ՀԱՅԿԱԿԱՆ ՍՍՀ ԳԻՏՈՒԹՅՈՒՆՆԵՐԻ ԱԿԱԴԵՄԻԱՅԻ ՏԵՂԵԿԱԳԻՐ
ИЗВЕСТИЯ АКАДЕМИИ НАУК АРМЯНСКОЙ ССР

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SET-THEORETICAL CANONICAL MODELS

In what follows ZF denotes a first order theory whose only non-logical symbol is the elementhood binary predicate $\in(x, y)$ and whose axioms are the usual Zermelo-Fraenkel settheoretical axioms (also denoted by ZF) of Extensionality (E), Powerset (P), Sumset (S), Infinity (I), Choice (C) and the axiom scheme of Replacement (R).

By a model $(M, \in)$ for $\in(x, y)$ we mean a domain M of individuals ("sets") and an assignment of *Truth* "1" or *Falsehood* "0" to every atomic formula $\in(a, b)$ where a and b are individuals ("sets") of M.

By a dyadic sequence we mean a (finite or transfinite) sequence whose terms are 0 or 1. Also, as expected, $\in(x, y)$ is more often denoted by $x \in y$.

Definition. *Let u be an ordinal number. A model $(M, \in)$ is called a set-theoretical canonical model of type u provided its domain M is a family $((d_i^j)_{i<u})_{j<u}$ of type u of dyadic sequences $(d_i^j)_{i<u}$ of type u and where $\in$ is defined by*:

$$(d_i^k)_{i<u} \in (d_i^j)_{i<u} \quad \textit{if and only if} \quad d_k^j = 1. \tag{1}$$

In what follows we refer to "set-theoretical canonical model" simply as "*canonical model*". Moreover, when no confusion is likely to arise, we denote $(d_i^j)_{i<u}$ by (d_i^j) and we denote a canonical model $(M, \in)$ by M.

Although $\omega_u = \aleph_u$ for every ordinal u, we use ω_u notation in the cases pertaining primarily to order and we use $\aleph_u$ notation in the cases pertaining primarily to cardinality. As usual, we denote ω_0 by ω. Also, in Propositions 5 and 6 we use the Generalized Continuum Hypothesis (GCH), i. e.,

$$2^{\aleph_u} = \aleph_{u+1} = \omega_{u+1} \tag{2}$$

Based on the above Definition, we prove:

Lemma 1. *In every canonical model every two distinct dyadic sequences represent distinct sets (individuals).*

Proof. Let $(d_i^j) \neq (d_i^h)$. Hence, for some ordinal k it must be the case that, say, $d_k^j = 1$ and $d_k^h = 0$. But then from (1) it follows that $(d_i^k) \in (d_i^j)$ whereas $(d_i^k) \bar{\in} (d_i^h)$. Thus, the sets (d_i^j) and (d_i^h) are distinct.

In view of Lemma 1 we have:

Proposition 1. *In every canonical model the axiom of Extensionality is valid.*

Next, we give a necessary and sufficient condition for the existence of a special kind of canonical models.

Proposition 2. *Let u be an ordinal number and $\bar{\bar{v}}$ be a cardinal number. Then there exists a canonical model M of type u whose domain consists of all dyadic sequences of type u each having less than $\bar{\bar{v}}$ ones, if and only if*

$$\bar{\bar{u}} = \sum_{\bar{\bar{c}} < \bar{\bar{v}}} \bar{\bar{u}}^{\bar{\bar{c}}}. \tag{3}$$

Proof. If M exists then by Definition 1, we have $\overline{\overline{M}} = \bar{\bar{u}}$. On the other hand, clearly, $\overline{\overline{M}}$ is equal to the cardinality of the set of all functions from a cardinal $\bar{\bar{c}}$ into u such that $\bar{\bar{c}} < \bar{\bar{v}}$. Hence, (3) holds.

Conversely, let (3) hold. Let M be the set of all functions from a cardinal $\bar{\bar{c}}$ into u such that $\bar{\bar{c}} < \bar{\bar{v}}$. But then from (3) it follows that $\overline{\overline{M}} = \bar{\bar{u}}$. Consequently, in view of Definition 1, it is clear that M can serve as a domain for a canonical model of type u.

