$, there is a function g, holomorphic on D, such that

$$|f(z)-g(z)|<\varepsilon\alpha(z),\ z\in E. \tag{1}$$

In this paper, we consider the problem of describing sets of approximation with speed α. We obtain new results, even for the case when D is a disc or the finite plane.

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A (relatively) closed set E in D is called a set of *uniform approximation* if E is a set of approximation with speed 1. Clearly, if E is a set of α-approximation, then E is, a fortiori, a set of uniform approximation. Thus, it is natural for us to consider only sets E which are sets of uniform approximation.

The first author has shown [1], that a necessary and sufficient condition in order that E be a set of uniform approximation is that $D^*\setminus E$ be connected and locally connected. Here $D^*=D\cup\{*\}$ denotes the Alexandrov compactification of D, where $*$ denotes the ideal point of D.

Actually, conditions on a set E of uniform approximation to be a set of α-approximation, depend exclusively on α and topological-metric properties of E^0. In this connection we need some notations and definitions.

1) For an arbitrary subset X of the Riemann sphere $\mathbf{C}^*$, we denote by X^0, $\overline{X}$ and ∂X respectively the interior, closure and boundary of X in $\mathbf{C}^*$.

2) Let E be a closed set in a domain D. Denote

$$\tilde{E}=(\overline{E^0})\cap D.$$

$\bar{E}$ is closed in D, and it is easy to show, using the well-known Tietze Extension Theorem [2, p. 134], that $\tilde{E}$ is a set of uniform approximation in D, if the set E is.

Now let α be a speed on E.

3) We say, that α is *log-superharmonic* if $\log \alpha|E^0$ is superharmonic and has a continuous superharmonic extension on some neighbourhood of the set $\bar{E}$.

4) Consider two extensions of $\alpha|\bar{E}$ on $\overline{(E^0)}$, $\bar{\alpha}$ and $\underline{\alpha}$, defined for $\zeta \in \partial E^0 \setminus \bar{E}$ by

$$\bar{\alpha}(\zeta) = \overline{\lim_{z\to\zeta}}\, \alpha(z),\ \underline{\alpha}(\zeta) = \underline{\lim_{z\to\zeta}}\, \alpha(z),\ z \in E^0.$$

Obviously, $\bar{\alpha}$ and $\underline{\alpha}$ are semicontinuous, non-negative and bounded. With the convention $\log 0 = -\infty$, the functions $\log \bar{\alpha}$ and $\log \underline{\alpha}$ are semicontinuous and upper-bounded.

5) We say, that the speed α is *inner-continuous*, if $\bar{\alpha} = \underline{\alpha}$. In this case, we use the notation α instead of $\bar{\alpha}$ or $\underline{\alpha}$.

6) For $z \in E^0$, we denote by μ_z the harmonic measure on ∂E^0 with respect to E^0. Let $\beta : \partial E^0 \to \{-\infty\} \cup \mathrm{R}$ be an upper-bounded and measurable function with respect to harmonic measure μ_z. The following condition on β is often used in this paper:

$$\inf_{z\in K\cap E^0} \int_{\partial E^0} \beta\, d\mu_z > -\infty \tag{2}$$

for each compact $K \subset D$.

Theorem 1A. *Let E be a set of approximation in a domain D with speed α. Then for* $\log \bar{\alpha}$ *condition* (2) *is satisfied.*

Theorem 1B. *Let D be a finitely connected domain and let E be a set of uniform approximation in D such that $\mathrm{C}^* \setminus E$ has no isolated points. Let α be a log-superharmonic speed on E, such that for* $\log \underline{\alpha}$ *condition* (2) *is satisfied. Then E is a set of approximation in D with speed α.*

The following consequence from these theorems is our main result.

Theorem 1. *Let D and E be as in Theorem* 1B *and let α be an inner-continuous and* log-*superharmonic speed on E. Then E is a set of approximation in D with speed α, if and only if condition* (2) *for* $\log \alpha$ *is satisfied.*

Let us say that the boundary of an open set G is a *curve at* ζ if there is a homeomorphism h of some neighbourhood V of ζ onto the unit disc $(|w|) < 1)$ with $h(\zeta) = 0$ and

$$V \cap \partial G = h^{-1}(\mathrm{Im}\, w = 0).$$

Theorem 2. *Suppose D, E and α are as in Theorem* 1 *and moreover that $D \cap \partial E^0$ is a curve at each of its points. Then E is a set of approximation with speed α if and only if*

$$\int_{\partial E^0} \log \alpha \, d\mu_z > -\infty \tag{3}$$

for each $z \in E^0$.

Now suppose D is a disc D_R with centre 0 of finite or infinite radius $0 < R \leqslant +\infty$. For $0 < t < R$ and $z \in E^0$, we write

$$\omega(z, t) = \mu_z(\partial E^0 \setminus D_t).$$

Let $v(z) = v(|z|)$ be a function on D_R which depends only on $|z|$. Then, if v is a convex increasing function of $\log|z|$, v is subharmonic on D_R (see Lemma 7). Conversely, if $v(z) = v(|z|)$ is subharmonic, then

$$v(t) = \frac{1}{2\pi} \int_0^{2\pi} v(te^{i\theta})\, d\theta$$

is a convex increasing function of $\log t$. Thus, if $\alpha(z) = \alpha(|z|)$, then α is a log-superharmonic speed if and only if $-\log\alpha(t)$ is a convex increasing function of $\log t$.

Theorem 3. *Let E be a set of approximation in D_R with asymptotic decreasing speed $\alpha(z) = \alpha(|z|)$. Then $\omega(z, R) = 0$ for each $z \in E^0$ and*

$$\sup_{z \in D_r \cap E^0} \int_0^R \log \alpha(t)\, d\omega(z, t) < \infty \tag{4}$$

for each $0 < r < R$. Conversely, let E be a set of uniform approximation in D_R, with $\omega(z, R) = 0$ for $z \in E^0$ and such that (4) is satisfied for some α, where $-\log\alpha$ is a log-convex function, tending to ∞ on $[0, R)$. Then E is a set of approximation in D_R with speed $\alpha(|z|)$.

A speed α on E is called an *asymptotic speed* if $\alpha(z) \to 0$ as $z \to \partial D$ on E. The set E is called a set of *asymptotic approximation* if E is a set of α-approximation for some asymptotic speed α. Obviously, such a speed α is inner-continuous and $\alpha = 0$ on $\partial D \cap \overline{E}$.

Corollary 1 [3]. *Let D and E be as in Theorem* 1. *Then, E is a set of asymptotic approximation if and only if*

$$\mu_z(\partial E^0 \setminus D) = 0$$

for each $z \in E^0$.

A set E of uniform approximation in D is called a set of *Carleman approximation* provided E is a set of α-approximation for every speed α.

