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SOME REMARKS ON STOCHASTIC TRANSFORMATIONS

This paper discusses some basic aspects of the first author's apprcach to stochasti c differential equations—the basic calculus, convergence, and the choice of operator decomposition into deterministic and random parts.

I. Adomian's iterative methods for solution of stochastic operator equations can be viewed as determining an inverse operator in series form. The solution of the linear equation $\mathrm{L}y=x$, for example, with L *a stochastic operator* and x a stochastic process defined on appropriate probability spaces, is written $y=\mathrm{L}^{-1}x$ where L^{-1} is given as a series if L can be given in the form $\mathrm{L}=L+R$ and where

i) L is non-random and possesses an inverse L^{-1}.
ii) the order of L is higher than that of R.

Let us consider briefly the simpler problem where x is a random variable and the operator is also a random variable. Thus consider the equation $ay=x$ where a, x are a pair of numerical valued independent random variables. The solution is the random variable $y=a^{-1}x$.

Because of the assumed independence of a and x, the expectation of y is given by $\langle y\rangle=\langle a^{-1}\rangle\langle x\rangle$. Let us inquire into the calculation of $\langle a^{-1}\rangle$ in the form of a series.

Choose a constant c, $-\infty<c<\infty$. We have

$$a^{-1}=\frac{1}{c+(a-c)}=$$

$$=\frac{1}{c}\frac{1}{1+\frac{a-c}{c}}=$$

$$=\frac{1}{c}\sum_{n=0}^{\infty}(-1)^n\left(\frac{a-c}{c}\right)^n. \tag{1}$$

This can be a useful result when we have convergence with probability one. Thus \

$$\langle a^{-1}\rangle=\frac{1}{c}\sum_{n=0}^{\infty}(-1)^n\left\langle\left(\frac{a-c}{c}\right)^n\right\rangle$$

and

$$<a^{-1}> \simeq \frac{1}{c} \sum_{n=0}^{N} (-1)^n < \left(\frac{a-c}{c}\right)^n > \tag{2}$$

so for $N < \infty$, we have an approximate expression for $<a^{-1}>$ in terms of moments of a.

Unfortunately (1) does not converge with probability one for all choices of the constant c. Such a case occurs for example if we have non-zero probabilities for both of the events $a > 0$ and $a < 0$.

When we consider linear *operators* instead of numbers, the situation is quite different and a series expansion of the mean of the inverse of a random operator proves to be fruitful in cases which are significant from the point of view of physics and other applications. Adomian has dealt with the *stochastic* (linear differential) *operator* $\mathbf{L}$ subject to the decomposition $\mathbf{L} = L + R$ and to conditions (i) and (ii).

Then, formally,

$$\mathbf{L}^{-1} = (L + R)^{-1} = [L(1 + L^{-1} R)]^{-1} = \sum_{n=0}^{\infty} (-1)^n (L^{-1} R)^n L^{-1}. \tag{3}$$

Because of requirement (ii), $L^{-1} R$ is an *integral* operator and under proper boundedness assumptions, the series given by (3) converges with probability one. As an example consider the equation $(L + \alpha)\, y = x$. We have $Ly = x - \alpha y$ or $y = L^{-1} x - L^{-1} \alpha y$ and finally

$$y = \mathbf{L}^{-1} x = \sum_{n=0}^{\infty} (-1)^n (L^{-1} \alpha)^n L^{-1} x$$

or

$$y = L^{-1} x - L^{-1} \alpha L^{-1} x + L^{-1} \alpha L^{-1} \alpha L^{-1} x - \cdots. \tag{4}$$

The first term is $L^{-1} x = \int_0^t e(t, \tau)\, x(\tau)\, d\tau$. The second term is

$$-L^{-1} \alpha L^{-1} x = -\int_0^t e(t, \tau)\, \alpha(\tau) \int_0^\tau e(\tau, \sigma)\, x(\sigma)\, d\sigma d\tau$$

or

$$-\int_0^t \int_0^\tau e(t, \tau)\, e(\tau, \gamma)\, \alpha(\tau)\, x(\gamma)\, d\gamma d\tau.$$

The third term is

$$\int_0^t \int_0^\tau \int_0^\gamma e(t, \tau)\, e(\tau, \gamma)\, e(\gamma, \sigma)\, \alpha(\tau)\, \alpha(\gamma)\, x(\sigma)\, d\gamma d\tau d\sigma$$

etc. If the Green's functions, i. e., the e's, are bounded in the interval, if the α is bounded a.s., and x is bounded a.s., they can be taken outside the integrals as bounds with the remaining n-fold integrals yielding an $n!$ in the denominator assuring convergence.

