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SOME APPLICATIONS OF ARAKÉLIAN'S APPROXIMATION THEOREMS TO THE THEORY OF CLUSTER SETS

The purpose of this short note is to display the great strength of Arakélian's generalizations of Mergelian's beautiful theorem [9] (see also [11]).

During the fifties F. Bagemihl and W. Seidel developed techniques for studying boundary behaviour based on approximation theory. The versatility of these techniques has been considerably enhanced by the introduction by K. Barth and W. Schneider of the "pole sweeping" method.

We believe that most of the cluster-set results proved thus far by these methods could be more easily obtained by the use of Arakélian's theorems. We content ourselves with two examples. First we prove an extended version of the Bagemihl-Seidel-Rudin theorems on the existence of holomorphic functions with prescribed asymptotic behaviour. Our second example, meant to show that Arakélian's theorem encompasses the pole sweeping technique, is an extended version of Schneider's theorem [12] on the existence of an unbounded holomorphic function bounded in a prescribed "Schneider noodle".

§ 1. Arakélian's theorems

Let $(\overline{C}, \chi)$ be the Riemann sphere endowed with the chordal metric. Let D be a proper domain in $\overline{C}$, and let D^* denote the one-point compactification of D. We denote by ∂D the boundary of D in $\overline{C}$. Following Arakélian we introduce the following notion.

1.1. Definition. Let E be a relatively closed subset of D. E is said to satisfy conditon (A) if for each $z \in D \setminus E$, there is a boundary curve $\gamma = \gamma_z$ in $D \setminus E$ which connects z to ∂D; i. e. there is a curve $\gamma(t)$, $0 \leqslant t < \infty$, in $D \setminus E$ such that $\gamma(0) = z$ and

$$\chi(\gamma(t), \partial D) \to 0, \text{ as } t \to \infty.$$

1.2. Definition (see [4, p. 422]). A domain $D_0 \subset D$ is simply connected with respect to D if each finite family of Jordan curves in D_0 which bounds in D also bounds in D_0.

1.3. Theorem. *A compact set $E \subset D$ satisfies condition (A) if and only if $D^* \setminus E$ is connected.*

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Proof. If E satisfies condition (A), then for each point $z \in D \setminus E$, there is a boundary curve $\gamma(z)$ as in Definition 1.1. Let ∞ be the ideal point of the one-point compactification D^*. Then $\overline{\gamma} = \gamma \cup \{\infty\}$, where $\overline{\gamma}$ denotes the closure of γ in D^*. Since γ is connected, so is its closure $\overline{\gamma}$, and thus

$$D^* \setminus E = \cup \{\overline{\gamma}(z) : z \in D \setminus E\}$$

is connected.

Conversely, suppose E is compact and $D^* \setminus E$ is connected. Let $z \in D \setminus E$ and let G be the component of $\overline{C} \setminus E$ which contains z. Then $\partial G \subset E$ and we claim that G meets ∂D. For otherwise $G \subset D \setminus E \subset D^* \setminus E$, and so G is open in $D^* \setminus E$. Moreover, since $\partial G \subset E$, then $\overline{G} \subset D$, where $\overline{G}$ denotes the closure of G in $\overline{C}$. Since $\overline{G}$ is compact, it is closed in D^* and so

$$G = \overline{G} \cap D^* \setminus E$$

is closed in $D^* \setminus E$. Thus G is both open and closed in $D^* \setminus E$, and since the latter is connected, it follows that $G = D^* \setminus E$. But then G contains the ideal point, which is absurd, since $G \subset D$. Hence G meets ∂D as claimed.

To conclude the proof, let γ^* be an arc in G from z to some point $z_1 \in G \cap \partial D$, and let z_0 be the first point of ∂D which γ^* meets. Now if we let γ be that portion of γ^* running from z to z_0, then γ satisfies the requirements of Definition 1.1. This completes the proof.

We present an example to show that the above theorem does not hold for (relatively) closed sets. Let E_1 be the graph of the curve

$$y = |(1/x) \sin (1/x)|, \ 0 < x \leqslant \pi^{-1}; \ E_2 = \{(0, y) : -1 \leqslant y < +\infty\},$$

and let E_3 be the straight line segment joining the points π^{-1} and $-i$. Now let $D = C$ and set $E = E_1 \cup E_2 \cup E_3$. Then $D^* \setminus E$ is connected, but E does not satisfy condition (A).