Corollary 2 [4—5]. *Let D and E be as in Theorem 1. Then, E is a set of Carleman approximation if and only if it satisfies*

Condition C: *For each compact K in D, there is a compact K' in D such that no component of E^0 meets both K and $D \setminus K'$.*

For $E \subset D_R$ and $0 < t < R$, we denote by $t\theta(t)$ the linear measure of $E^0 \cap \partial D_t$, and set

$$\tilde{\theta}(t) = \sup_{t < s} \theta(s).$$

Corollary 3. *Let* $-\log\alpha$ *be a log-convex function on* $[0, R)$ *and let* E *be a set of uniform approximation in* D_R *such that*

$$\int_0^R \log\alpha(t)\, d\bar{\theta}(t) < +\infty. \tag{5}$$

Then E *is a set of approximation with speed* $\alpha(|z|)$, *if* $\bar{\theta}(R-0)=0$.

Corollary 4. *Let* $\alpha(z)=\alpha(|z|)$ *be a speed decreasing to zero on* $\mathbf{C}$ *and let* E *be a set of uniform approximation in* $\mathbf{C}$ *such that for some* $0<k<1$,

$$\int_1^\infty \log\alpha(t)\left\{t\theta(kt)\exp\left[\pi\int_1^{kt}\frac{d\rho}{\rho\theta(\rho)}\right]\right\}^{-1} dt > -\infty. \tag{6}$$

Then, E *is a set of approximation with speed* α.

Corollary 5 [6]. *Let* $\alpha(z)=\alpha(|z|)$ *be a decreasing speed on* $\mathbf{C}$ *satisfying*

$$\int_1^\infty \frac{\log\alpha(t)}{t^{3/2}}\, dt > -\infty. \tag{7}$$

Then, each set of uniform approximaton by entire functions is a set of approximation with speed α. *If the integral* (7) *diverges, then the conclusion fails.*

We shall say that E^0 is *sectorial* in D_R if there are continuous functions θ_1 and θ_2 on $(0, R)$ with

$$\theta_1 < \theta_2 < \theta_1 + 2\pi$$

and

$$E^0 = \{z\colon\ \theta_1(|z|) < \arg z < \theta_2(|z|)\},$$

for some branch of $\arg z$.

Corollary 6 [6]. *Let* $-\log\alpha$ *be a log-convex function on* $[0, R)$ *and suppose* E *is a set of uniform approximation on* D_R *such that* E^0 *is sectorial and for some* $0 \leqslant t_1 < R$,

$$\int_{t_1}^R \log\alpha(t)\left\{t\theta(t)\exp\left[\pi\int_{t_1}^{t}\frac{d\rho}{\rho\theta(\rho)}\right]\right\}^{-1} dt > -\infty. \tag{8}$$

Then E *is a set of approximation with speed* α, *if* (42) *holds.*

II. Remarks

1. Theorem 1 is no longer true if the complement of $\bar{E}$ is allowed to have infinitely many components (even if they are non-degenerate). We wish to thank A. A. Gonchar for furnishing us with the following example.

The set $E=\bar{E}$, which will be constructed as the complement of certain "champagne bubbles" (see [7]) in the unit disc D_1, has the property that $\mu_z(\partial E \setminus D_1) = 0$, for all $z \in E^0$, and so by Lemma 9 below, satisfies (2) for some asymptotic speed α. However, E is also a set of uniqueness, in the sense that if $f(z) \to 0$ as $z \to \partial D_1$, $z \in E$, then $f=0$, for any f continuous on E and holomorphic on E^0. Clearly, E cannot be a set of asymptotic approximation in the domain D obtained from D_1 by removing the centers of the "champagne bubbles".

To construct E we need the following fact (see [8]). For given $\varepsilon > 0$, $0 \leqslant \varphi' < \varphi'' < 2$, and $0 < r < 1$, there exists a set σ_ε,

$$\sigma_\varepsilon \subset S = \{te^{i\varphi\pi} : t \in (r, 1), \ \varphi \in (\varphi', \ \varphi'')\},$$

where σ_ε is a union of disjoint closed discs accumulating to $\gamma = \bar{S} \setminus D_1$ such that $\Gamma_\varepsilon = \partial\sigma_\varepsilon \cap D_1$ has length less than ε and $\mu_z(\gamma, D_1 \setminus \sigma_\varepsilon) = 0$.

It is easy to see that there exist $\varepsilon_n \searrow 0$ such that

$$\sum_1^\infty \varepsilon_n 2^{n(p+1)} < [\log(p+2)]^p,$$

for arbitrary $p = 0, 1, 2, \cdots$. We construct as above $\sigma_n = \sigma_{\varepsilon_n}$ for $S = S_n$, with $r = 2^{-n}$, $\varphi' = 2^{-n}$ and $\varphi'' = 2^{-n+1}$. Also, set $\Gamma_n = \Gamma_{\varepsilon_n}$ and $\Gamma_0 = \sigma_0 = \varnothing$. Moreover, we denote by σ_{-n}, S_{-n} etc. the reflection of the sets σ_n, S_n etc. with respect to the real axis. We set

$$E = D_1 \setminus \bigcup_{-\infty}^{+\infty} \sigma_n^0.$$

Now let f be continuous on E and holomorphic on E^0 and suppose $f(z) \to 0$ as $z \to \partial D$, $z \in E$. Putting $f(z) = 0$, for $|z| \geqslant 1$, we have

$$f^{(p)}(x) = \sum_{-\infty}^{\infty} \frac{p!}{2\pi i} \int_{\Gamma_n} \frac{f(t)}{(t-x)^{p+1}} \, dt, \quad x \in [0, 2]$$

for $p = 0, 1, 2, \cdots$. Since, $|t - x| > 2^{-|n|}$ for $t \in \Gamma_n$ and $x \in [0, 2]$, we have

$$|f^{(p)}(x)| \leqslant p! \sum_1^\infty \varepsilon_n 2^{n(p+1)} < [(p+2)\log(p+2)]^p.$$

Now by the well-known Carleman-Denjoy theorem on quasianalytic functions, $f(x) = 0$, for $x \in [0, 1)$, and so $f = 0$. Thus E is a set of uniqueness and the example is complete.

2. If $\mathbf{C}^* \setminus \tilde{E}$ has an isolated point a, then it may still be possible to approximate with a certain asymptotic speed α near a, however condition (2) is inappropriate in this situation, for it completely ignores the behaviour of α near a, and it certainly is not possible to approximate arbitrarily fast near an isolated point a.

3. In Theorem 1, we assume that $\log \alpha$ is superharmonic on $\tilde{E}$. In fact, the remark following lemma 5 shows that we may restrict our attention to functions $\log \alpha$ superharmonic on E^0 (and inner-continuous on E).

4. Corollary 1 was first proved for the unit disc [9]. A. Stray [3] extended the result to quite general domains including certain infinitely connected domains.

5. Condition C was introduced in [4] where it was shown to be necessary for Carleman approximation. Nersesian [5] showed that for arbitrary domains, the conidtion is also sufficient.

