ՀԱՅԿԱԿԱՆ ՍՍՀ ԳԻՏՈՒԹՅՈՒՆՆԵՐԻ ԱԿԱԴԵՄԻԱՅԻ ՏԵՂԵԿԱԳԻՐ
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ON THE DEGREE OF CHEBYSHEV APPROXIMATION ON SETS WITH SEVERAL COMPONENTS

1°. Let K be a compact set in the z-plane with p, $1 < p < \infty$, simply connected disjoint components $I_1, I_2, \cdots, I_p$ none of which is a single point. For a connected set K the degree of best approximation (in the Chebyshev sense)

$$E_n(f, K) = E_n(f) = \inf_{q \in P_n} \sup_{z \in K} |f - q| = \inf_{q \in P_n} \| f - q \|_K$$

P_n = polynomials of degree $\leqslant n$] is determined by the regularity properties of f. In the case under consideration here a new phenomenon arises, because the geometrical nature of K plays a decisive role. To study the influence of the geometry of K we consider approximation of functions which have very regular behavior on K. To be precise, we restrict ourselves to functions in the class R:

Definition 1. $f \in R$ if and only if

a) $f(z)$ is defined on K.

b) On every connected component I_j of K $f(z)$ is equal to the restriction of an entire function $h_j(z)$ to I_j.

c) Not all $h_j(z)$ are identical.

The relevant geometric properties of K find their analytic expression in the properties of the Green's function $k(z)$ of the complement K' of K in the completed z-plane with pole at ∞. We extend the definition of $k(z)$ to all complex z by setting

$$k(z) = 0 \qquad (z \in K).$$

For $0 < \alpha < \infty$, let

$$D_\alpha = \{z | k(z) < \alpha\}.$$

If α is sufficiently small the set D_α will consist of p disjoint regions. Each one of these regions will contain exactly one I_j. As α increases there will be values α' of α such that $D_{\alpha'+\varepsilon}$ has fewer components than $D_{\alpha'-\varepsilon}$ for every ε, $0 < \varepsilon < \alpha'$. The values α' for which this occurs can be characterized as those positive numbers for which the level line $k(z) = \alpha'$ contains a critical point of $k(z)$ (i. e. a point where $\frac{\partial k}{\partial x} = \frac{\partial k}{\partial y} = 0$). It is well khown that there are $p-1$ such points, if counted with proper multiplicity [1, p. 32]. For simplicity we assume that each

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of these points is of multiplicity one. Since $k(z)$ is harmonic, this means that in terms of a suitably chosen new variable

$$\zeta = \xi + i\eta = e^{i\gamma}(z - z_0).$$

$$k(z) = k(z_0) + c(z_0)(\xi^2 - \eta^2) + \text{higher order terms} \quad (c(z_0) > 0) \quad (1)$$

near the critical point z_0.

It is an immediate consequence of the definition of the class R that $f \in R$ has a holomorphic extension to D_α for sufficiently small α (the holomorphic functions defining $f(z)$ in distinct components of D_α are the $h_j(z)$ entering into the definition of $f \in R$). By requirement c) of Definition 1 there must be a β, $0 < \beta < \infty$, such that $f(z)$ is holomorphic in D_β, but not in D_α with $\alpha > \beta$.

Theorem 1. *Suppose there are constants α_1, A_1 such that for $0 < \alpha < \alpha_1$ the length of ∂D_α, the boundary of D_α, is less than A_1. Suppose also that at all critical points $k(z)$ has an expansion of the form* (1).

If $f \in R$ is holomorphic in D_β, but not in D_α with $\alpha > \beta$, then one can find a non-negative integer $q = q(f)$ such that

$$E_n(f) \geqslant A_1 n^{-q-\frac{1}{2}} e^{-n\beta}. \quad (2)$$

Remarks. 1. Critical points of higher multiplicity can be investigated in the same way and a similar statement holds in their presence.

2. The hypothesis about the length of the level curves ∂D_α is satisfied if and only if each I_j has a rectifiable boundary. Under more stringent conditions on K it is possible to show that Theorem 1 gives the correct order of magnitude of $E_n(f)$:

A function of a real variable is said to be in class C^{m+}, if its m^{th} derivative satisfies a Lipschitz condition with some positive exponent. The boundary of K is in C^{m+} if the boundary of K consists of rectifiable curves such that the coordinate functions are C^{m+} functions of arc length.

