ՀԱՅԿԱԿԱՆ ՍՍՀ ԳԻՏՈՒԹՅՈՒՆՆԵՐԻ ԱԿԱԴԵՄԻԱՅԻ ՏԵՂԵԿԱԳԻՐ
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SIGNAL PROCESSING IN A RANDOMLY TIME VARYING SYSTEM

Abstract: Signal processing systems, which are deterministic with inexactly specified parameters, or, stochastic because system parameters vary randomly in time, are treated in terms of a stochastic transformation or operator theory which determines statistical measures of the system output in terms of statistical measures of the system input, and stochastic Green's functions, or system functions, involving statistics of the random parameters. The earliest application was the determination of the spectral density of a randomly sampled random process. Extensions permit the determination of two point correlation functions of the system output for systems involving any linear operations, e. g., for the solution process of an ordinary or partial differential equation with stochastic process coefficients, boundary values, or forcing functions without the usual averaging, perturbation, or special process assumptions. Optimal processing or control can be considered as applications of the theory.

Randomly time varying systems arise in a number of ways. Practically all radio transmission channels can be treated as randomly time varying linear filters. Adaptive systems may be viewed as stochastic systems as parameters adapt to a changing environment. A random signal process may be sampled at random intervals of time for various reasons, e. g., in a multiloop system, we may have random sampling because of varying time delays for control computations for a particular feedback loop. In early work of the author [1], and in still earlier classified work, expressions were found for the first time for the spectral density of a randomly sampled random radar signal process. It became clear that the processing which resulted in the random sampling could be viewed as a „stochastic filter“ of a very general type [1]. It is unfortunate in this connection that terminology such as stochastic filter, stochastic system, stochastic control system, stochastic matrix, and stochastic differential equation have been widely used in much less general and appropriate connections). The expression for the spectral density of the randomly sampled random process reduced to known results for sampled processes when the sampling became regular, i. e., where the system became deterministic. This same investigation determined the correlation of the solution process of a first order differential equation of the form $\dot{y}+\xi y=\eta$ where $\xi(t)$ and $\eta(t)$ were Gaussian stochastic processes and compared the result with Tikhonov's earlier solution when it became available in translation. Basically, the investigation [1, 2, 3, 4] was an attempt to combine linear system theory or linear operator theory and the theory of probability and stochastic processes to consider systems involving stochastic behavior—i. e., a theory of „stochastic systems" or „stochastic operators". Thus, the random sampling operator could be viewed as a stochastic operator which randomly transformed (sampled) the original random process. The concept of a „stochastic Green's function" was introduced [1, 2, 3, 4] for desired statistical measures of the output process; References 4 or 7 suggest and develop an iterative procedure for systems modelled by stochastic differential equations involving stochastic process coefficients $a_\nu(t, \omega)$, $t \in T$, $\omega \in (\Omega, F, \mu)$, $\nu=1, 2, \cdots, n$. Such a case is more interesting than cases where randomness appears only in the forcing function or the boundary conditions since we have a stochastic (differential) operator. More recent work [5, 8, 9, 10] deals with stochastic partial differential operators as well and consequently a great number of physical applications. Physically the operator may represent a filter, a communication channel, a measurement, or a scattering medium. We write $y=\mathrm{H}[x]$ where H is the stochastic operator, x is the input process and y the output process. Thus the response $y(\xi)=\mathrm{H}x(\eta)=\int_{-\infty}^{\infty} h(\xi, \eta)\, x(\eta)\, dy$ where h is a random Green's function and y and x are random functions i. e., the output and input of a stochastic filter. Then, if the input and the parameters of H are statistically independent, the two point correlation $R_y(t_1, t_2)$ for the stochastic process y can be given in terms of the correlation R_x of the stochastic process input x if an appropriate quantity called a stochastic Green's function (for correlations) can be defined. The result holds also for the random function solution y of the differential equation $\mathrm{L}y=x$ where L is given by $\sum_{\nu=0}^{n} a_\nu(t, \omega)\, d^\nu/dt^\nu$. Supposing $\mathrm{L}=L+\mathrm{R}$ where L is the deterministic part of the operator and R the random part, i. e., each coefficient process $a_\nu=\langle a_\nu\rangle+\alpha_\nu$ where α_ν is a zero mean random process, then $L=\sum_{\nu=0}^{n}\langle a_\nu(t)\rangle d^\nu/dt^\nu$ and

$$\mathrm{R}=\sum_{\nu=0}^{n} \alpha_\nu(t)\, d^\nu/dt^\nu, \text{ (or } \sum \alpha_\nu(t, \omega)\, d^\nu/dt^\nu).$$