It is clear that assumption (ii) is necessary as shown by the example:

$$Ly = [a\ d/dt]\ y$$

where a is a random variable. An attempt to invert L using the decomposition

$$Ly = c\frac{dy}{dt} + (c-a)\frac{dy}{dt}$$

where c is a constant meets exactly the same difficulties as occurred in the example with numbers.

Actually the situations causing difficulty are contrary to Adomian's assumption that the coefficient of the highest order derivative is deterministic and greater than zero, so it cannot fluctuate through zero to negative values. In other words, these cases are subsumed in the equation $Ry = x$ where $\mathrm{L} \equiv R$ and $L = 0$ which can be solved by adding and subtracting $Ly = x$ to write $Ly + Ry - Ly = x$ or $Ly = x - Ry + Ly$ and finally $y = L^{-1}x - L^{-1}Ry + y$. Requirement (ii) already made this case a pathological case. If L has a deterministic part L whose order is higher than that of R, there are no difficulties and convergence is assured. The series representation is of direct use when we solve $\mathrm{L}y = x$ with L and x statistically independent.

If $\mathrm{L} = \sum_{\nu=0}^{n} a_\nu(t, \omega)\, d^\nu/dt^\nu$ then it is convenient to take

$$L = \sum_{\nu=0}^{n} \langle a_\nu(t, \omega)\rangle\, d^\nu/dt^\nu$$

and

$$R = \sum_{\nu=0}^{n-1} \alpha_\nu(t,\omega)\, d^\nu/dt^\nu$$

where

$$a_\nu = \langle a_\nu\rangle + \alpha_\nu \quad \text{for } \nu\ 1, 2, \cdots, n-1,\ a_n > 0,\ \alpha_n = 0.$$

II. A question which naturally arises is whether it is necessary or optimal that $L = \langle \mathrm{L}\rangle$. Adomian lets $\mathrm{L} = \sum_{\nu=0}^{n} a_\nu(t, \omega)\, d^\nu/dt^\nu$ and $a_\nu = \langle a_\nu\rangle + \alpha_\nu(t, \omega)$ for $\nu = 0, 1, \cdots, n-1$. Then $L = \sum_{\nu=0}^{n} \langle a_\nu\rangle d^\nu/dt^\nu$ and $R = \sum_{\nu=0}^{n-1} \alpha_\nu(t, \omega)\, d^\nu/dt^\nu$. He points out that $L = \langle \mathrm{L}\rangle$ is convenient rather than necessary but does not go into it further. Suppose then $L \neq \langle \mathrm{L}\rangle$. Then we write

$$Ly + (\mathrm{L} - L)\, y = x$$

where

$$\mathbf{L}-L=R.$$

Since $L \neq \langle \mathbf{L} \rangle$, $\langle R \rangle$ is no longer zero, an assumption made in Adomian's work. If this mean value $\langle R \rangle$ is not to be incorporated into L, it is necessary to treat it like the Ry term. Thus let $L_2 = \langle R \rangle$. Then

$$y = L^{-1} x - L^{-1} Ry - L^{-1} L_2 y.$$

The same situation arises if it is difficult to get a Green's function for the actual L operator even if $L = \langle \mathbf{L} \rangle$. Then we can split L into $L_1 + L_2$ where L_1 is easily invertible and write

$$y = L_1^{-1} x - L_1^{-1} Ry - L_1^{-1} L_2 y.$$

Both situations make each individual y_i more complicated by adding terms. Thus we get