1.4. Theorem. *A domain $D_0 \subset D$ is simply connected with respect to D if and only if $D^* \setminus D_0$ is connected.*

Proof. Suppose first that D_0 is simply connected with respect to D. Then if $\{D_n\}$ is a normal exhaustion [4, p. 587] (see also [8, p. 300]) of D_0, and if some component of $\overline{C} \setminus D_n$ is in D, then it is also in D_0. Thus we may assume that no component of $\overline{C} \setminus D_n$ is in D. Hence each component of $\overline{C} \setminus D_n$ meets ∂D, and so $D^* \setminus D_n$ is connected. Now D^* is compact Hausdorff and $D^* \setminus D_n$ is a nested sequence of continua. It follows that

$$D^* \setminus D_0 = \bigcap_{n=1}^{\infty} (D^* \setminus D_n)$$

is a continuum, and in particular is connected.

Suppose conversely that $D^*\setminus D_0$ is connected and $\alpha_1,\cdots,\alpha_n$ is a finite family of Jordan curves in D_0 which bounds in D i. e. there is a domain G such that

$$\partial G=\alpha_1+\alpha_2+\cdots+\alpha_n\subset D_0,$$

and G is precompact in D. We must show that $G\subset D_0$. Since G is precompact in D, $\overline{G}$ is compact in D, and

$$\overline{G}\cap(D^*\setminus D_0)=G\cap(D^*\setminus D_0)$$

is closed in $D^*\setminus D_0$. Also G is open in D^* and so $G\cap(D^*\setminus D_0)$ is open in $D^*\setminus D_0$. Since $D^*\setminus D_0$ is connected and $G\cap D_0\neq\varnothing$, it follows that $G\cap(D^*\setminus D_0)=\varnothing$. Thus $G\subset D_0$, and the proof is complete.

Motivated by the preceding theorems and the theorem to follow, we now extend Definition 1.2.

1.5. Definition. A subset $E\subset D$ is said to be simply connected with respect to D if and only if $D^*\setminus D$ is connected.

1.6. Definition. A subset $E\subset D$ is said to be a set of uniform approximation (by functions holomorphic in D) provided that for each function g continuous on E and holomorphic on E° and for each $\varepsilon>0$, there exists a function f holomorphic in D such that

$$|f(z)-g(x)|<\varepsilon, \text{ for all } z\in E.$$

The following theorem of Arakélian generalizes Mergelian's theorem to arbitrary domains.

1.7. Theorem [1]. *A compact set $E\subset D$ is a set of uniform approximation if and only if E is simply connected with respect to D.*

Let us denote by ∞ the ideal point in D^*.

1.8. Definition. A set $E\subset D$ is in Arakélian's class $K(D)$ if E satisfies condition (A) and for each neighborhood U of ∞, there is a neighborhood V of ∞ such that each point

$$z\in(D\setminus E)\cap V$$

can be connected to ∞ by a boundary curve γ in U (compare Definition 1.1).

We state another theorem of Arakélian which extends Mergelian's theorem to closed sets.

1.9. Theorem [1]. *A (relatively) closed set $E\subset D$ is a set of uniform approximation if and only if $E\in K(D)$.*

Let us now consider a much stronger sort of approximation.

1.10. Definition. A set $E\subset D$ is a set of tangential approximation (by functions holomorphic in D) provided that for each function g continuous on E and holomorphic on E° and each positive continuous function $\varepsilon(t)$, $0<t<1$, there is a function f holomorphic in D such that, for all $z\in E$

$$|f(z)-g(z)|<\varepsilon(\chi(z,\partial D)).$$

The following result is also due to Arakèlian [1].

1.11. Theorem. *Let E be a closed subset of D such that $E \in K(D)$ and $E^{\circ} = \varnothing$. Then E is a set of tangential approximation.*

With additional hypotheses it is sometimes possible to achieve tangential approximation even though E° may be quite large [7].

§ 2. Applications

In this section, with the aid of Arakélian's theorems, we extend some known results on cluster sets. Moreover our proofs will be much shorter than the earlier proofs. We begin with the following two theorems originally proved by Bagemihl and Seidel [2, 3].

2.1. Theorem. *Let $D = (|z| < R)$, $R \leqslant +\infty$; let $\alpha_1, \alpha_2, \cdots$, be a countable family of disjoint simple boundary curves, and let g be defined and continuous on each α_n. Then there exists a function f holomorphic in D such that for each n*

$$|f(\alpha_n(t)) - g(\alpha_n(t))| \to 0, \text{ as } t \to \infty.$$

Proof. Let $0 < r_1 < r_2 < \cdots < r_n < \cdots < R$, $r_n \to R$, and set $E = \bigcup_n \{\alpha_n \cap (|z| \geqslant r_n)\}$. Then E satisfies the conditions in Arakélian's Theorem 1.11, and g is continuous on E. Thus f exists as claimed.