Adomian has defined the „stochastic Green's function", when the the chosen „statistical measure" is the two-point correlation, as the kernel G_H of the integral

$$R_y(t_1, t_2) = \iint G_H(t_1, t_2, \sigma_1, \sigma_2)\, R_x(\sigma_1, \sigma_2)\, d\sigma_1\, d\sigma_2.$$

In earlier work, the quantity G_H was given in terms of a random Green's function $h(t, \tau)$ by the relation

$$G_H(t_1, t_2, \sigma_1, \sigma_2) = \langle h(t_1, \sigma_1)\, \dot{h}(t_2, \sigma_2)\rangle.$$

Later work [8] has further defined the random Green's function above by

$$h(t, \sigma) = G(t, \sigma) - \int \Gamma(t, \tau)\, G(\tau, \sigma)\, d\tau$$

with Γ a resolvent kernel, introduced by Sibul [5], which allows a convenient reformulation of the iterative approach. It is defined by

$$\Gamma(t, \tau) = \sum_{m=0}^{\infty} (-1)^m K_{m+1}(t, \tau),$$

where

$$K_m(t, \tau) = \int K(t, \tau_1)\, K_{m-1}(\tau_1, \tau)\, d\tau_1$$

with $K_1 = K$. Supposing G and K are known, Γ and h can be determined, and G_H can then be calculated as

$$G_H = G(t_1, \sigma_1)\, \dot{G}(t_2, \sigma_2) - \int \langle \Gamma(t_1, \tau_1)\rangle\, G(\tau_1, \sigma_1)\, \dot{G}(t_2, \sigma_2)\, d\tau_1 -$$

$$- \int \langle \dot{\Gamma}(t_2, \tau_2)\rangle\, G(t_1, \sigma_1)\, \dot{G}(\tau_2, \sigma_2)\, d\tau_2 +$$

$$+ \iint \langle \Gamma(t_1, \tau_1)\, \dot{\Gamma}(t_2, \tau_2)\rangle\, G(\tau_1, \sigma_1)\, \dot{G}(\tau_2, \sigma_2)\, d\tau_1\, d\tau_2.$$

The quantity $G(t, \tau)$ is the ordinary or deterministic Green's function for the operator L obtained by separating a stochastic operator $\mathbf{L}$ into a deterministic part L and a random part $\mathbf{R}$. The quantity K is the Green's function corresponding to $L^{-1}\mathbf{R} = G(t, \tau) \sum_{\nu=0}^{n} \alpha_\nu(t, \omega)\, d^\nu/dt^\nu$ and can be obtained by repeated integrations by parts or by use of Green's formula in terms of the adjoint operator as

$$K(t, \tau) = \sum_{\nu=0}^{n} (-1)^\nu d^\nu/dt^\nu\, [\alpha_\nu(\tau)\, G(t, \tau)]$$

where α_ν is the random fluctuation in each coefficient process. If $\mathbf{R}$ involves no derivatives, i. e., if $\mathbf{L} = L + \alpha(t)$, then $K = G(t, \tau)\, \alpha(\tau)$.

Thus the stochastic Green's functions for any desired statistical measure can be computed, e. g., the two point correlation $R(t_1, t_2)$ above for the general nonstationary cases, or $R(\tau)$ for the very special case of stationary transformation of a stationary process, or $\Phi(f)$, the spectral density. In the random sampling example of the author's dissertation, $\Phi_y(f) = \int K(s, f)\,\Phi_x(s)\,ds$ gave the spectral density of a randomly sampled random process whose spectral density before sampling was Φ_x. The stochastic Green's function $K(s, f)$ was calculated for any probability law for the sampling.