$$y_0 = L_1^{-1} x$$

$$y_1 = - L_1^{-1} Ry_0 - L_1^{-1} L_2 y_0$$

$$y_2 = - L_1^{-1} Ry_1 - L_1^{-1} L_2 y_1$$

$$y_3 = - L_1^{-1} Ry_2 - L_1^{-1} L_2 y_2$$

$$\vdots$$

Then

$$y_1 = - L_1^{-1} RL^{-1} x - L_1^{-1} L_2 L_1^{-1} x$$

$$y_2 = - L_1^{-1} R [- L_1^{-1} RL_1^{-1} x - L_1^{-1} L_2 L_1^{-1} x] -$$

$$- L_1^{-1} L_2 [- L_1^{-1} RL_1^{-1} x - L_1^{-1} L_2 L_1^{-1} x] =$$

$$= (L_1^{-1} R)(L_1^{-1} R)(L_1^{-1} x) + (L_1^{-1} R)(L_1^{-1} L_2)(L_1^{-1} x) +$$

$$+ (L_1^{-1} L_2)(L_1^{-1} R)(L_1^{-1} x) + (L_1^{-1} L_2)(L_1^{-1} L_2)(L_1^{-1} x).$$

Thus we see the addition of the last three terms to y_2 because of trying to use a simple L_1 or equivalently because of not using $L = \langle \mathbf{L} \rangle$ and it is easy to see what happens with $y_3, y_4, \cdots$.

III. As a final remark, with $y = L^{-1} x - L^{-1} Ry$, we can clearly write $y_{n+1} = L^{-1} x - L^{-1} Ry_n$ as a method of successive approximations. Thus $y_1 = L^{-1} x - L^{-1} R y_0$, $y_2 = L^{-1} x - L^{-1} R y_1$, $y_3 = L^{-1} x - L^{-1} Ry_2$, etc.

Then each y_i is the complete answer to that degree of approximation. For example, the second term of y_3 integrates the result of operating with R on the entire y_2, i. e., three terms. Although the final result is the same, Adomian's procedure (iterative method) is to think of y as a decomposition into $y_0 + y_1 + \cdots$ and identify y_0 as $L^{-1} x$. If we call $\Phi_n = y_0 + y_1 +, \cdots, y_{n-1}$ as the approximation to n terms each y_i is the result of operating on only the preceding term.

IV. The calculus is that of L_p spaces. The integral operator H in $y = Hx$ is an a.s. bounded linear stochastic operator from an abstract set χ into itself, mapping $x(t, \omega)$ to $y(t, \omega)$ such that $Ly(t, \omega) = x(t, \omega)$ a. s. The a. s. boundedness of H follows from the conditions:

(i) $F(t, \omega) = \int_0^t l(t, \tau) x(\tau, \omega) d\tau$ is bounded a. s.

(ii) $\alpha_\nu (t, \omega) = a_\nu (t, \omega) - \langle a_\nu (t, \omega) \rangle$ are bounded a. s. for all ν from 0 to $n-1$.

The $\langle a_\nu \rangle$ are continuous on T; the derivatives of α_ν are bounded a.s. to appropriate orders; and $l(t, \tau)$ and its k-th derivatives for $0 \leqslant k \leqslant n-1$ are jointly continuous in t, τ over TxT. The set χ consists of all real stochastic processes on T. If $\chi_1 \subset \chi$ consists of all real stochastic processes $x(t, \omega)$ with $\omega \varepsilon \Omega$ in the probability space (Ω, B, μ), it follows that $H: \chi_1 \to \chi_1$ with $Hx(t, \omega) = y(t, \omega)$ a.s.

The expectation $\langle x(t, \omega) \rangle$ of $x(t, \omega)$ on χ_1 is given as the almost sure Lebesgue integral over Ω for each $t \varepsilon T$ if the integral exists, i. e.,

$$\langle x_t(\omega) \rangle = \int_\Omega x_t(\omega) \, d\mu(\omega)$$

and we denote by $L_1(\chi_1)$ the set of all equivalence classes of real valued random variables x_t whose expectation $\langle x_t(\omega) \rangle$ exists. Hence $x_t(\omega) \varepsilon L_1(\chi_1)$ if $\langle x_t(\omega) \rangle < \infty$ a.s.