The above theorem is a special case of the next theorem, but we thought it worthwhile to present the preceding proof because of its brevity. The following theorem was originally proved for monotonic boundary curves in domains bounded by finitely many Jordan curves.

2.2. Theorem. *Let D be a proper domain of the Riemann sphere; let α_n, $n = 1, 2, \cdots$, be a countable family of disjoint simple boundary curves; and let g be defined and continuous on each α_n. Then there exists a function f holomorphic on D such that for each n*

$$|f(\alpha_n(t)) - g(\alpha_n(t))| \to 0, \text{ as } t \to \infty.$$

Proof. Let $\{D_k\}$ be a normal exhaustion of D; i. e. $D = D_1 \cup D_2 \cup \cdots \cup D_k \cup \cdots$, where each D_k is bounded by finitely many Jordan curves and $\overline{D}_k \subset D_{k+1}$, $k = 1, 2, \cdots$. We may also assume that each D_k has the property that each component of $\overline{C} \setminus \overline{D}_k$ meets ∂D (i. e. D_k is simply connected with respect to D).

Let β_n be the tail end of α_n starting from the last point at which α_n leaves $\overline{D}_n$, and set $E = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_n \cup \cdots$. We will show that E satisfies the conditions of Arakélian's Theorem on tangential approximation. First of all

$$E \cap \overline{D}_n = \overline{D}_n \cap \bigcup_{k=1}^{n} \beta_k$$

is closed (perhaps empty) and thus E is closed in D and nowhere dense.

For $\varepsilon > 0$, we write

$$V_\varepsilon(\partial D) = \{z \in D : \chi(z, \partial D) < \varepsilon\}.$$

To show that $E \in K(D)$ it will be sufficient to show that for each $\varepsilon > 0$, there is a $\delta > 0$ such that for all z in

$$(D \setminus E) \cap V_\delta(\partial D), \tag{1}$$

we can find a boundary curve $\gamma = \gamma(z)$,

$$\gamma(z) \subset (D \setminus E) \cap V_\varepsilon(\partial D),$$

which connects z to ∂D.

Suppose ε and z are given as above. We claim there exists an N such that

$$n > N \Rightarrow \beta_n \subset V_\varepsilon(\partial D). \tag{2}$$

Indeed

$$D \setminus V_\varepsilon(\partial D) = \{z \in D : \chi(z, \partial D) \geq \varepsilon\} = \{z \in \overline{C} : \chi(z, \overline{C} \setminus D) \geq \varepsilon\}$$

is closed in $\overline{C}$ and hence compact. This set is therefore contained in some D_k, and we may choose $N = k+1$ in (2).

For $k = 1, 2, \cdots, N$, let z_k be the last point at which α_k leaves $\overline{D}_N$, and let α_k^N be the initial part of α_k running from $\alpha_k(0)$ to z_k (if $\alpha_k \cap \overline{D}_N = \emptyset$, we may set $\alpha_k^N = \emptyset$). We now choose $N_1 > N$ so that

$$D_{N_1} \supset \bigcup_{k=1}^{N} \alpha_k^N.$$

Let $\delta > 0$ be chosen so that

$$V_\delta(\partial D) \subset D \setminus \overline{D}_{N_1}.$$

Suppose z lies in (1), and let G be the component of $\overline{C} \setminus \overline{D}_{n1}$ which contains z. By the way in which the exhaustion $\{D_n\}$ was chosen, it follows that G meets ∂D. Let γ^* be an arc from z to ∂D in G. if $\gamma^* \cap E = \emptyset$, we are through. If not, there is a first point $p \in \gamma^*$ which is in E. Thus p lies in some β_n. Let $\lambda = \chi(p, E \setminus \beta_n) > 0$, and let q be the first point of γ^* for which $\chi(q, p) \leqslant \lambda/2$.

There are two cases to consider. If $n > N$, then $\beta_n \subset V_\varepsilon(\partial D)$. Thus we may easily construct the required curve $\gamma = \gamma(z) \subset V_\varepsilon(\partial D)$ by running along γ^* from z to q. Then γ stays close to β_n and follows β_n out to ∂D in such a way as to remain in $V_\varepsilon(\partial D)$ and to remain disjoint from each β_k, $k = 1, 2 \cdots$.

If $n \leqslant N$, then $p \in \beta_n \setminus \alpha_n^N$. This means that p is past (further along) z_n on β_n and so β_n will never return to $\overline{D}_N$ after it passes p. Thus we may again construct γ by going along γ^* from z to q, and then staying near β_n as it runs out to ∂D, being careful to avoid each β_k and to remain outside of $\overline{D}_N$ and therefore in $V_\varepsilon(\partial D)$. Thus $E \in K(D)$, and the theorem follows from Arakélian's Theorem on tangential approximation.