Proposition 3. *Let M be a canonical model of ordinal type u whose domain consists of all dyadic sequences of type u each having less than $\bar{\bar{v}}$ ones. Then every set of M has less than $\bar{\bar{v}}$ elements. Moreover, for every collection of less than $\bar{\bar{v}}$ sets*

$$(d_i^a)_{i<u},\ (d_i^b)_{i<u},\ (d_i^c)_{i<u}, \cdots \tag{4}$$

of M there exists a set $(d_i^j)_{i<u}$ of M whose elements are precisely the sets of M which are listed in (4).

Proof. The fact that every element of M has less than $\bar{\bar{v}}$ elements follows directly from (1).

Now, let us consider the dyadic sequence $(d_i^j)_{i<u}$ such that

$$d_i^j = 1 \ \textit{ if and only if } \ i = a,\ i = b,\ i = c, \cdots \tag{5}$$

But then from (5) it follows that $(d_i^j)_{i<u}$ is a dyadic sequence of type u having less than $\bar{\bar{v}}$ ones. Hence, $(d_i^j)_{i<u}$ is a set of model M.

On the other hand, from (1) and (5) it follows that $(d_i^k) \in (d_i^j)$ if and only if $d_k^j = 1$, if and only if $k = a$, $k = b$, $k = c, \cdots$ Thus, by (1), the elements of (d_i^j) are precisely the sets of M which are listed in (4).

Below, we give an example of a canonical model in which, except for the axiom of Infinity, all other axioms of ZF are valid.

Since $\aleph_0 = \sum_{n<\omega} \aleph_0^n$ from (3) it follows that a canonical model such as described in the following Proposition, exists.

Proposition 4. *Every canonical model A of type ω whose domain consists of all dyadic sequences of type ω each having less than $\aleph_0$ (i. e., no or only finitely many) ones is a model for $ZF—I$.*

Proof. Clearly, to prove the Proposition, in view of Proposition 1, it is enough to show that axioms P, S, C and axiom scheme R are valid in a canonical model A described in the Proposition and that axiom I is not valid in A.

Let s be a set in A.

By Proposition 3 we see that in A there exist only finitely many subsets $s_1, \cdots, s_n$ of s. But then, from Proposition 3 it follows that in A there exists a set $P_A(s)$ whose elements are precisely $s_1, \cdots, s_n$. Thus, in A every set s has a powerset $P_A(s)$. Hence, axiom P is valid in A.

Similarly, by Proposition 3 we see that in A there exist no or only finitely many elements $e_1, \cdots, e_k$ of elements of s. But then, from Proposition 3 it follows that in A there exists a set $U_A s$ which is the emtpy set (the zero sequence of type ω) or whose elements are precisely $e_1, \cdots, e_k$. Thus, in A every set s has a sumset $U_A s$. Hence, axiom S is valid in A.

Let $P(x, y)$ be a set-theoretical binary predicate functional in x on s in A. By Proposition 3 we see that in A there exist no or only finitely many sets $c_1, \cdots, c_m$ such that $P(a_i, c_i)$ is true in A for some element a_i of s. But then, from Proposition 3 it follows that in A there exists a set which is the empty set or whose elements are precisely $c_1, \cdots, c_m$. Thus, axiom scheme R is valid in A.

Let d be a disjointed (i. e., whose elements are pairwise disjoint) nonempty set in A. By Proposition 3 we see that in A there exist no or only finitely many sets $r_1, \cdots, r_n$ which can be unique representatives of elements of d. But then from Proposition 3 it follows that in A there exists a set which is the empty set or whose elements are precisely $r_1, \cdots, r_n$. Thus, in A there exists a choice set of d. Hence axiom C is valid in A.

On the other hand, from Proposition 3 it follows that in A there exists no set t such that $\varnothing \in t$ and if $x \in t$ then $(x \cup \{x\}) \in t$, where $\varnothing$ is the zero sequence of type ω. Hence, in A axiom I is not valid.