The expectations of x_t^k and $|x_t|^k$ for all k are called the k-th moment and the k-th absolute moment of the r. v. $x_t(\omega)$ for each $t \varepsilon T$. Then $L_p(\chi_1)$ for $1 < p \leqslant \infty$ denotes equivalence classes of real valued r. v. $x_t(\omega) \varepsilon \chi_1$ such that $\langle |x_t|^p \rangle < \infty$ a.s. for $t \varepsilon T$. The set of random variables $x(t_1, \omega), x(t_2, \omega), \cdots$ of the process $x(t, \omega)$ generate a linear vector space $V_{x(t, \omega)}$ on R and the correlation function $R_x(t_1, t_2)$ defines an inner product over the space $V_{x(t, \omega)}$ given by

$$R_x(t_1, t_2) = \langle x(t_1, \omega) x(t_2, \omega) \rangle$$
$$= (x(t_1, \omega), x(t_2, \omega))$$

and the norm $\|x(t, \omega)\| = (x(t_2, \omega), x(t_2, \omega))^{1/2}$. The metric $\rho(x(t_1, \omega), x(t_2, \omega)) = \|x(t_1, \omega) - x(t_2, \omega)\|$ and we have a Hilbert space $L_2(\Omega, R)$

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Ռ. Վ. ՀԱՄԲԱՐՁՈՒՄՅԱՆ, Ջ. ԱԴՈՄՅԱՆ. Մի քանի դիտողություն՝ ստոխաստիկ ձևափոխությունների վերաբերյալ (*ամփոփում*)

Աշխատանքում դիտարկվում են ստոխաստիկ դիֆերենցիալ հավասարումների Ջ. Ադոմյանի տեսության որոշ հարցեր, այդ թվում՝ օպերատորների շարքերի զուգամիտության և օպերատորները ստոխաստիկ դետերմինիստիկ մասերի վերլուծելու խնդիրները։

Р. В. АМБАРЦУМЯН, Дж. АДОМЯН. *Несколько замечаний о стохастических преобразованиях* (резюме)

В работе обсуждаются некоторые вопросы, относящиеся к подходу первого из авторов к стохастическим дифференциальным уравнениям, в т. ч. сходимость рядов операторов и выбор разложения оператора на детерминистическую и случайную части.

REFERENCES

1. *G. Adomian*. Random Operator Equations in Mathematical Physics. J. Math. Phys. Vol. 11, № 3, March, 1970, 1069—1084.
2. *G. Adomian*. ed. Stochastic Systems Analysis, Abblied Stochastic Processes, Acad. Press, 1980, 1—18.
3. *G. Adomian*. A Constructive Method for Linear and Nonlinear Stochastic Partial. Differential Equations. Applied Nonlinear Analysis, Academic Press, 1980, 13—23.
4. *G. Adomian*. On tbe Modeling and Analysis, of Nonlinear Stochastic Systems, Proceedings of the Second International Conference on Mathematical Modeling, St. Louis, Missouri, July 11—13, 1979. (To appear late 1980 as a book).
5. *G. Adomian* and *K. Malakian*. Operator Theoretic Solution of Stochastic Systems, Journal of Mathematical Analysis and Applications, vol. 76, no. 1. July 1980, 183—201.
6. *G. Adomian* and *K. Malakian*. Self-Correcting Appoximate Solutions by the Iterative Method for Nonlinear Stochastic Differential Equations, Journal of Mathematical Analysis and Applications, vol. 76, no. 2, August 1980, 309—327.
7. *G. Adomian* and *K. Malakian*. Recent Results in Stochastic Equations—The Nonlinear Case. Nonlinear Equations in Abstract Spaces, ed. by V. Lakshmikantham, Academic Press, 1978, 3—23.
8. *G. Adomian* and *K. Malakian*. On the Existence of Solutions for Linear and Nonlinear Stochastic Operator Equations, Journal of Mathematical Analysis and Applications, vol. 62, no. 2, February 1978, 229—235.
9. *G. Adomian* and *K. Malakian*. Linear Stochastic Partial Differential Equations, Journal of Mathematical Analysis and Applications, vol. 77, no. 2. October 1980, 505—512.
10. *G. Adomian* and *K. Malakian*. Stochastic Analysis, International Journal of Mathematical Modeling, to appear January 1981.