Theorem 2. *Suppose that the hypotheses of Theorem 1 hold and that in addition ∂K consists of Jordan curves and simple arcs in C^{2+}.*

Then one can find polynomials $p_n(x)$ such that

$$\|f - p_n\|_K < A_2 n^{-q-\frac{1}{2}} e^{-n\beta},$$

where q is the same integer as in Theorem 1.

Notation. The letter A will be used for a positive number which may depend on f and K, but not on n. The value of A may change from one occurrence to the next.

2°. Proof of Theorem 1. Theorem 1 is proved by estimating $\int_{\partial D_\alpha} f(z)\varphi(z)\,dz$ in two different ways, where $\varphi(z)$ is a suitable bounded holomorphic function in K' (the complement of K), with a zero of order $n+2$ at infinity.

First we construct a suitable $\varphi(z)$. Let $\omega_j(z)$ be the harmonic measure of ∂I_j with respect to K'. We assume that φ is of the form

$$\varphi(z,\ n+2) = \varphi(z) = \exp\{u(z) + i\bar{u}(z)\}$$

where

$$u(z) = -(n+2)\,k(z) + \textstyle\sum_{j=1}^{p} \lambda_j\,\omega_j(z) \quad (\lambda_j \in \mathbf{R},\ |\lambda_j| < A).$$

Here $\bar{u}$ must be the conjugate harmonic function of u defined by

$$u(z) + i\bar{u}(z) = \int_{z_0}^{z} \left\{ \frac{\partial u}{\partial x}(\zeta) - i\,\frac{\partial u}{\partial y}(\zeta) \right\} d\zeta + \text{const.} \tag{2.1}$$

In general this will not define a single-valued φ, since the integral in (2.1) depends on the path of integration. Integrals along different paths with the same end points will differ by

$$\sum m_l\,\pi_l(u) \ (m_l \in \mathbf{Z},\ \pi_l(u) = \text{period of } u + i\bar{u} = \int_{C_l} \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right) d\zeta,$$

where C_l is a curve encircling I_l once in the positive sense, but not encircling any other I_j. $\pi_l(u)$ is an imaginary number. The function $\varphi(z)$ is single-valued, if and only if

$$\pi_l(u) = 2\pi i\, q_l \qquad (q_l \in \mathbf{Z}). \tag{2.2}$$

If we put

$$\pi_l(k) = i\nu_l \qquad \pi_l(\omega_j) = iP_{l_j},$$

e can rewrite (2.2) as

$$\textstyle\sum_{j=1}^{p} P_{l_j}\lambda_j = (n+2)\,\nu_l + 2\pi q_l = \kappa_l \quad (l = 1, 2, \cdots, p). \tag{2.3}$$

H. Widom [4, p. 142] proved that the equations

$$\begin{aligned} &\textstyle\sum_{j=1}^{p} P_{l_j}\lambda_l = c_l \qquad (l = 1, 2 \cdots p) \\ &\lambda_1 + \lambda_2 + \cdots + \lambda_p = 0 \end{aligned} \tag{2.4}$$

have a unique solution, if and only if

$$c_1 + c_2 + \cdots + c_p = 0.$$

Since, by Cauchy's Theorem,

$$i\sum \nu_l(k) = \int_{|\zeta| = R} \left(\frac{\partial k}{\partial x} - i\frac{\partial k}{\partial y}\right) d\zeta = 2\pi i,$$

it follows that (2.3) has a unique solution if and only if the integers q_l satisfy

$$2\pi(n + 2 + q_1 + q_2 \cdots + q_p) = \kappa_1 + \kappa_2 \cdots + \kappa_p = 0.$$

By linearity, we can therefore find real constants Q_{jl}, depending only on the P_{l_j}, such that

$$\lambda_j = \sum_{l=1}^{p-1} Q_{jl} \varkappa_l \qquad (j=1, 2, \cdots, p-1). \tag{2.5}$$

By adjusting the integers q_l we can make $|\varkappa_l| < A$ and so, by (2.4) and (2.5)

$$|\lambda_j| < A \qquad (j = 1, 2, \cdots, p).$$

With this choice of the λ, φ is a bounded, holomorphic function in K'.