6. By considering the nature of the prime ends of $D \cap \partial E^0$ it is possible to relax the condition that $D \cap \partial E^0$ be a curve at each of its points, in the statement of Theorem 2.

7. Of course, if E^0 is connected, it is the same to assert that (3) holds at each point of E^0 as to assert that it holds at some point in E^0.

8. In [6], Corollary 6 was proved for the case that $R = +\infty$.

9. The notion of a completely regular point for the Dirichlet problem is not new; the name "completely regular" perhaps is. Brelot [10] has shown that a regular point need not be completely regular.

In Lemma 3 the condition that ∂G be a curve at the point ζ can be relaxed by considering the prime end structure of ∂G. It follows that the hypothesis in Theorem 2 can also be relaxed. However, conditions (2) and (3) are not in general equivalent as the following example shows.

Example: There is a set E of uniform approximation in C and a log-superharmonic function α such that condition (3) holds but condition (2) fails .

Let y_j be a sequence of positive numbers decreasing to zero and choose δ_j so small that the rectangles

$$E_j = \{z = x + iy : |x| \leqslant j,\ |y - y_j| \leqslant \delta_j\}$$

are disjoint. We may construct a continuous positive function α on C which decreases to zero so rapidly that for $z_j = iy_j$

$$H_{E_j^0}(z_j) \leqslant -j, \tag{9}$$

where the left-member denotes the solution of the Dirichlet problem on E_j^0 with boundary values $\log \alpha$. By Lemma 9 below, we may assume that α is log-superharmonic.

Now let

$$S_j = \{ z : |x| \leqslant \varepsilon_j,\ y_{j+1} \leqslant y \leqslant y_j\}$$

be a thin rectangle connecting E_j to E_{j+1}, where ε_j will be specified shortly. Set

$$E = \overline{\bigcup_1^{\infty} (E_j \cup S_j)}.$$

Then E is a set of uniform approximation. We may choose ε_j decreasing to zero so rapidly that

$$\int_{\partial E^0} \log \alpha \, d\mu_{z_1} > -\infty, \tag{10}$$

and thus $H_{E^0} = H_{E^0}^{\log \alpha}$ exists and is finite valued. We may further insist that ε_j decrease so rapidly that for each j,

$$H_{E^0}|E_j^0$$

is bounded. Thus, by Lemma 2 below, $H_{E_j^0}$ and H_{E^0} have the same boundary values on $\partial E_j \cap \partial E$. Hence by making the sequence ε_j still smaller, if necessary, we have that for each j

$$|H_{E_j^0}(z_j) - H_{E^0}(z_j)| < 1. \tag{11}$$

From (9) and (11), we have

$$\int_{\partial E^0} \log \alpha \, d\mu_{z_j} \leqslant -j + 1. \tag{12}$$

Thus, from (12), (2) fails. But (11) shows that

$$H_{E^0}(z_1) = \int_{\partial E^0} \log \alpha \, d\mu_{z_1} > -\infty,$$

and since E^0 is connected, H_{E^0} is finite for all $z \in E^0$. Hence (3) holds.

10. The problem of asymptotic approximation by *entire* functions was investigated by Carleman, Roth, Lavrentiev—Keldysh, Keldysh, Mergelian, Dzrbashian, Kaplan, and Arakelian. This problem for more general domains with special conditions imposed on the speed α and set E has been considered by Brown—Gauthier, Roth, Nersesian, Scheinberg. The speeds α considered are particular cases of the speeds we consider.

In all of the works just cited, as well as in the present work, α is assumed to be positive on E. If α is allowed to assume the value zero, then the problem becomes not only one of approximation but also of simultaneous interpolation. This question has been considered by Gauthier—Hengartner, Nersesian, Rubel—Venkateswaran, and Sinclair.

III. Preliminary lemmas

In this section, G will denote an open set in $\mathbf{C}^*$ having non-constant positive superharmonic functions. Thus, we may consider the generalized Dirichlet problem on G. Let φ be a function on ∂G. If the generalized solution on G with boundary values φ exists we denote it by H_G^φ, or simply H_G, if φ is clear from context, and in this case, we say that φ is *resolutive* for G. A well known theorem of Brelot states that φ is resolutive if and only if it is integrable with respect to harmonic measure, and in this case

$$H_G^\varphi(z) = \int_{\partial G} \varphi \, d\mu_z. \tag{13}$$

Lemma 1 [11, p. 23]. *Let φ be resolutive for G. Let V be an open set and define*

$$\psi = \begin{cases} \varphi & \text{on } \partial G \cap V, \\ H_G^\varphi & \text{on } \partial V \cap G. \end{cases} \tag{14}$$

Then ψ is resolutive for $V \cap G$ and

$$H^{\psi}_{V\cap G} = H^{\varphi}_{G} | V \cap G. \tag{15}$$

We recall that a boundary point $\zeta \in \partial G$ is called *regular* if

$$\lim_{z \to \zeta} H^{\varphi}_{G}(z) = \varphi(\zeta) \tag{16}$$

for each φ continuous on ∂G.

Lemma 2. *Suppose $\zeta \in \partial G$ is a regular point and φ is resolutive and continuous at ζ. Then* (16) *holds if and only if H^{φ}_{G} is bounded near ζ.*

Proof: Suppose H^{φ}_{G} is bounded in some neighbourhood V of ζ. Let ψ be given by (14). Since regularity is a local property, ζ is also regular for $V \cap G$, and since ψ is bounded and continuons at ζ, it is well-known that

$$\lim_{z \to \zeta} H^{\psi}_{V\cap G}(z) = \psi(\zeta).$$

Thus, (16) follows from Lemma 1.

The converse is trivial.

We shall say that a point $\zeta \in \partial G$ is *completely regular* if (16) holds for each resolutive function φ which is continuous at ζ.

Lemma 3. *Suppose ∂G is a curve at ζ. Then ζ is a completely regular point.*

Proof: Suppose φ is resolutive and continuous at ζ. Since ∂G is a curve at ζ, there is some neighbourhood W of ζ such that $W \cap G$ is homeomorphic to a disc. To show (16) we may assume that $\varphi \geqslant 0$, for if not we may write $\varphi = \varphi^{+} - \varphi^{-}$ and consider φ^{+} and φ^{-} separately.

Since H^{φ}_{G} is non-negative, it follows that $H^{\varphi}_{G}|W \cap G$ has finite asymptotic values at a dense set of points on ∂G near ζ (see [12, Th IV. 14]). Thus, we may construct a neighbourhood V of ζ such that H^{φ}_{G} is bounded on $G \cap \partial V$. Since φ is continuous at ζ, we may also assume that φ is bounded on $\partial G \cap V$. Thus, from Lemma 1, it follows that H^{φ}_{G} is bounbed in $G \cap V$ and from Lemma 2, we have (16). This completes the proof of Lemma 3.

