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OBTAINING FIRST AND SECOND ORDER STATISTICS $\langle y \rangle$ and $R_y(t_1, t_2)$ IN STOCHASTIC DIFFERENTIAL EQUATIONS FOR THE NONLINEAR CASE

Previous papers [1, 2] by the author have recently developed a convergent method of successive approximation for determination of the first and second order statistics, $\langle y \rangle$ and $R_y(t_1, t_2)$ for the solution of the nonhomogeneous nonlinear stochastic differential equation

$$\mathbf{L} y + n(y, \dot{y}, \cdots) = x(t) \tag{1}$$

where $x(t, \omega)$, $t \in T$, $\omega \in (\Omega, J, \mu)$, a probability space, is a stochastic process; L is a linear stochastic operator [3, 4, 5] given by $\mathbf{L} = \sum_{\nu=0}^{n} a_\nu(t, \omega)\, d^\nu/dt^\nu$, an nth order differential operator with stochastic coefficients $a_\nu(t, \omega)$, $t \in T$,

$$\omega \in (\Omega, J, \mu), \text{ for } \nu = 0, 1, \cdots, n-1 \text{ and } n(y, \dot{y}, \cdots) =$$

$$= \sum_{\mu=0}^{m_m} b_\mu(t, \omega)\, (y^{(\mu)})^{m_\mu}$$

representing a nonlinear operator with stochastic coefficients b_μ on $T \times \Omega$, where $y^{(\mu)}$ of course represents the μ th derivative. The processes may be complex and defined on different probability speces as well but this will not be shown here.

We assume that the operator $\mathbf{L}$ is separable into a sum of deterministic and random parts, L and R respectively, i. e., $\mathbf{L} = L + R$. Assuming $a_n > 0$ and is deterministic, while a_ν for $\nu = 0, 1, \cdots, n-1$ may be stochastic, $a_\nu = \langle a_\nu \rangle + \alpha_\nu(t, \omega)$ where α_ν is a zero-mean random process, thus $L = \sum_{\nu=0}^{n} \langle a_\nu \rangle\, d^\nu/dt^\nu$ and $R = \sum_{\nu=0}^{n-1} \alpha_\nu(t, \omega)\, d^\nu/dt^\nu$. We assume that $\langle a_\nu \rangle$ exists and is continuous on T. The development also assumes that L is invertible, i. e., a Green's function $e(t, \tau)$ exists. Further, $\langle b_\mu \rangle$ exists and the random or fluctuating parts of a_ν and $b_{\mu'}$ given by α_ν and $\beta_{\mu'}$ are continuous, a. e. We assume the input x is bounded on T and that derivatives of a_ν and b_μ to the necessary order are bounded, and finally that the input process and coefficient processes are statistically independent. Now we write (1) as

$$y(t, \omega) = L^{-1} x(t, \omega) - L^{-1} R y(t, \omega) - L^{-1} n(y, \dot{y}, \cdots) \tag{2}$$

or in terms of the Green's function

$$y(t,\omega)=\int_0^t e(t,\tau)\,x(\tau,\omega)\,d\tau-\int_0^t e(t,\tau)\sum_{\nu=0}^{n-1}\alpha_\nu(\tau,\omega)\,(d^\nu y(\tau,\omega)/d\tau^\nu)\,d\tau-$$

$$-\int_0^t e(t,\tau)\sum_{\mu=0}^{m}b_\mu(\tau,\omega)\,(d^\mu y(\tau,\omega)/d\tau^\mu)^{m_\mu}\,d\tau. \qquad (3)$$

For the case of zero initial conditions [1, 3], Adomian's „stochastic Green's theorem" allows writing the second term in terms of an adjoint operator as

$$\int_0^t R^1_\tau[e(t,\tau)]\,y(\tau,\omega)\,d\tau$$

where

$$R^1_\tau[e(t,\tau)]=\sum_{k=0}^{n-1}(-1)^k(d^k/d\tau^k)[\alpha_k(\tau)\,e(t,\tau)]=K(t,\tau,\omega).$$

This term has also been given elsewhere in terms of a „stochastic resolvent kernel" by Sibul [6] and by Adomian [3]. Letting

$$y=\sum_{i=0}^{\infty}(-i)^i y_i \text{ where } y_0=F(t)=L^{-1}x(t)+\Phi \text{ where } \Phi$$

satisfies the homogeneous equation $Ly=0$, and expanding n in a Taylor series, we obtain an iterative solution [1] with each term y_i for $i\geqslant 1$ depending on the *previous* y_{i-1}. Since we eventually work back to y_0, ensemble averages will separate-because of the statistical independence of x from the coefficient processes without closure approximations or white noise assumptions.