Since $\varphi(z)$ has a zero of order $n+2$ at ∞, an application of Cauchy's Theorem yields

$$\int_{\partial D_\alpha} z^j \varphi(z)\, dz = \int_{|z|=R} z^j \varphi(z)\, dz = 0 \quad (R > R_0;\ j=0, 1, 2, \cdots, n).$$

Therefore, if $q(z)$ is the polynomial of degree $\leqslant n$ which best approximates $f(z)$ on $\overline{D_\alpha}$, $\alpha > 0$,

$$\left| \int_{\partial D_\alpha} f(z)\, \varphi(z)\, dz \right| = \left| \int_{\partial D_\alpha} (f(z) - q(z))\, \varphi(z)\, dz \right|$$

$$\leqslant E_n(f, \overline{D_\alpha}) \int_{\partial D_\alpha} |\varphi(z)|\, |dz|.$$

Since $|\varphi(z)| < A$ and, for $\alpha < \alpha_1$, $\int_{\partial D_\alpha} |dz| < A$, it follows that

$$E_n(f, \overline{D_\alpha}) > A \left| \int_{\partial D_\alpha} f(z)\, \varphi(z)\, dz \right| \qquad (\alpha < \alpha_1). \tag{2.6}$$

By a well-known result of J. L. Walsh [3, p. 80, 81] the polynomials of best approximation to $f(z)$ in K, $p_n(z)$, converge uniformly to $f(z)$ in $D_{\beta-\varepsilon}$ $(0 < \varepsilon < \beta)$. By the two-constant theorem applied to the subharmonic function $\log |f(z) - p_n(z)|$ in $D_{\beta-\varepsilon} \setminus K$

$$\log |f(z) - p_n(z)| < \log E_n(f, K) \left(1 - \frac{k(z)}{\beta - \varepsilon}\right) \qquad (n > n_0).$$

Therefore in $\overline{D_\alpha}$ with

$$\alpha \leqslant (\beta - \varepsilon) / |\log E_n(f)| = \alpha_2,$$

$$E_n(f, \overline{D_\alpha}) \leqslant \|f - p_n\|_{\overline{D_\alpha}} < e E_n(f).$$

Hence, using (2.6)

$$E_n(f) > A \left| \int_{\partial D_\alpha} f(z)\, \varphi(z)\, dz \right| \qquad (\alpha \leqslant \alpha_2). \tag{2.7}$$

By the definition of β, $f(z)$ must have a singular point on ∂D_β. This can only happen for $f \in R$, if there are critical points of $k(z)$ on

∂D_β. D_β is the union of at most p disjoint regions Δ_ν $(\nu = 1, 2, \cdots)$, say. By our assumptions each critical point of $k(z)$ on ∂D_β is a common boundary point of exactly two Δ, s. There are no other points which belong to the boundary of more than one Δ. If z_0 is a critical point of $k(z)$, then there is exactly one line L through z_0 in the direction of steepest increase of k, namely the ξ-axis, in the notation of (1). Draw all such lines L through critical points on ∂D_β. For sufficiently small ε these lines divide $D_{\beta+\varepsilon}$ into subregions Δ'_ν $(\nu=1, 2, \cdots)$. Each Δ'_ν contains one and only one Δ_ν and there are exactly two Δ'_ν which have a given line L in common. If $f(z) = h_\mu(z)$, say, in Δ_ν, then we have, by Cauchy's Theorem,

$$\left|\int_{\partial D_\alpha} f(z)\,\varphi(z)\,dz\right| = \left|\sum_\nu \int_{\Delta'_\nu} h_\mu(z)\,\varphi(z)\,dz\right| \qquad (\alpha < \beta). \tag{2.8}$$

On the level line $\partial D_{\beta+\varepsilon}$ (of length $\leqslant A$)

$$|h_j(z)| < A \ (j = 1, 2, \cdots, p) \text{ and } |\varphi(z)| < Ae^{-n(\beta+\varepsilon)}.$$

Therefore

$$\left|\int_{\partial\Delta'_\nu \cap \partial D_{\beta+\varepsilon}} h_\mu(z)\,\varphi(z)\,dz\right| < Ae^{-n(\beta+\varepsilon)}. \tag{2.9}$$