If v is superharmonic on G, then by the Riesz theorem, we may associate a unique non-negative Borel measure ν to v which we call the Riesz measure for v. Without loss of generality, we shall assume that $\infty \notin G$.

Lemma 4. *Suppose v is superharmonic on G and continuous at some point $z_0 \in G$. Let ν be the Riesz measure for v. Then, the logarithmic potential*

$$P(z, A) = \int_{A} \log \frac{1}{|\zeta - z|} \, d\nu(\zeta)$$

is continuous at z_0, where A is any Borel set whose closure is compact in G.

Proof: It is enough to suppose $\overline{A}$ in a sufficiently small open neigbourhood U of z_0 with compact closure in G. Then, by the Riesz theorem, $P(z, U)$ is continuous at z_0 and finite valued, for $z \in U$. Since, on each component of G, $P(z, A)$ is either superharmonic or identically infinite, it follows that $P(z, A)$ is superharmonic for all $z \in G$, and finite valued since it is dominated by $P(z, U)$. In particular, $P(z, A)$ (and $P(z, U \setminus A)$) is lower semicontinuous. Thus, we have only to show upper semicontinuity at z_0.

$$\overline{\lim_{z \to z_0}} P(z, A) = \overline{\lim_{z \to z_0}} [P(z, U) - P(z, U \setminus A)] =$$
$$= P(z_0, U) - \underline{\lim_{z \to z_0}} P(z, U \setminus A) \leqslant$$
$$\leqslant P(z_0, U) - P(z_0, U \setminus A) = P(z_0, A).$$

Lemma 5. *Let $D = D_R$ for some $0 < R \leqslant +\infty$, and suppose E is (relatively) closed in D and $D^* \setminus E$ is connected and locally connected. Suppose v is superharmonic on E, continuous on E, and harmonic on E^0. Then for each constant $\varepsilon > 0$, there exists a function h, harmonic on E such that*

$$|h(z) - v(z)| < \varepsilon, \ z \in E.$$

Remark. Recently, Gauthier, Hengartner and Labreche [13] have obtained a more general result in which it is not necessary to assume that v is superharmonic on E (see also [14], [15]). Since their proof is different and not yet available, we present our own.

Proof: Since $D^* \setminus E$ is connected and locally connected, we may construct an exhaustion of D, $D_j \nearrow D$, such that $D^* \setminus (E \cup \overline{D}_j)$ is also connected and locally connected (see [16, Lemma 3]).

By assumption, there is an open set $U \supset E$ such that v is superharmonic on $\overline{U} \cap D$. Set $D_{-1} = D_0 = \varnothing$ and

$$\sigma_j = (D_j \setminus D_{j-1}) \cap U.$$

Since the Riesz measure for v has no mass on E^0, $P(z, \sigma_j)$ is harmonic on $E^0 \cup \overline{D}_{j-2}$. By Lemma 4, $P(z, \sigma_j)$ is continuous on E. Suppose first that $R < \infty$. Then $P(z, \sigma_j)$ is also continuous on $\overline{E} \cup \overline{D}_{j-2}$ and harmonic on the interior of this same set. By the theorem of Keldysh [17] (see also [18]), there is a function h_j, harmonic on $\overline{E} \cup \overline{D}_{j-2}$ with

$$|h_j(z) - P(z, \sigma_j)| < \varepsilon 2^{-j}, \ z \in \overline{E} \cup \overline{D}_{j-2}. \tag{17}$$

For each j, there is a function v_j, harmonic on $U \cap D_j$ such that for $z \in U \cap D_j$,

$$v(z) = P(z, U \cap D_j) + v_j(z) =$$
$$= \sum_{k=1}^{j} P(z, \sigma_k) + v_j(z). \tag{18}$$

Now set

$$h(z) = v(z) + \sum_{k=1}^{\infty} \{h_k(z) - P(z, \sigma_k)\}, \tag{19}$$

which by (17) converges on a neighbourhood of E. From (18), we can write

$$h(z)=\sum_{k=1}^{j} h_k(z)+v_j(z)+\sum_{k=j+1}^{\infty}[h_k(z)-P(z,\sigma_k)],$$

from which it is clear that h is harmonic on E. From (19) and (17), we have

$$|h(z)-v(z)|<\varepsilon,\ z\in E, \tag{20}$$

and the proof is complete in case $R<\infty$.

If $R=\infty$, we choose a point $z_0\in \mathbf{C}\setminus \overline{E}$ and merely replace $P(z,\sigma_j)$ by $P(z,\sigma_j)+\nu(\sigma_j)\log|z-z_0|$ in (17) and (19). Let $\Delta(z_0)$ be a disc centered at z_0 such that $\overline{\Delta}\cap E=\varnothing$. Then, instead of (17) we have

$$|h_j(z)-P(z,\sigma_j)-\nu(\sigma_j)\log|z-z_0||<\varepsilon\cdot 2^{-j}$$

for $z\in(\overline{E}\cap\overline{D}_{j-2})\setminus\Delta$. Again h will be harmonic on E and satisfy (20). This completes the proof of Lemma 5.

Lemma 6 [19, 20]. *Let E be a set of uniform approximation in a domain D and let ω be a continuous, bounded, zero-free function on E, holomorphic on E^0. Then E is a set of approximation with speed $|\omega|$.*

To prove the Lemma it is enough to construct a function φ, holomorphic on D and satisfying

$$0<|\varphi|<|\omega| \text{ on } E, \tag{21}$$

for then, the $|\omega|$—approximation of the function f follows from the uniform approximation of f/φ on E.

We may assume that $|\omega|<1$ on E. There exists g, holomorphic on D, such that

$$|2/\omega-g|<1 \text{ on } E,$$

and hence

$$|1/g|<|\omega| \text{ on } E. \tag{22}$$

Let $(z_j)_1^N$ $(N\leqslant\infty)$ be the zeros of g, $z_j\overline{\in}E$, taken according to their multiplicities. Now construct a function b, holomorphic in D, $0<|b|<1$ on E and $b(z_j)=0$ for each j. Then from (22) it follows that $\varphi=b/g$ satisfies (21).

To construct b, consider a sequence $(E_j)_1^N$ of sets of uniform approximation in D. $z_j\overline{\in}E_j$ and $E_j\supset E$ for each j. If $N<\infty$, we put $E_j=E$. If $N=\infty$, we assume, that

$$\bigcup_{j=1}^{\infty} E_j=D.$$

Since $E_j\cup\{z_j\}$ is a set of uniform approximation in D, for each j there exists a function b_j, holomorphic in D, $b_j(z_j)=0$, such that

$$|b_j-1|<2^{-j} \text{ on } E_j.$$

The function

$$b=4^{-1}\prod_{j=1}^{N} b_j$$

satisfies to all our conditions.

Notice, that the existence of b may be easily proved by using results on simultaneons approximation and interpolation (see [21]).