The objective of this paper is to show further details of the determination of the mean solution $\langle y\rangle$ and the twopoint correlation function $R_y(t_1,t_2)$, or the covariance $K_y(t_1,t_2)$, for $y(t,\omega)$. To obtain these statistical measures [3], y must, of course, be averaged over the space Ω. (If the probability space are different for α_ν, β_μ, x, then the averaging must be carried out over the appropriate spaces). Similarly to obtain $R_y(t_1,t_2)$, we find $\langle y(t_1)\,y(t_2)\rangle$ or $\langle y(t_1)\,\overset{*}{y}(t_2)\rangle$ in the complex case. To correspond to earlier notation from the linear case, assume for the moment that $n=0$ and in terms of the stochastic resolvent kernel Γ

$$y(t,\omega)-F(t,\omega)-\int_0^t K(t,\tau,\omega)\,y(\tau,\omega)\,d\tau=$$

$$=F(t,\omega)-\int_0^t\Gamma(t,\tau,\omega)\,F(\tau,\omega)\,d\tau$$

where

$$\Gamma(t, \tau, \omega) = \sum_{m=1}^{\infty} (-1)^{m-1} K_m(t, \tau, \omega),$$

$$K_m(t, \tau, \omega) = \int_0^t K(t, \tau_1, \omega) K_{m-1}(\tau_1, \tau, \omega)\, d\tau_1,$$

$$K_1 = K.$$

Now including the nonlinear term

$$\langle y \rangle = \langle F(t) \rangle - \int_0^t \langle \Gamma(t, \tau) \rangle \langle F(\tau) \rangle d\tau -$$

$$- \langle \int_0^t e(t, \tau)\, n(y, \dot{y}, \cdots)\, d\tau \rangle.$$

Let's examine the complicating last term for an example, say $n = \beta y^2$. The term becomes

$$- \langle \int_0^t e(t, \tau)\, \beta y^2(\tau)\, d\tau \rangle = - \langle L^{-1}\, \beta y^2 \rangle$$

$L^{-1}\, \beta y^2$ adds to the previous linear solution the terms

$$L^{-1}\, \beta\, (y_0^2 + y_1^2 + y_2^2 + \cdots - 2 y_0 y_1 + 2 y_0 y_2 \cdots - 2 y_1 y_2 + 2 y_1 y_3 \cdots).$$

Thus the nonlinear term adds to $\langle y \rangle$ a series

$$- \langle \int_0^t e(t, \tau)\, \beta y_0^2(\tau)\, d\tau \rangle + \cdots =$$

$$= - \int_0^t e(t, \tau) \langle \beta(\tau) \rangle \int_0^{\tau} \int_0^{\gamma} e(\tau, \gamma)\, e(\gamma, \sigma) \langle x(\gamma)\, x(\sigma) \rangle\, d\sigma\, d\gamma\, d\tau + \cdots$$

where the separation of averages takes place as before.

Now consider $R_y(t_1, t_2) = \langle y(t_1)\, \dot{y}(t_2) \rangle$

$$R_y(t_1, t_2) = \langle [F(t_1, \omega) - \int_0^{t_1} \Gamma(t_1, \tau_1, \omega) F(\tau_1, \omega)\, d\tau_1 -$$

$$- \int_0^{t_1} e(t_1, \tau_1)\, n(y(\tau_1), \dot{y}(\tau_1), \cdots)\, d\tau_1] \cdot [F(t_2, \omega) -$$

$$- \int_0^{t_2} \Gamma(t_2, \tau_2, \omega) F(\tau_2, \omega)\, d\tau_2 -$$

$$-\int_0^{t_2} e(t_2, \tau_2)\, n(y(\tau_2), \dot{y}(\tau_2), \cdots)\, d\tau_2]^* >.$$