The contribution to (2.8) of a line segment L through the critical point z_0 is

$$\int_L \{h_\mu(z) - h_\rho(z)\}\,\varphi(z)\,dz = J(L) \tag{2.10}$$

where $f = h_\mu$ and $f = h_\rho$, respectively, in the two Δ' having L as their common boundary. By (1), near z_0, $k(z) + i\tilde{k}(z)$, expressed in terms of $\zeta = \xi + i\eta$ has the form

$$k(z) + i\tilde{k}(z) = \beta + i\tilde{\beta} + c(\xi^2 - \eta^2 + 2i\xi\eta) + \text{higher order terms.}$$

In particular, on L

$$k(z) + i\tilde{k}(z) = \beta + i\tilde{\beta} + c\xi^2 + \text{higher order terms in } \xi.$$

Expanding in powers of ξ

$$h_\mu(z) - h_\rho(z) = b_s\xi^s + b_{s+1}\xi^{s+1} + \cdots \ (b_s \neq 0), \tag{2.11}$$

$$e^{\Sigma\lambda_j(\omega_j(z) + i\tilde{\omega}_j(z))} = a_0 - a_1\xi + \cdots \qquad (a_0 \neq 0).$$

Here the b's are independent of n, the a's depend on n (via the λ, s), but $|a_j| < A_j$ uniformly in n. Substituting these power series, the integral (2.10) takes the form

$$J(L) = e^{-i\gamma} \int_{-a}^{b} e^{-(n+2)\beta - i(n+2)\tilde{\beta} - (n+2)[c\xi^2 + \cdots]} (a_0 b_s \xi^s + \cdots)\, d\xi.$$

By applying Laplace's method [2, p. 78] to the integral $J(L)$,

$$J(L) = \begin{cases} e^{-i\gamma} C_s [(n+2)c]^{-\frac{s+1}{2}} e^{-(n+2)(\beta + i\tilde{\beta})} a_0 b_s + R \ (s \text{ even}) \\ R \qquad\qquad (s \text{ odd}) \end{cases} \tag{2.12}$$

where

$$|R| < A(n+2)^{-\frac{s+2}{2}} e^{-n\beta}.$$

Remembering the definition of a_0 we can write the leading term in (2.12)

$$e^{-i\gamma} C_s [(n+2)c]^{-\frac{s+1}{2}} e^{-n\beta} |\varphi(z_0) e^{n\beta}| b_s. \tag{2.13}$$

Since $|\varphi(z_0) e^{n\beta}|$ lies between two positive numbers independent of n, Theorem 1 follows from (2.12), if z_0 is the only critical point on ∂D_β and $s = 2q$ is even.

Suppose now that several critical points $z_1, z_2, \cdots$ lie on ∂D_β, and suppose that the lowest power ξ^{s_j} occurring in the corresponding expansions (2.11) is ξ^s, s even.

Note that one can find a function $\psi(z)$, holomorphic and bounded in the complement K' of K, which satisfies

$$\psi(z_j) = e^{i\alpha_j} \quad (j = 1, 2, \cdots, p-1;\ \alpha_j \text{ given, real}).$$

(E. g., if $w = \psi_k(z)$ maps the complement of I_p conformally onto $|w| < 1$, $\psi_k(z_k) = 0$, one can choose ψ of the form $\psi(z) = \sum_k \gamma_k \prod_{j \neq k} \psi_j(z)$).

If $\varphi(z)$ is replaced by $\varphi(z)\psi(z)$, the leading contribution of each z_j to (2.8) is obtained from those z_j for which $s_j = s$ and each such z_j contributes a leading term

$$C_s e^{-i\gamma_j} [(n+2) c(z_j)]^{-\frac{s+1}{2}} e^{-n\beta} (\varphi(z_j) e^{n\beta}) \psi(z_j) b_s(z_j).$$

By suitable choice of the $\psi(z_j)$ all these terms can be given equal argument, so that no cancellations between them are possible and Theorem 1 follows in this case with $q = \dfrac{1}{2} s$.