Recent work [8] has shown that in the event perturbation theory is adequate to deal with the randomness involved, the results for perturbation theory are easily specialized from the general expression for $G_H(t_1, t_2, \tau_1, \tau_2)$ by letting $\mathrm{R} = \epsilon \mathrm{L}_1$, with $\langle \mathrm{L}_1 \rangle = 0$. However, we are not limited to perturbation results nor special processes such as white noise and we make no closure approximations in the Boguliubov or hierarchy manner [6]. Thus for the stochastic differential equation $\mathrm{L}y = x$ where $x(t, \omega')$, $t \in T$, $\omega' \in \Omega'$ is a stochastic process (Ω' has a σ-algebra defined and a measure) and $\mathrm{L}(t, \omega) = \sum_{\nu=0}^{n} a_\nu(t, \omega)\, d^\nu/dt^\nu$ is an operator involving the stochastic process coefficient $a_\nu(t, \omega)$, $t \in T$, $\omega \in \Omega$, the expected solution $\langle y \rangle = \int_\Omega \int_{\Omega'} y(t, \omega, \omega')\, d\mu(\omega)\, d\mu'(\omega')$ where μ and μ' are the appropriate measures on Ω and Ω' [6]. $\langle y \rangle$ is not in general, except for singular measures, equal to $L^{-1}\langle x \rangle$ where $\langle x \rangle = \int_{\Omega'} x(t, \omega')\, d\mu'(\omega')$ and $L[\cdot] = \langle \mathrm{L}(t, \omega)[\cdot] \rangle = \int_\Omega \mathrm{L}(t, \omega)[\cdot]\, d\mu(\omega)$. Equivalently, $\langle \mathrm{L}y \rangle \neq \langle \mathrm{L} \rangle \langle y \rangle$, obviously, and, similarly, quantities of the form $\langle \mathrm{R}L^{-1}\mathrm{R}L^{-1}\mathrm{R} \cdots \mathrm{R}L^{-1}\mathrm{R}y \rangle$ cannot be separated into $\langle \mathrm{R}L^{-1}\mathrm{R}L^{-1} \cdot \mathrm{R} \cdots \mathrm{R}L^{-1}\mathrm{R} \rangle \langle y \rangle$ and the error in making such an approximation is now determinable.

One can further consider a linear system with input $x(t)$, whose output state vector $y(t)$ is n dimensional, described by the system equation $\dot{y} = f(y, x, t) = A(t)\,y + x(t)$ where A is an $n \times n$ stochastic matrix (not the usual probability transition matrices but matrices with randomly time varying elements $a_{ij}(t, \omega)$, $t \in T$, $\omega \in (\Omega, \mathrm{F}, \mu)$, a probability space) $x(t)$ is the product of an $n \times r$ stochastic matrix $B(t)$ with a $r \times 1$ matrix $u(t)$, a convenient formulation for stochastic control applications. The deterministic Green's function G will now become a Green's matrix for the deterministic part of the operator, i. e., a state transition matrix. This work is being published.

Finally, generalizations to partial stochastic differential equations and wave equations with stochastic d'Alembertian operator are possible Consider the scalar wave equation

$$\nabla^2 y(\bar{r}, t, \omega) - \frac{\partial^2}{\partial t^2}\left[\frac{1}{c^2} + \alpha(\bar{r}, t, \omega)\right] y(\bar{r}, t, \omega) = x(\bar{r}, t, \omega)$$

where $t \in T$ represents time, $\bar{r} \in R^3$, $\omega \in \Omega$ on a probability space (Ω, F, μ). The quantities x and α, and consequently y, are all stochastic processes (s. p.) dependent on space position and time, i. e., random fields. The deterministic operator L is given by the ordinary d'Alembertian $\nabla^2-(1/c^2)\,\partial^2/\partial t^2$ and the random part of the stochastic operator by $R=(\partial^2/\partial t^2)\,\alpha$.