For the linear case this gave

$$R_y(t_1, t_2) = R_{FF}(t_1, t_2) - \int_0^{t_1} < \Gamma(t_1, \tau_1, \omega)\, R_{FF}(\tau_1, t_2)\, d\tau_1 -$$

$$-\int_0^{t_2} < \overset{*}{\Gamma}(t_2, \tau_2, \omega) > R_{FF}(t_1, \tau_2)\, d\tau_2 +$$

$$+\int_0^{t_1}\int_0^{t_2} < \Gamma(t_1, \tau_1, \omega)\, \overset{*}{\Gamma}(t_2, \tau_2, \omega) > R_{FF}(\tau_1, \tau_2)\, d\tau_1 d\tau_2$$

which, as earlier work has shown [4] reduces to the result for perturbation theory *where perturbation theory is applicable.* In the nonlinear case, we obtain nine terms, five of which involve n.

$$R_y(t_1, t_2) = < F(t_1)\overset{*}{F}(t_2) > - \int_0^{t_2} < \overset{*}{\Gamma}(t_2, \tau_2) > < F(t_1)\overset{*}{F}(\tau_2) > d\tau_2 -$$

$$-\int_0^{t_1} < \Gamma(t_1, \tau_1) > < F(\tau_1)\overset{*}{F}(t_2) > d\tau_1 -$$

$$-\int_0^{t_1}\int_0^{t_2} < \Gamma(t_1, \tau_1)\, \overset{*}{\Gamma}(t_2, \tau_2) > < F(\tau_1)\overset{*}{F}(\tau_2) > d\tau_1 d\tau_2 -$$

$$- < \int_0^{t_2} F(t_1)\, \overset{*}{e}(t_2, \tau_2)\, \overset{*}{n}(y(\tau_2), \cdots)\, d\tau_2 > -$$

$$- < \int_0^{t_1} \overset{*}{F}(t_2)\, e(t_1, \tau_1)\, n(y(\tau_1), \cdots)\, d\tau_1 > +$$

$$+ < \int_0^{t_1} \Gamma(t_1, \tau_1)\, F(\tau_1)\, d\tau_1 \int_0^{t_2} \overset{*}{e}(t_2, \tau_2)\, \overset{*}{n}(y(\tau_2), \cdots)\, d\tau_2 > +$$

$$+ < \int_0^{t_1} e(t_1, \tau_1)\, n(y(\tau_1), \cdots)\, d\tau_1 \int_0^{t_2} \overset{*}{\Gamma}(t_2, \tau_2)\, \overset{*}{F}(\tau_2)\, d\tau_2 > +$$

$$+ \int_0^{t_1} e(t_1, \tau_1)\, n(y(\tau_1), \cdots)\, d\tau_1 \int_0^{t_2} \overset{*}{e}(t_2, \tau_2)\, \overset{*}{n}(y(\tau_2), \cdots)\, d\tau_2 >.$$

Thus the iterative process will, in the nonlinear case, add the terms (before averaging)

$$-\int_0^{t_2} F(t_1)\overset{*}{e}(t_2, \tau_2)\overset{*}{n}(y_0(\tau_2),\cdots)\,d\tau_2 -$$

$$-\int_0^{t_1} \overset{*}{F}(t_2)\,e(t_1, \tau_1)\,n(y_0(\tau_1),\cdots)\,d\tau_1 +$$

$$+\int_0^{t_1}\Gamma(t_1, \tau_1)\,F(\tau_1)\,d\tau_1\int_0^{t_2}\overset{*}{e}(t_2, \tau_2)\,\overset{*}{n}(y_0(\tau_2)\cdots)\,d\tau_2 +$$

$$+\int_0^{t_2}\Gamma(t_2, \tau_2)\,\overset{*}{F}(\tau_2)\,d\tau_2\int_0^{t_1} e(t_1, \tau_1)\,n(y_0(\tau_1),\cdots)\,d\tau_1 +$$

$$+\int_0^{t_1} e(t_1, \tau_1)\,n(y_0(\tau_1)\cdots)\,d\tau_1\int_0^{t_2}\overset{*}{e}(t_2, \tau_2)\,\overset{*}{n}(y_0(\tau_2)\cdots)\,d\tau_2$$

+ higher terms involving $y_1, y_2, \cdots$.