It remains to prove the Theorem in the case $s = \min s_j =$ odd integer. Again we replace $\varphi(z)$ by $\varphi(z)\psi(z)$, but this time we choose $\psi(z)$ so that

$$\psi(z_j) = 0, \ \psi'(z_j) = e^{i\alpha_j} \ (z_j \in \partial D_\beta).$$

It is again easy to see that such a ψ exists, which is bounded and holomorphic in K'. The principal contributions to the new integral (2.8) are now of the form

$$C_s \sum_j [(n+2)\, c(z_j)]^{-\frac{s+2}{2}} e^{-n\beta} (\varphi(z_j)\, e^{n\beta})\, \psi'(z_j)\, b_s(z_j)\, e^{-i\gamma(z_j)}$$

and cancellations between terms are impossible, if the $\psi'(z_j)$ are suitable chosen. This proves the Theorem with

$$q = \frac{1}{2}(s+1),$$

if s is odd.

3°. Proof of Theorem 2. Let $\varphi(z) = \varphi(z, n+2)$ be the function defined in the proof of Theorem 1. The function

$$\psi(z) = 1/\varphi(z, n)$$

is holomorphic in K' except for a pole at ∞ where

$$\psi(z) = Az^n + \text{lower powers of } z \qquad (|z| \geq R). \tag{3.1}$$

Also

$$A_1 e^{n\alpha} < |\psi(z)| < A_2 e^{n\alpha} \qquad (z \in \partial D_\alpha;\ \alpha > 0). \tag{3.2}$$

Let

$$q(z) = \frac{1}{2\pi i} \int\limits_{|\zeta| = R} \frac{\psi(\zeta)}{\zeta - z}\, d\zeta \qquad (|z| < R).$$

By (3.1),

$$q(z) = \sum_{\nu=0}^{\infty} \frac{1}{2\pi i} \int\limits_{|\zeta| = R} \psi(\zeta)\, z^\nu \zeta^{-\nu-1}\, d\zeta = \text{polynomial of degree } n \text{ in } z.$$

For z outside $\overline{D}_\alpha$, by the residue theorem,

$$q(z) - \psi(z) = \frac{1}{2\pi i} \int\limits_{\partial D_\alpha} \frac{\psi(\zeta)\, d\zeta}{\zeta - z}.$$

It follows from results of H. Widom [4, Lemma 11.2] that the integral on the right hand side of (3.1) is uniformly bounded for $z \in K'$ and all n. Therefore

$$|q(z) - \psi(z)| < A \qquad (z \in K',\ n = 1, 2, \cdots). \tag{3.3}$$

Consider now

$$p(z) = \frac{1}{2\pi i} \int\limits_{\partial D_\alpha} \frac{f(\zeta)\,(q(\zeta) - q(z))}{q(\zeta)\,(\zeta - z)}\, d\zeta$$

where α is so small that ∂D_α consists of p disjoint curves C_j each of which surrounds one I_j. Since $(q(\zeta) - q(z))/(\zeta - z)$ is a polynomial of degree $n-1$ in z, $p(z)$ is a polynomial of degree $n-1$. By Cauchy's formula

$$f(z) - p(z) = \frac{1}{2\pi i} \int\limits_{\partial D_\alpha} \frac{f(\zeta)\, d\zeta}{\zeta - z} - p(z) = \frac{q(z)}{2\pi i} \int\limits_{\partial D_\alpha} \frac{f(\zeta)\, d\zeta}{q(\zeta)\,(\zeta - z)} =$$

$$= \sum_{j=1}^{p} \left\{ \frac{q(z)}{2\pi i} \int\limits_{C_j} \frac{h_j(\zeta)\,(\psi(\zeta) - q(\zeta))\, d\zeta}{q(\zeta)\,\psi(\zeta)\,(\zeta - z)} + \frac{q(z)}{2\pi i} \int\limits_{C_j} \frac{h_j(\zeta)\, d\zeta}{\psi(\zeta)\,(\zeta - z)} \right\} \quad (z \in D_\alpha). \tag{3.4}$$