Lemma 7. [22, p. 18]. *If φ is a convex increasing function of u and $u=u(z)$ is subharmonic, then $\varphi(u(z))$ is also subharmonic.*

Lemma 8. *Let α be a positive function decreasing to zero on $[0, R)$. Then, there exist functions α_- and α_+ also decreasing to zero, with $0<\alpha_-<\alpha<R\alpha_+$, and such that $-\log\alpha_-$ and $-\log\alpha_+$ are both log-convex.*

Proof: Set $P=-\log\alpha$. Choose any sequence of points r_j increasing to R. It is easy to construct a piecewise linear convex function P_- with nodes at r_j and with $P_->P$. Then we put $\alpha_-=\exp(-P_-)$.

To construct α_+, we may assume $R=1$. We set

$$A=\{z=x+iy : 0\leqslant x<1,\ P(x)\leqslant y\}$$

and denote by $\hat{A}$ the closed convex hull of A. Now set for $0\leqslant x<1$,

$$P_+(x)=\min\{y : x+iy\in\hat{A}\}.$$

Clearly, P_+ is convex. It is easy to see that $\hat{A}$ does not meet the line $x=1$, and so P_+ tends to ∞. Finally, we put $\alpha_+=\exp(-P_+)$.

Lemma 9. *Let E^0 be an open subset of the unit disc D_1 such that $\mu_z(\partial E^0\setminus D_1)=0$, for each $z\in E^0$. Then, there exists a log-superharmonic asymptotic speed α on D_1, such that* (2) *is satisfied for* $\log\alpha$.

Proof: By Lemma 8, it is sufficient to construct a positive function $\alpha(t)$ decreasing to zero as $t\uparrow 1$, and $\log\alpha$ satisfying (2).

Set, for $s<t<1$

$$W_s(t)=\sup_{\substack{|z|<s\\ z\in E^0}}\omega(z, t),$$

and $W_s(t)=0$ if $E^0\cap D_s=\varnothing$. By Dini's theorem, $W_s(t)\searrow 0$ as $t\nearrow 1$.

Now set $t_0=0$ and choose t_1 so close to 1 that $W_0(t_1)<2^{-1}$. Having chosen $t_0, t_1,\cdots, t_{j-1}$ such that

$$iW_{t_{i-1}}(t_i)<2^{-i},\quad i=0, 1,\cdots, j-1,$$

we may again, by Dini's theorem, choose t_j so near to 1 that

$$jW_{t_{j-1}}(t_j)<2^{-j}. \tag{23}$$

Thus, by induction, we have a sequence $t_j\nearrow 1$ and satisfying (23).

Now define

$$\alpha(t)=e^{-j},\ t_j\leqslant t<t_{j+1}.$$

Then, setting $A_j=(D_{t_{j+1}}\setminus D_{t_j})\cap E^0$, we have

$$\int_{\partial E^0}\log\alpha d\mu_z=\int_{D_{t_{i+1}}\cap\partial E^0}+\sum_{j=i+1}^{\infty}\int_{A_j}.$$

If $z\in D_{t_i}$, then for $j\geqslant i+1$,

$$\int\limits_{\lambda_j} > -j\omega(z, t_j) \geqslant -jW_{t_{j-1}}(t_j) > -2^{-j}.$$

Thus $\log \alpha$ satisfies (2) and the proof is complete.

Lemma 10. *If $\alpha(t)$ is decreasing for $0 \leqslant t < R$, and if $\beta(z, t)$ is also decreasing and uniformly bounded for $z \in E^0$, $\beta(z, R-0) \equiv 0$, then for each $0 < r < R$,*

$$\sup_{z \in D_r \cap E^0} \int\limits_0^R \log \alpha(t)\, d\beta(z, t) < +\infty \tag{24}$$

if and only if

$$\inf_{z \in D_r \cap E^0} \int\limits_0^R \beta(z, t)\, d(\log \alpha(t)) > -\infty. \tag{25}$$

Proof:

$$\int\limits_0^R \log \alpha(t)\, d\beta(z, t) = \lim_{T \nearrow R} \int\limits_0^T \log \alpha(t)\, d\beta(z, t) =$$

$$= (\log \alpha(t))\, \beta(z, t) \Big|_0^R - \lim_{T \nearrow R} \int\limits_0^T \beta(z, t)\, d(\log \alpha(t)).$$

The Lemma follows, since $(\log \alpha(t))\, \beta(z, t)$ is upper bounded, independently of z.

IV. Proofs

1. Proof of Theorem 1A. Suppose E is a set of α-approximation in D. Then, for some $\zeta_0 \in D \setminus E$ and for the function $f(z) = (z-\zeta_0)^{-1}$ and $\varepsilon > 0$, consider a function $g = g_\varepsilon$ satisfying (1). Now choose ε_1 and ε_2 positive such that $\varepsilon_1 + \varepsilon_2 < 1$ and $g_{\varepsilon_1} \neq g_{\varepsilon_2}$. Then $\varphi = g_{\varepsilon_1} - g_{\varepsilon_2}$ is holomorphic in D and $\varphi \not\equiv 0$. From (1) we have

$$|\varphi| < \alpha \text{ on } E. \tag{26}$$

If $E^0 \neq \varnothing$, then it follows from (26), that

$$\overline{\lim_{\substack{z \to \zeta \\ z \in E^0}}} \log |\varphi(z)| \leqslant \log \bar{\alpha}(\zeta),\ \zeta \in \partial E^0. \tag{27}$$

For each n, the function $\beta_n = n \vee \log \bar{\alpha}$ is bounded semicontinuous on ∂E^0 and therefore resolutive. Since $\log|\varphi|$ is subharmonic on E^0 and by (27) has boundary values $\leqslant \beta_n$, then we have, using (13) and the definition of the generalized solution to the Dirichlet problem, that

$$\log|\varphi(z)| \leqslant \int\limits_{\partial E^0} \beta_n\, d\mu_z,\ z \in E^0.$$

Letting here $n \to -\infty$, we have

$$\log|\varphi(z)| \leqslant v(z) = \int\limits_{\partial E^0} \log \bar{\alpha}\, d\mu_z,\ z \in E^0. \tag{28}$$

The integral converges since $\log \bar{\alpha}$ is upper bounded and since $\varphi \neq 0$.

It follows from (28) that there is an exhaustion $K_j \nearrow D$ and sequence $M_j > -\infty$, such that for each j

$$v(z) > M_j, \quad z \in E^0 \cap \partial K_j. \tag{29}$$

From (29) and Lemma 1, it follows that v is bounded on $K_j \cap E^0$ for each j. This completes the proof of Theorem 1A.