Litting $L^{-1}x=F$ we write the above as

$$y(\bar{r}, t, \omega)=F(\bar{r}, t, \omega)+L^{-1}(\partial^2/\partial t^2)\,\alpha(\bar{r}, t, \omega)\,y(\bar{r}, t, \omega)$$

where L^{-1} is the inverse of the operator $\nabla^2-(1/c^2)(\partial^2/\partial t^2)$. Denoting the Green's function by $G(\bar{r}, t, \tau)$ or $G(t, \tau)$*, the last term is rewritten as $\int\int G(t, \tau)\,(\partial^2/\partial t^2)\,\alpha(\bar{r}, \tau)\,y(\bar{r}, \tau)\,dv d\tau$, i. e., the random operator R is $-(\partial^2/\partial t^2)\,\alpha(t)$.

After integrating twice by parts we can write**

$$y(\bar{r}, t, \omega)=F(\bar{r}, t, \omega)+\int_{-\infty}^{\infty}\int_{v}[\partial^2 G(\bar{r}, t, \tau)/\partial\tau^2]\,\alpha(\bar{r}, \tau, \omega)\,y(\bar{r}, \tau, \omega)\,dv d\tau$$

or more simply

$$y(t)=F(t)+\int\int[\partial^2 G(t, \tau)/\partial\tau^2]\,\alpha(\tau)\,y(\tau)\,dv d\tau.$$

(Suppressing the ω and $\bar{r}$ for notational convenience) if quantities

$$G(t, \tau)\frac{\partial}{\partial\tau}\alpha(\tau)\,y(\tau)$$

and

$$\frac{\partial G(t, \tau)}{\partial\tau}\alpha(\tau)\,y(\tau)$$

vanish as $t\to\pm\infty$ which we suppose does happen either because of the initial conditions (G and G' zero) or because α is a reducible-to-stationary stochastic process.

We write $\Gamma(t, \tau)=\sum_{m=0}^{\infty}(-1)^m K_{m+1}(t, \tau)$ with $K_1=K$ as before,

$$K(t, \tau)=\frac{\partial^2 G(t, \tau)}{\partial\tau^2}\,\alpha(\tau),$$

$$K_2(t, \tau)=\int\int K(t, \tau_1)\,K(\tau_1, \tau)\,dv d\tau_1=$$

The Green's function, for L^{-1}, is found from $\square^3 G=\delta(x_1-x_1')\,\delta(x_2-x_2')\,\delta(x_3-x_3')\,\delta(t-t')$ where $G(x_1, x_2, x_3, t)$, or $G(\bar{r}, t)$ satisfies the equation and initial conditions $G(\bar{r}, 0)=G_t(\bar{r}, 0)=0$ and such that G is bounded for all t as either x_1, x_2, $x_3\to\pm\infty$.

** Products of generalized functions are undefined unless they are in different dimensions, but we always do an integration in between so the iteration procedure here will still be valid.

$$= \iint \frac{\partial^2 G(t, \tau_1)}{\partial \tau_1^2} \alpha(\tau_1) \frac{\partial^2 G(\tau_1, \tau)}{\partial \tau^2} \alpha(\tau)\, dv d\tau_1,$$

$$K_3(t, \tau) = \iint K(t, \tau_1) K_2(\tau_1, \tau)\, dv d\tau_1, \quad \text{etc.}$$

$$\Gamma(t, \tau) = K(t, \tau) - K_2(t, \tau) + K_3(t, \tau) + \cdots =$$

$$= K(t, \tau) - \iint K(t, \tau_1) K(\tau_1, \tau)\, dv_1 d\tau_1 + \iiiint K(t, \tau_1) K(\tau_1, \tau_2) \times$$

$$\times K(\tau_2, \tau)\, dv_1 dv_2 d\tau_1 d\tau_2 - \cdots =$$

$$= \frac{\partial^2 G(t, \tau)}{\partial \tau^2} \alpha(\tau) - \iint \frac{\partial^2 G(t, \tau_1)}{\partial \tau_1^2} \frac{\partial^2 G(\tau_1, \tau)}{\partial \tau^2} \alpha(\tau_1) \alpha(\tau)\, dv_1 d\tau_1 +$$

$$+ \iiiint \frac{\partial^2 G(t, \tau_1)}{\partial \tau_1^2} \frac{\partial^2 G(\tau_1, \tau_2)}{\partial \tau_2^2} \frac{\partial^2 G(\tau_2, \tau)}{\partial \tau^2} \alpha(\tau_1) \alpha(\tau_2) \alpha(\tau) \times$$