By (3.3) and (3.2), $q(z) \neq 0$ outside D_γ for $n > n_0(z)$. We can therefore deform the contours C_j into the contours $\partial\Delta'$, used in the proof of Theorem 1. On $\partial\Delta'_\nu$, $|\zeta - z| > A$, $|h_j(\zeta)| < A$. By (3.2) and (3.3)

$$|\psi(\zeta)| > A_1 e^{n\beta}, \quad |q(\zeta)| > A e^{n\beta}.$$

On the parts of $\partial\Delta'$, which lie on $\partial D_{\beta+\varepsilon}$

$$|\psi(\zeta)| > A_1 e^{n(\beta+\varepsilon)}.$$

Therefore it is an easy consequence of (3.4) that

$$|f(z) - p(z)| < \Big\{ Ae^{-2n\beta} + Ae^{-n(\beta+\varepsilon)} + \\ + \sum_{z_0} \left| \frac{3}{2\pi} \int_L (h_\mu(\zeta) - h_\nu(\zeta)) \frac{d\zeta}{\psi(\zeta)(\zeta - z)} \right| \Big\} |q(z)| \tag{3.5}$$

where the summation is over all critical points z_0 lying on ∂D_β. Each integral is estimated by Laplace's method and yields a contribution which is less than

$$An^{-\frac{s+1}{2}} e^{-n\beta} |q(z)| \qquad (s \text{ even}) \tag{3.6}$$

and less than

$$An^{-\frac{s+2}{2}} e^{-n\beta} |q(z)| \qquad (s \text{ odd}). \tag{3.7}$$

By (3.2) and (3.3)

$$|q(z)| < A \qquad (z \in \partial D_{1/n}) \tag{3.8}$$

and Theorem 2 follows from (3.5), (3.6), (3.7), (3.8) and the maximum principle.

Վ. Գ. Ի. ՖՈՒՔՍ. Մի քանի կոմպոնենտներով բազմությունների վրա Չեբիշևյան մոտարկման աստիճանի մասին *(ամփոփում)*

Դիցուք K-ն կոմպակտ է C-ում $I_1, \cdots, I_p$ միակապ կոմպոնենտներով $(2 < p < \infty)$. f ֆունկցիան սահմանված է K-ի վրա հետևյալ հավասարություններով

$$f(z) = h_j(z), \; j = 1, 2, \cdots, p$$

($z \in I_j$, h_j-ն ամբողջ ֆունկցիա է)։

Դիցուք k-ն $\overline{C} \setminus K$-ի Գրինի ֆունկցիան է, անվերջ հեռու բևեռով։

Եթե h_j ֆունկցիաներից գոնե երկուսը տարբեր են, ապա գոյություն ունի այնպիսի β $(0 < \beta < \infty)$, որ f-ը թույլ է տալիս անալիտիկ շարունակություն մինչև $D_\beta = \{z | k(z) < \beta\}$, բայց ոչ D_α, երբ $\alpha > \beta$.

Ապացուցված է, որ K-ի և f-ի վրա որոշ պայմանների դեպքում տեղի ունի

$$A_1 n^{q-\frac{1}{2}} e^{-\beta n} < E_n(f) = \inf_p \sup_{z \in K} |f(z) - p(z)| < A_2 n^{q-\frac{1}{2}} e^{-\beta n}$$

գնահատականը, որտեղ p-ն n-ից ոչ բարձր կարգի բազմանդամ է, $q = q(f)$ որոշ ոչ բացասական հաստատուն է, իսկ A_1, A_2-ը n-ից անկախ հաստատուններ են։

В. Г. И. ФУКС. *О степени Чебышевской аппроксимации на множествах с несколькими компонентами* (резюме)

Пусть K-компакт в C с p односвязными компонентами $I_1, \cdots, I_p$ $(2 \leq p < \infty)$. Функция f определена на K равенствами $f(z) = h_j(z)$ $(z \in I_j$, h_j — целая функция) $j = 1, 2, \cdots, p$.

Функция k— функция Грина для $\bar{C} \setminus K$ с полюсом в бесконечности.

Если хотя бы две из функций h_j различны, то существует такое β $(0 < \beta < \infty)$, что f позволяет аналитическое продолжение в $D_\beta = \{z | k(z) < \beta\}$, но не в D_α при $\alpha > \beta$.

Доказывается, что при некоторых условиях на K и f имеет место оценка

$$A_1 n^{-q-\frac{1}{2}} e^{-\beta n} < E_n(f) = \inf_p \sup_{z \in K} |f(z) - p(z)| < A_2 n^{-q-\frac{1}{2}} e^{-\beta n},$$

где p — полином степени не выше n, $q = q(f)$ — некоторая неотрицательная константа, A_1, A_2 — константы, не зависящие от n.

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