2. Proof of Theorem 1B. Using Tietze's Extension Theorem, we may be sure, that the set $\bar{E}$ is a set of uniform approximation in D. Using for $\tilde{E}$ Lemma 6 and again Tietze's Theorem, we see, that to prove Theorem 1B, it is enough to construct a function ω, continuous on $\tilde{E}$ and holomorphic on E^0, such that

$$0 < |\omega| < \alpha \text{ on } \bar{E}. \tag{30}$$

Let $K_1, K_2, \cdots, K_n$ be the complementary components of D, which meet with the closure of $\bar{E}$. Suppose that for each $j = 1, 2, \cdots, n$ we can construct a neighbourhood V_j of K_j and a function ω_j holomorphic on $D_j = \mathbf{C}^* \setminus K_j$ and satisfying

$$0 < |\omega_j| < \alpha \text{ on } \bar{E}_j = \tilde{E} \cap \bar{V}_j. \tag{31}$$

Then, for some suitably small constant $\lambda > 0$, we have that

$$\omega = \lambda \omega_1 \omega_2 \cdots \omega_n$$

is holomorphic on D and satisfies (30) on $\tilde{E}$. Thus we have only to construct each ω_j, $j = 1, 2, \cdots, n$.

Fix j then. We shall consider two cases depending on whether K_j meets with $\overline{D \setminus E}$ or not. Suppose first that

$$K_j \cap (\overline{D \setminus E}) = \varnothing.$$

Then, there is an open Jordan neighbourhood V_j of K_j such that

$$D_j \cap \bar{V}_j = E^0 \cap \bar{V}_j = \bar{E} \cap \bar{V}_j = \tilde{E}_j \subset E^0.$$

By the hypothesis of the Theorem, K_j cannot be a singleton. Putting

$$v(z) = \int_{\partial E^0} \log \alpha \, d\mu_z, \quad z \in E^0,$$

we may write

$$v|\tilde{E} = v_1 - v_2,$$

where v_1 is harmonic in D_j and v_2 is harmonic in V_j. Thus, v_2 is bounded on K_j and so for some constant λ,

$$v_1 - \lambda < v \text{ on } \tilde{E}_j.$$

Since v is the greatest harmonic minorant of $\log\alpha$ on E^0, we also have

$$v_1-\lambda<\log\alpha \text{ on } \tilde{E}_j.$$

Now write $h_j=v_1-\lambda$ and let $\bar{h}_j$ be a harmonic conjugate of h_j. Then $\omega_j=\exp(h_j+i\bar{h}_j)$ is holomorphic on D_j and satisfies (31).

Consider now the second case: that is,

$$K_j\cap(\overline{D\setminus E})\neq\varnothing.$$

In this case, there exists a neighbourhood V_j of K_j such that $\tilde{E}\cap\bar{V}_j=\tilde{E}_j$ is a set of uniform approximation in D_j.

By mapping D_j conformally to D_R for some $0<R\leqslant+\infty$, it follows from Lemma 5, that there is a function h_j, harmonic on $\tilde{E}_j$ with

$$|h_j(z)-v(z)|<1,\ z\in\tilde{E}_j, \tag{32}$$

where

$$v(z)=\begin{cases}\displaystyle\int_{\partial E^0}\log\alpha\,d\mu_z, & z\in E^0,\\ \log\alpha, & z\in U\setminus E^0,\end{cases} \tag{33}$$

and U is the domain of definition of α. By a result in [16] (see also [15]), there is a function u_j, harmonic on D_j, with

$$|u_j-h_j|<1,\quad z\in\tilde{E}_j. \tag{34}$$

Now let $\tilde{u}_j$ be a harmonic conjugate of u_j and set

$$\omega_j=\exp(u_j-2+i\bar{u}_j).$$

Then, from (32), (33), and (34), it follows that ω_j satisfies (31).

This completes the proof of Theorem 1B.

3. Proof of Theorem 1. Theorem 1 is an immediate consequence of Theorem 1A and Theorem 1B.

4. Proof of Theorem 2. Suppose the hypotheses of Theorem 2 are satisfied. By (3), $\log\alpha$ is resolutive. Thus, (2) for $\log\alpha$ follows from Lemmas 2 and 3. Hence Theorem 2 follows from Theorem 1.

5. Proof of Theorem 3. Let E be a set of α-asymptotic approximation. Then $\log\alpha=-\infty$ on $\partial E^0\setminus D$ and so by Theorem 1A, $\omega(z,R)=0$ for each $z\in E^0$. If α is decreasing, then we have the identity

$$\int_0^R\log\alpha(t)\,d\omega(z,t)=-\int_{\partial E^0}\log\alpha\,d\mu_z$$

for $z\in E^0$, and (4) follows from Theorem 1A.

To show the converse, we remark first, that it follows from the conditions on α, that α is decreasing on some interval $[t_0,R)$. We may assume $t_0=0$ and then $\alpha(|z|)$ is a log-superharmonic speed on D_R, satisfying (4). By the above mentioned identity, α satisfies the conditions of Theorem 1B for $D=D_R$. Since E is a set of uniform approximation in D_R, it follows that $C^*\setminus\tilde{E}$ is connected. Then from Theorem 1B it follows that E is a set of α-approximation in D_R.

6. Proof of Corollary 1. Suppose E is a set of α-approximation for some asymptotic speed α. Then $\log\alpha=-\infty$ on $\partial E^0\setminus D$ and by Theorem 1A, for $\log\alpha$ condition (2) is satisfied. This implies that $\mu_z(\partial E^0\setminus D))=0$ for $z\in E^0$.

Conversely, let $\mu_z(\partial E^0\setminus D)=0$ for $z\in E^0$. To prove, that E is a set of asymptotic approximation, it is enough by Theorem 1B, to construct an asymptotic log-superharmonic speed α on some neighbourhood of $\bar{E}$, such that for $\log\alpha$ condition (2) is satisfied.

Denote by $K_1,\cdots, K_n$ the complementary components of D, which meet with the closure of $\bar{E}$. We may assume, that no K_j can be a singleton. By conformally mapping $C^*\setminus K_j$ onto the unit disc D_1, it follows from Lemma 9, that there is a log-superharmonic asymptotic speed α_j on $C^*\setminus K_j$, such that for $\log\alpha_j$, condition (2) is satisfied. Then obviously the speed

$$\alpha=\alpha_1\wedge\alpha_2\wedge\cdots\wedge\alpha_n$$

is asymptotic and log-superharmonic on D, such that condition (2) for $\log\alpha$ satisfied.

7. Proof of Corollary 2. Suppose first that α is a log-superharmonic speed on E. Fix K compact in D and let K' be a compact set associated to K as in Conidtion C. Then, for $z\in K\cap E^0$, $\mu_z(\partial E^0\setminus\setminus K')=0$ and so

$$\int_{\partial E^0}\log\alpha\, d\mu_z=\int_{K'\cap\partial E^0}\log\alpha\, d\mu_z\geqslant\inf_{K'\cap\partial E^0}\log\alpha>-\infty.$$

Thus, (2) holds for each log-superharmonic speed, and so by Theorem 1, E is a set of α-approximation for each log-superharmonic speed α.