Thus, Proposition 4 is proved.

Below, under the assumption of GCH, we give an example of a canonical model in which, except for the axiom of Powerset, all other axioms of ZF are valid.

Let us observe that (2) implies $\aleph_1 = \sum_{\bar{k} < \omega_1} \aleph_1^{\bar{k}}$. Therefore, from (3) it follows that a canonical model such as described in the following Proposition, exists.

Proposition 5. *Every canonical model B of type ω_1 whose domain consists of all dyadic sequences of type ω_1 each having less than $\aleph_1$ ones, is a model for $ZF - P$.*

Proof. Clearly, to prove the Proposition, in view of Proposition 1, it is enough to show that axioms S, C, I and axiom scheme R are valid in a canonical model B described in the Proposition and that axiom P is not valid in B.

Let h be a set in B such that h has $\aleph_0$ ones. By Proposition 3, in view of (2), we see that in B there exist $2^{\aleph_0} = \aleph_1$ subsets of h. However, since every set in B has less than $\aleph_1$ ones, the set h has no powerset in B. Hence axiom P is not valid in B.

Let s be a set in B.

Let us recall that the product of two cardinals each less than $\aleph_1$ is less than $\aleph_1$. Thus, by Proposition 3 we see that in B there exist no or only less than $\aleph_1$ elements $e_1, \cdots$ of elements of s. But then, from Proposition 3 it follows that in B the sumset of s exists. Hence, axiom S is valid in B.

Let $P(x, y)$ be a set-theoretical binary predicate functional in x on s in B. By Proposition 3 we see that in B there exist no or only less than $\aleph_1$ sets $c_1, \cdots$ such that $P(a_i, c_i)$ is true in B for some element a_i of s. But then, from Proposition 3 it follows that axiom scheme R is valid in B.

Let us observe that if x is a set in B then in view of Proposition 3 both $\{x\}$ and $x \cup \{x\}$ is a set in B. But then since $\aleph_0 < \aleph_1$, from Proposition 3 it follows that in B there exists a set whose elements are denumerably many sets $\varnothing$, $\{\varnothing\}$, $\{\varnothing, \{\varnothing\}\}, \cdots$, where $\varnothing$ is the zero sequence of type ω_1. Hence axiom I is valid in B.

Let d be a nonempty disjointed set in B. By Proposition 3 we see that in B there exist no or only less than $\aleph_1$ sets $r_1, \cdots$ which can be unique representatives of elements of d. But then from Proposition 3 it follows that axiom C is valid in B.

Thus, Proposition 5 is proved.

Below, again under the assumption of GCH, we give an example of a canonical model in which, except for the axiom of Sumset, all other axioms of ZF are valid.

Let us observe that (2) implies $\aleph_{\omega+1} = \sum\limits_{\bar{k} < \omega_\omega} \aleph_{\omega+1}^{\bar{k}}$. Therefore, from (3) it follows that a canonical model such as described in the following Proposition, exists.

P r o p o s i t i o n 6. *Every canonical model G of type $\omega_{\omega+1}$ whose domain consists of all dyadic sequences of type $\omega_{\omega+1}$ each having less than $\aleph_\omega$ ones, is a model for $ZF-S$.*

P r o o f. Clearly, to prove the Proposition, in view of Proposition 1, it is enough to show that axioms P, C, I and axiom scheme R are valid in a canonical model G described in the Proposition and that axiom S is not valid in G.

Since $\aleph_i < \aleph_\omega$ for every $i < \omega$, in view of Proposition 3, we see that each of the following denumerebly many sets

$$\aleph_0, \aleph_1, \cdots, \aleph_i, \cdots \quad with \quad i < \omega \tag{6}$$

is a set in G. Also since $\aleph_0 < \aleph_\omega$, again by Proposition 3 we see that in G there exists a set g whose elements are the sets listed in (6).

However, since $\aleph_\omega = \bigcup_{i<\omega} \aleph_i$ and since every set in G has less than $\aleph_\omega$ ones, the set g has no sumset in G. Hence axiom S is not valid in G.

Let s be a set in G.