Under appropriate hypotheses it is also possible to specify boundary behaviour on uncountably many boundary curves. The following theorem is due to Bagemihl and Seidel [3] and Rudin [10]. The following version is stated in [5, p. 163].

2.3. Theorem. *Let g be an arbitrary continuous function in* $|z|<1$, *let F be a set of first category on* $|z|=1$, *and let* $\{L(\theta)\}$ *be a family of mutually disjoint boundary curves terminating at the points* $e^{i\theta}\in F$ *and such that* $\{L(\theta)\}$ *is homeomorphic to a family of radial segments* $\{\lambda(\theta)\}$ *terminating at F. That is, there exists a homeomorphism* ψ *of* $|z|\leq 1$ *onto itself which leaves each point* of $|z|=1$ *fixed and such that* $L(\theta)=\psi(\lambda(\theta))$. *Then there exists a function f, holomorphic in* $|z|<1$, *such that for every* $e^{i\theta}\in F$

$$f(z)-g(z)\to 0, \text{ as } z\to e^{i\theta} \text{ on } L(\theta).$$

Proof. $F\subset\bigcup_{n=1}^{\infty}F_n$, where each F_n is closed and nowhere dense in $|z|=1$. Set

$$E=\bigcup_{n=1}^{\infty}E_n,$$

where

$$E_n=\psi\{\lambda(\theta)\cap(|z|\geq 1-1/n):\ e^{i\theta}\in F_n\}.$$

Then E satisfies the hypotheses of Arakélian's Theorem on tangential approximation, and so f exists as claimed.

By a similar argument we could also prove the analogous theorem of Bagemihl and Seidel [3] on tresses. Also, we believe that the above theorems remain valid on arbitrary open Riemann surfaces, however it seems that the approximation theory for proving such theorems (easily) has yet to be developed.

The following theorem is due to Schneider [12] although he restricted his attention to monotonic (in modulus) boundary curves for simplicity. The original proof made use of the pole sweeping method.

2.4. Theorem. *Let* $D=(|z|<R)$, $R\leq+\infty$; *and let* α *and* β *be two simple boundary arcs disjoint except for their common initial point, say* $\alpha(0)=\beta(0)=0$. *Let* E_0, E_1 *be the two domains into which* $\alpha\cup\beta$ *separates D. Then there exists a function f, holomorphic in D, such that f is bounded in* E_0 *and unbounded in* E_1.

Proof. Let $\{z_n\}$ be a sequence in E_1 such that $|z_n|\to R$, as $n\to\infty$, and set $E=E_0\cup\{z_n\}$. We define a continuous function g on E by setting g equal to zero on E_0 and $g(z_n)=n$, for $n=1, 2\cdots$.

As in the proof of Theorem 2.2 (but much more easily in this case) we see that E is relatively closed in D and is in $K(D)$. From Arakélian's Theorem 1.9, there exists a function f, holomorphic in D, such that $|f(z)-g(z)|<1$, for all $z\in E$. The proof is complete.

Remark: After submitting this paper, we have noticed that Theorem 2.2 of the present paper is essentially the same as Theorem 9 in W. Kaplan's paper, Approximation by entire functions, Mich. Math. J., 3 (1955—1956), 43—52.

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Պ. ԳՈԹՅԵ, Վ. ԶԱՅԴԵԼ. Առաքելյանի մոտարկման թեորեմների որոշ կիրառություններ եզրային բազմությունների տեսության մեջ *(ամփոփում)*

Հոդվածում ցույց է տրվում, որ տվյալ տիրույթում անալիտիկ ֆունկցիաներով շոշափումային մոտավորության թեորեմները պարզեցնում են անալիտիկ ֆունկցիաների եզրային վարքի վերաբերյալ որոշ հարցերի հետազոտումը։

Մոտարկումային թեորեմների օգնությամբ միասնական ձևով ապացուցվում են Բագեմիլի և Զեյդելի, Ռուդինի, Շնեյդերի թեորեմների ընդհանրացված տարբերակները եզրային բազմությունների տեսությունից։

П. ГОТЬЕ, В. ЗЕЙДЕЛЬ. *Некоторые приложения аппроксимационных теорем Аракеляна в теории предельных множеств* (резюме)

В заметке показывается, что теоремы о возможности касательного приближения аналитическими в данной области функциями упрощают изучение некоторых вопросов граничного поведения аналитических функций. С помощью аппроксимационных теорем единым способом доказываются обобщенные варианты некоторых теорем Багемиля и Зейделя, Рудина, Шнейдера из теории предельных множеств.

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