Now if α_0 is an arbitrary speed, then by Lemma 8 and the technique used in Corollary 1, we may construct a log-superharmonic speed α on E with $0<\alpha\leqslant\alpha_0$. Since we have already shown that E is a set of α-approximation, it is, all the more, a set of α_0-approximation.

Suppose, conversely, that E fails to satisfy Condition C. Then, for some $K\subset D$ and some sequence $K_j\nearrow D$, there are components E_j^0 of E^0 which meet both K and $D\setminus K_j$. Choose a sequence $z_j\in E_j^0\cap K$. It is easy to construct a speed α which decreases to zero so rapidly that

$$\int_{\partial E_j^0}\log\alpha\, d\mu_{z_j}<-j,\ j=1,2,\cdots.$$

Hence by Theorem 1A, E is not a set of α-approximation, and so, not a set of Carleman approximation. This completes the proof of Corollary 2.

8. Proof of Corollary 3. From Lemma 10 we have

$$\int_0^R \tilde{\theta}(t)\, d(\log \alpha(t)) > -\infty. \tag{35}$$

For $z \in D_r \cap E^0$, $r < t$, it is clear that $\omega(z; t)$ is less than the harmonic measure of $E^0 \cap \partial D_t$ with respect to the open set $E^0 \cap D_t$. This, in turn, is less than the harmonic measure of the same set with respect to D_t. This last quantity is dominated by $\lambda(r)\theta(t)$, for some constant $\lambda(r)$. Thus, $\omega(z, t) \leqslant \lambda(r)\, \tilde{\theta}(t)$. Hence,

$$\int_0^R \omega(z, t)\, d(\log \alpha) \geqslant \int_0^r \omega(z, t)\, d(\log \alpha) + \lambda(r) \int_r^R \tilde{\theta}(t)\, d(\log \alpha).$$

Hence from (35) and Lemma 10, (5) is satisfied, and so Corollary 3 follows from Theorem 3.

9. Proof of Corollary 4. First, we shall show that we may assume that $-\log \alpha(t)$ is log-convex. Set $\gamma = (\log c)^{-1}$ for $c > 1$ and denote

$$\log \alpha_c(r) = \gamma \int_1^{cr} \log \alpha(t) \frac{dt}{t}.$$

Obviously, $-\log \alpha_c(r)$ is a convex function of $\log r$. Assuming $\alpha < 1$, we have for $r > 1$:

$$\log \alpha_c(r) < \gamma \int_r^{cr} \log \alpha(t) \frac{dt}{t} < \log \alpha(r).$$

Integrating by parts, and since $\log \alpha_c$ is upper-bounded, we have that (6) holds for α_c and k', if and only if

$$\int_1^{\infty} \exp\left[-\pi \int_1^{k't} \frac{d\rho}{\rho\theta(\rho)}\right] \log \alpha(ct) \frac{dt}{t} > -\infty.$$

Since

$$\frac{dt}{t} \leqslant 2\pi \frac{dt}{t\theta(k'\, t)},$$

the finiteness of the last integral follows trom (6) after the substitution $ct \to t$, if we choose $c > 1$ such that $k' = ck < 1$. Thus, in proving Corollary 4, we may assume, that $-\log \alpha$ is an increasing log-convex function.

We may also assume that E is not a set of Carleman approximation, for otherwise E is trivially a set of α-approximation. Thus, by Corollary 2, there is an r_0, $0 < r_0 < \infty$, such that for each $t > r_0$ some component of E^0 meets both ∂D_{r_0} and ∂D_t.

Fix $r > r_0$ and suppose $z \in D_r \cap E^0$. Let D be the component of $E^0 \cup D_r$ which contains $z=0$. Then D is unbounded. Let Δ_t denote the component of $D \cap D_t$ which contains $z=0$ and let $t\theta(t)$ be its linear measure. Following Tsuji, we define $\theta^*(t)$ as follows. If $\partial D_t \not\subset E^0$, then we put $\theta^*(t)=\theta(t)$ and if $\partial D_t \subset E^0$, we put $\theta^*(t)=\infty$. Since E is a set of uniform approximation, and we may assume that $E \neq \mathbf{C}$, there is some t_0 such that $\theta^*(t)=\theta(t)$ for $t > t_0$. Let $u_t(z)$ be the harmonic measure of $\partial\Delta_t \cap \partial D_t$ with respect to Δ_t. Then, Tsuji has shown [12, p. 116], that if $z \in D_r \cap E^0$ and $r < kt/2$, $0 < k < 1$, then

$$u_t(z) \leqslant \frac{9}{\sqrt{1-k}} \exp\left[-\pi \int_{2r}^{kt} \frac{d\rho}{\rho\,\theta^*(\rho)}\right].$$

Since $\theta^*(t)=\theta(t)$ for $t > t_0$ and we may choose $2r > kt_0$, we have

$$u_t(z) \leqslant \lambda_r \exp\left[-\pi \int_{t_0}^{kt} \frac{d\rho}{\rho\theta(\rho)}\right] \tag{36}$$

for $t > (2/k)\,r$, where λ_r is some constant. Since $\omega(z, t) = \mu_z(\partial E^0 \setminus D_t)$, it is clear that $\omega(z, t)$ is dominated by the harmonic measure of $\partial D \setminus D_t$ with respect to the domain D. This in turn is dominated by $u_t(z)$. Thus, from (36), we have

$$\omega(z, t) \leqslant \lambda_r \exp\left[-\pi \int_{t_0}^{kt} \frac{d\rho}{\rho\theta(\rho)}\right]. \tag{37}$$

From (6) and Lemma 10, we have that

$$\int_1^{\infty} \exp\left[-\pi \int_{t_0}^{kt} \frac{d\rho}{\rho\theta(\rho)}\right] d(\log \alpha) > -\infty. \tag{38}$$

Hence from (38), (37) and Lemma 10, we have (4) and so Corollary 4 follows from Theorem 3.

10. P r o o f o f C o r o l l a r y 5. Suppose α satisfies (7). There exists a set E_0 of uniform approximation, such that $E \subset E_0$ and for E_0, $|\theta(t)-2\pi| < t^{-1}$, $t \geqslant 1$.

Then conditions (6) and (7) are equivalent, and it follows from Corollary 4, that E_0 is a set of α-approximation. Then so is the set E.

Now let E is a set of α-approximation by entire functions for some decreasing asymptotic speed $\alpha(|z|)$. Then by Theorem 3, α satisfies (4). Using Lemma 10 (for $\beta = \omega$), we have from (25):

$$\inf_{z \in D_r \cap E^0} \int_0^\infty \omega(z, t)\, d(\log \alpha(t)) > -\infty \tag{39}$$

for each $0 < r < \infty$.

Suppose for the set E, that

$$\omega(z, t) > \frac{\lambda_r}{\sqrt{t}} \tag{40}$$

for $z \in D_r \cap E^0$ and $t > cr$, where $0 < r < \infty$, c and λ_r are some constants. It follows from (39) and (40) that

$$\int^\infty \frac{1}{\sqrt{t}}\, d(\log \alpha(t)) > -\infty.$$

Hence α satisfies (7).