$$\times dv_1 dv_2 d\tau_1 d\tau_2 - \cdots .$$

Thus we can determine the s. g. t. (stochastic Green's function) either for the spectral density s. m. (statistical measure) if it exists, or immediately the more general two point correlation (and mutual coherence functions) thus

$$R_y(t_1, t_2) = \iint G_H(t_1, t_2, \sigma_1, \sigma_2) R_x(\sigma_1, \sigma_2)\, d\sigma_1 d\sigma_2$$

where G_H is found from $h(t, \tau)$, the random Green's function.

The first term of G_H (which we do not write out) shows the results for waves propagating in a *deterministic* medium. The other terms of G_H involving statistics of Γ show the effects of spectral spreading due to the stochastic medium. These are the terms lost by a monochromatic assumption. The calculation for a specific case presents considerable difficulty but can be made knowing the statistics (i. e., s. m.) of α (such as correlation if α is gaussian). This problem has been considered substantially only by Sibul*).

The simplest calculation of the spectral spreading should result (if one determines first that stationarity exists in the solution process or wave function—the necessary and sufficient conditions will be discussed in another paper) if the spectral density is calculated from [1]

$$\Phi_y(f) = \int K_H(s, f)\, \Phi_x(s)\, ds$$

where the stochastic Green's function K_H is

$$K_H(s, f) = \iiint \langle h(t, \tau)\, h^*(t+\beta, \tau+\alpha) \rangle \exp\{2\pi i f \beta\} \times$$

$$\times \exp\{-2\pi i s \alpha\}\, d\tau d\beta d\alpha.$$

* A following paper by Sibul and Adomian will calculate the spectral spreading and the mutual coherence and related quantities.

In the more general nonstationary case, we make the time domain iterative treatment, and if we assume gaussian behavior for the index of refraction, we observe the odd terms vanish in the series (terms involving products of odd numbers of α's) and the even terms are negative. Thus in forming products $y(t_1)\, y(t_2)$ for correlations, the contribution of the spectral spreading or non-monochromatic terms of G_H (i. e., the last three of the four term expression) are all positive.

Our procedure involves no assumption of statistical independence of the solution s. p. or wave function and the stochastic index of refraction and makes no closure approximations [6].

The first application of this work was the processing of a signal by a „stochastic filter" which randomly sampled the signal at intervals of time governed by a probability law. Work on optimization of stochastic systems and numerous other applications is immediately suggested.

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Գ. ԱԴՈՄՅԱՆ. Ազդանշանի ձևափոխումը ժամանակի ընթացքում պատահական ձևով փոփոխվող սիստեմներում *(ամփոփում)*

Դիտարկվում է սիստեմի ելքում պրոցեսի բաշխումը նկարագրելու խնդիրը սիստեմի մուտքում պրոցեսի բաշխման և սիստեմը բնութագրող պատահական Գրինի ֆունկցիայի տերմիններով։ Խնդիրը բերվում է ստոխաստիկ դիֆերենցիալ հավասարման լուծմանը։ Ստացված է նաև երկկետանի կորելյացիոն ֆունկցիայի արտահայտությունը։

Г. АДОМЯН. *Преобразование сигнала в системах, случайно меняющихся во времени* (резюме)

С помощью теории стохастических операторов рассматриваются преобразующие сигнал детерминистические системы; системы, параметры которых определены неточно или стохастически, а также системы, параметры которых случайно меняются во времени.

Решается задача определения распределения процесса на выходе системы в терминах распределения процесса на ее входе и стохастической функции Грина, описывающей работу системы. Впервые эта теория была применена при определении спектральной плотности случайной выборки значений случайного процесса. В дальнейшем удалось получить двухточечную функцию корреляции для процесса на выходе системы, включающей любые линейные операции, т. е. для решения обычного или с частными производными дифференциального уравнения с коэффициентами, граничными условиями или правыми частями, задаваемыми случайными процессами. При этом удается избежать предположений о частном виде процессов, а также процедур возмущения усреднения. Оптимальное преобразование процесса или его регулирование могут рассматриваться как применения этой теории.

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