Suppose as an example again $n(y(\tau)) = \beta y^2(\tau)$ we will obtain (after averaging them) the additional terms to $R_y(t_1, t_2)$ of

$$-\int_0^{t_2} F(t_1)\,\overset{*}{e}(t_2, \tau_2)\,\beta y_0^2(\tau_2)\,d\tau_2 -$$

$$-\int_0^{t_1}\overset{*}{F}(t_2)\,e(t_1, \tau_1)\,\beta y_0^2(\tau_1)\,d\tau_1 +$$

$$+\int_0^{t_1}\Gamma(t_1, \tau_1)\,F(\tau_1)\,d\tau_1\int_0^{t_2}\overset{*}{e}(t_2, \tau_2)\,\beta y_0^2(\tau_2)\,d\tau_2 +$$

$$+\int_0^{t_2}\Gamma(t_2, \tau_2)\,\overset{*}{F}(\tau_2)\,d\tau_2\int_0^{t_1} e(t_1, \tau_1)\,\beta y_0^2(\tau_1)\,d\tau_1 +$$

$$+\int_0^{t_1} e(t_1, \tau_1)\,\beta y_0^2(\tau_1)\,d\tau_1\int_0^{t_2}\overset{*}{e}(t_2, \tau_2)\,\beta y_0^2(\tau_2)\,d\tau_2$$

and the higher terms of the iteration [1]. We see averages will all separate.

In the linear case, $\langle y\rangle$ was a two-fold integral of $y(t, \omega)$ over the measure spaces of the x process and the coefficients a_ν. In the nonlinear case, we have an additional integration because of the $b_\mu(t, \omega)$.

However, because of the statistical independence of coefficients and input and the iteration back to $F(t)$, the separations will occur as before the n term gives us an integral over the measure spece of the b_μ as discussed earlier [1]. A number of special cases have been considered and will appear elsewhere; the complete theory with applications will appear in a book edited by Richard Bellman.

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Գ. Ա. ԱԴՈՄՅԱՆ. Ոչ գծային ստոխաստիկ դիֆերենցիալ հավասարումների առաջին և երկրորդ կարգի $\langle y \rangle$ և $R_y(t_1, t_2)$ ստատիստիկաների ստացումը *(ամփոփում)*

Հոդվածում իտերացիայի մեթոդով ստացված են երկու կորելյացիայի ֆունկցիաներ $R_y(t_1, t_2)$ *և* $\langle y \rangle$ *միջին լուծումը ընդհանուր ոչ գծային ստոխաստիկ դիֆերենցիալ հավասարման համար։*

Այդ մեթոդը հանդիսանում է Ադոմյանի (գծային հավասարման համար) մեթոդի ընդհանուրացումը։

Г. А. АДОМЯН. *Получение статистик первого и второго порядка,* $\langle y \rangle$ *и* $R_y(t_1, t_2)$ *для нелинейных стохастических дифференциальных уравнений* (резюме)

Для общего нелинейного стохастического дифференциального уравнения среднее решение $\langle y \rangle$ и две точечные функции корреляции $R_y(t_1, t_2)$ получены методом итерации, обобщающего метод Адомяна для линейного стохастического уравнения.

REFERENCES

1. *G. Adomian.* Nonlinear Stochastic Differential Equations, Jour. of Mathematical Analysis and Applications, in publication.
2. *G. Adomian.* Stochastic Systems Modelling, Third International Congress of Cybernetics and Systems, Bucharest, Romania, 25—29 August, 1975 (Proceedings to be published in book form).
3. *G. Adomian.* Random Operator Equations in Mathematical Physics I, Journal of Mathematical Physics, vol. 11, no. 3. pp. 1069—1084, March, 1970.
4. *G. Adomian.* Random Operator Equations in Mathematical Physics II, Journal of Mathematical Physics, vol. 12, no. 9, pp. 1944—48, September, 1971.
5. *G. Adomian.* The Closure Approximation in the Hierarchy Equations, Journal of Statistical Physics, vol. 3, no. 2, Summer, 1971.
6. *L. H. Sibul.* Application of Linear Stochastic Operator Theory, Ph. D. Dissertation, Pennsylvania State University, 1968.