It is easy to construct a set E of uniform approximation, satisfying (40). For instance, E is such a set, if

$$E \supset E_0 = \{z : z = x + iy,\ y \leqslant x^2\}.$$

11. Proof of Corollary 6. We shall estimate $\omega(z, t)$ using the Ahlfors distortion theorem.

Fix $z \in D_r \cap E^0$, and suppose $r < t < R$, and t is very close to R. We map E^0 to a strip S_E by the log-function. We then map the strip S_E to the strip $S_\pi = \{X + iY : |Y| < \pi/2\}$ so that $\partial E^0 \setminus D_R$ corresponds to $X = +\infty$. Consider the image L_t of $E^0 \cap \partial D_t$ in S_π and denote by $u_1(t)$ the minimum value of X for points on L_t. Then, by the Ahlfors distortion theorem (see [23, p. 101])

$$u_1(t) > \lambda + \pi \int_{t_1}^{t} \frac{d\rho}{\rho\theta(\rho)}, \tag{41}$$

for some constant λ, provided

$$\int_{t_1}^{R} \frac{dt}{t\theta(t)} = \infty. \tag{42}$$

This last formula is obvious if $R = +\infty$.

From (41), we have that $\omega(z, t)$ is dominated by the harmonic measure of $\partial S_\pi \cap \{X > u_1(t)\}$. This we estimate by mapping S_π to a half-plane H so that $\omega(z, t)$ is dominated by the harmonic measure of $H \cap \{|W| > \exp[u_1(t)]\}$. Since we may assume that the image of z is of module less than $\exp[u_1(t)]$, this in turn is bounded by the harmonic measure of the half-circle of radius $\exp[u_1(t)]$ with respect to the half-disc. Thus, (see [23, p. 49]), for $z \in D_r \cap E^0$

$$\omega(z, t) \leqslant \lambda_r \exp[-u_1(t)]$$

for some constant λ_r.

Hence, from (41), we have

$$\omega(z, t) \leqslant \lambda_r \exp\left[-\pi \int_{t_1}^{t} \frac{dt}{t\theta(t)}\right].$$

Corollary 6 then follows from Lemma 10 by the same reasoning as was used in the proofs of Corollaries 3, 4, and 5.

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Ն. Հ. ԱՌԱՔԵԼՅԱՆ. Պ. Մ. ԳՈԹՅԵ. Հոլոմորֆ ֆունկցիաներով շոշափումային մոտավորության մասին *(ամփոփում)*

Դիցուք D-ն տիրույթ է C^*-ում և E-ն D-ի հարաբերական փակ, սեփական ենթաբազմություն է։ Թող $\alpha > 0$ ֆունկցիան լինի անընդհատ E-ի վրա և սահմանափակ։ Անվանենք E-ն α-մոտավորության բազմություն (D-ում), եթե ամեն մի $\varepsilon > 0$ թվի և կամայական f ֆունկցիայի համար, որն անընդհատ է E-ի վրա և հոլոմորֆ E^c-ում, գոյություն ունի D-ում հոլոմորֆ այնպիսի g ֆունկցիա, որ

$$|f(z) - g(z)| < \varepsilon\alpha(z), \quad z \in E.$$

$\alpha = 1$ դեպքում E-ն կանվանենք հավասարաչափ մոտավորության բազմություն: Վերջիններիս նկարագրությունը տրված է [1]-ում։

Ներկա աշխատանքում հետազոտվում է α-մոտավորության բազմությունների նկարագրման խնդիրը։

Հիմնական թեորեմ 1-ում դիտարկված է այն դեպքը, երբ D-ն վերջավոր կապանի է, իսկ E-ն D-ում հավասարաչափ մոտավորության այնպիսի բազմություն է, որ $C^* \setminus E$-ն չունի մեկուսացված կետեր։

α-ից պահանջվում է, որ նրա նեղացումը E^0-ի վրա ունենա անընդհատ շարունակություն $\overline{(E^0)}$-ի վրա, ընդ որում $\log\alpha$-ն լինի սուպերհարմոնիկ $(\overline{E^0}) \cap D$-ի վրա։ Նշված պայմանների դեպքում, որպեսզի E-ն լինի α-մոտավորության բազմություն, անհրաժեշտ է ու բավարար, որ

$$\inf_{z \in K \cap E^0} \int_{\partial E^0} \log\alpha \, d\mu_z > -\infty$$

ամեն մի $K \subset D$ կոմպակտի համար, որտեղ μ_z-ը ∂E^0-ի հարմոնիկ չափն է E^0-ի նկատմամբ։

Նշված արդյունքը միավորում է չափական և երկրաչափական բնույթի մի շարք հայտնի հայտանիշներ, որոնք վերաբերվում են ամբողջ և անալիտիկ ֆունկցիաներով ասիմպտոտական մոտավորություններին (երբ $\alpha(z) \to 0$, $z \to \partial D$ դեպքում) և բերում է այդ բնույթի նոր հայտանիշների (տես թեորեմներ 2—3-ը և հետևանքներ 1—6-ը)։

Н. У. Аракелян, П. М. Готье. О касательном приближении голоморфными функциями (резюме).

Пусть D — область в C^* и E — относительно замкнутое собственное подмножество D. Пусть $\alpha > 0$ — непрерывная на E и ограниченная функция. Назовем E *множеством*

α- приближения (в D), если для каждого $\varepsilon > 0$ и произвольной функции f, непрерывной на E и голоморфной в E^0, существует такая голоморфная в D функция g, что

$$|f(z) - g(z)| < \varepsilon\alpha(z), \quad z \in E.$$

В случае $\alpha = 1$ будем называть E *множеством равномерного приближения*. Описание таких E дано в [1].

В настоящей работе исследуется задача описания множеств α-приближения. В основной теореме 1 рассматривается случай, когда D — конечносвязная область, а E — множество равномерного приближения в D, такое, что $C^*\setminus E$ не имеет изолированных точек. Относительно α предполагается, что ее сужение на E^0 допускает непрерывное продолжение на $\overline{(E^0)}$, причем $\log \alpha$ — супергармоническая на $\overline{(E^0)} \cap D$. Тогда для того, чтобы E было множеством α-приближения, необходимо и достаточно, чтобы

$$\inf_{z \in K \cap E^0} \int_{\partial E^0} \log \alpha \, d\mu_z > -\infty$$

для каждого компакта $K \subset D$, где μ_z — гармоническая мера ∂E^0 относительно E^0.

Отмеченный результат объединяет ряд известных метрических и геометрических критериев об асимптотическом приближении целыми и аналитическими функциями (когда $\alpha(z) \to 0$ при $z \to \partial D$) и приводит к новым критериям такого характера (см теоремы 2—3 и следствия 1—6).

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