Let us observe that in view of (2) we have $2^{\aleph_i} = \aleph_{i+1} < \aleph_\omega$ for every $i < \omega$. Thus, in G there exist only I less than $\aleph_\omega$ subsets $s_1, \cdots$ of s. But then from Proposition 3 it follows that in G the powerset of s exists. Hence axiom P is valid in G.

Let $P(x, y)$ be a set-theoretical binary predicate functional in x on s in G. By Proposition 3 we see that in G there exist no or only less than $\aleph_\omega$ sets $c_i, \cdots$ such that $P(a_i, c_i)$ is true in G for some element a_i of s. But then, from Proposition 3 it follows that axiom scheme R is valid in G.

Again, as in the case of the proof of Proposition 5, it can readily be verified that axioms I and C are valid in G.

Thus, Proposition 6 is proved.

Based on the assumption of the existence of a strongly inaccessible cardinal $\aleph_s$, we give below an example of a canonical model in which all axioms of ZF are valid.

Since for a strongly inaccessible cardinal $\aleph_s$ we have $\aleph_s = \sum_{\xi<\omega_s} \aleph_s^{\xi}$, from (3) it follows that a canonical model such as described in the following Proposition, exists.

P r o p o s i t i o n 7. *Let $\aleph_s$ be a strongly inaccessible cardinal. Every canonical model H of type $\aleph_s$ whose domain consists of all dyadic sequences of type $\aleph_s$ each having less than $\aleph_s$ ones, is a model for ZF.*

P r o o f. Since $\aleph_s$ is a strongly cardinal, $2^{\aleph_u} < \aleph_s$ for every $u < s$. But then, as in the proof of Proposition 6, we see that axiom P is valid in H.

Again, since $\aleph_s$ is a strongly inaccessible cardinal, $\bigcup_{i<v} \overline{\overline{c_i}} < \aleph_s$ for $v < \aleph_s$ and $c_i < \aleph_s$ for every $i < v$. But then, as in the proof of Proposition 4, or that of Proposition 5, we see that axiom S is valid in H.

As in the case of the proof of Proposition 5, it can readily be verified that axioms E, I, C and axiom scheme R are also valid in H.

Thus, Proposition 7 is proved.

R e m a r k. We observe that the independence of each of the axioms I, P, S from the remaining axioms of ZF is easily established by means of the canonical models A, B, G. Also (under the assumption of the existence of a strongly inaccessible cardinal) the consistency of the axioms of ZF is readily established by means of the canonical model H.

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Ա. ԱԲՅԱՆ. Բազմությունների տեսության կանոնական տիպարներ *(ամփոփում)*

Սահմանվում է կանոնական տիպարի գաղափարը բազմությունների տեսության աքսիոմատիկ սիստեմների համար։ Ապացուցվում է, որ

1) եթե Ցերմելո-Ֆրենկելի տեսության համապատասխանող ենթասիստեմները անհակասելի են, ապա կարելի է հիմնավորել անվերջության աքսիոմի անկախությունը, բազմությունների գումարի աքսիոմի անկախությունը և ենթաբազմությունների բազմության աքսիոմի անկախությունը Ցերմելո-Ֆրենկելի սիստեմի մնացած աքսիոմներից կանոնական տիպարների միջոցով.

2) եթե Ցերմելո-Ֆրենկելի սիստեմը անհակասելի է ու գոյություն ունի ոչ հասանելի կարդինալ թիվ, ապա գոյություն ունի Ցերմելո-Ֆրենկելի սիստեմի կանոնական մի տիպար։

А. АБИЯН. *Канонические модели теории множеств* (резюме)

Вводится понятие канонической модели для аксиоматических систем теории множеств. Доказывается, что

(1) если соответствующие подсистемы теории Цермело-Френкеля непротиворечивы, то при помощи канонических моделей можно установить независимость аксиомы бесконечности, независимость аксиомы суммы множеств и независимость аксиомы множества подмножеств от остальных аксиом системы Цермело-Френкеля,

(2) если система Цермело-Френкеля непротиворечива и существует недостижимое кардинальное число, то существует каноническая модель системы Цермело-Френкеля.