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MEAN VALUES AND CURVATURES

We divide this exposition into two parts. Part 1 refers to the mean value of the Euler—Poincaré characteristic of the intersection of two convex hypersufaces in E_4. Part 11 deals with the definition of q-th total absolute curvatures of a compact n-dimensional variety imbedded in euclidean space of $n+N$ dimensions, extending some results given in [10].

I. On convex bodies in E_4

1. Introduction. Let K be a convex body in 4-dimensional euclidean space E_4 and let W_i ($i=0, 1, 2, 3, 4$) be its Minkowski's *Quermass integrale* (see for instance Bonnesen—Fenchel [1]). Recall that

$$W_0 = V = \text{volume of } K$$
$$4W_1 = F = \text{area of } \partial K \tag{1.1}$$
$$W_4 = \pi^2/2,$$

and, if K has sufficiently smooth boundary, we have also

$$4W_2 = M_1 = \text{first mean curvature} = \frac{1}{3}\int_{\partial K}\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)d\sigma$$

$$4W_3 = M_2 = 2\text{ th mean curvature} = \frac{1}{3}\int_{\partial K}\left(\frac{1}{R_1R_2}+\frac{1}{R_1R_3}+\frac{1}{R_2R_3}\right)d\sigma \tag{1.2}$$

where R_i are the principal radii of curvature and $d\sigma$ is the element of area of ∂K.

For instance, if $K=$ sphere of radius r, we have

$$V=\frac{1}{2}\pi^2r^4, \quad F=2\pi^2r^3, \quad M_1=2\pi^2r^2, \quad M_2=2\pi^2r. \tag{1.3}$$

We will use throughout the invariants V, F, M_1, M_2 because they have a more geometrical meaning; however we do not assume smoothness to ∂K, so that as definition of M_1, M_2 we take $M_1=4W_2$, $M_2=4W_3$.

The invariants V, F, M_1, M_2 are not independent. They are related by certain inequalities which may be written in the following symmetrical form (following Hadwiger [6]).

$$W_\alpha^{\beta-\gamma}\, W_\beta^{\gamma-\alpha}\, W_\gamma^{\alpha-\beta} \geqslant 1, \quad 0 \leqslant \alpha \leqslant \beta \leqslant \gamma \leqslant 4. \tag{1.4}$$

In explicit form and using the invariants V, F, M_1, M_2 the inequalities (1.4) give the following non-independent inequalities

$$F^2 > 4VM_1, \quad F^3 \geqslant 16\, V^2 M_2, \quad F^4 \geqslant 128\, \pi^2 V^3,$$

$$M_1^3 > 4\, VM_2, \quad M_1^2 \geqslant 2\, \pi^3 V, \quad M_2^4 \geqslant 32\, \pi^6 V,$$

$$M_1^2 > FM_2, \quad M_1^3 > 2\, \pi^2 F^2, \quad M_2^3 > 4\, \pi^4 F,$$

$$M_2^2 \geqslant 2\, \pi^2 M_1.$$

We will represent throughout the paper by O_i the volume of the i-dimensional unit sphere, that is

$$O_i = \frac{2\pi^{\frac{i+1}{2}}}{\Gamma\left(\frac{i+1}{2}\right)} \tag{1.6}$$

or instance

$$O_0 = 2,\ O_1 = 2\pi,\ O_2 = 4\pi,\ O_3 = 2\pi^2,\ O_4 = \frac{8}{3}\pi^2,\ O_5 = \pi^3. \tag{1.7}$$

2. Mean value of $\chi\,(\partial K \cap g\partial K)$. Let G be the group of isometries of E_4. For any $g \in G$ we represent by $g\partial K$ the image of ∂K by the isometry g. Let dg denote the invariant volume element of G (=kinematic density for E_4). Assume the convex body K fixed and consider the intersections $\partial K \cap g\partial K$, $g \in G$. Then, Federer [5] and Chern [2] have proved the following integral formula

$$\int_G \chi\,(\partial K \cap g\partial K)\; dg = 64\, \pi^2 F M_2 \tag{2.1}$$

where $\chi\,(\partial K \cap g\partial K)$ denote the Euler—Poincaré characteristic of the surface $\partial K \cap g\partial K$.

On the other side, the so-called fundamental kinematic formula of integral geometry, gives

$$\int_{K \cap gK \neq \varnothing} dg = 8\pi^2 \left(4\pi^2 V + 2\, FM_2 + \frac{3}{2}\, M_1^2 \right) \tag{2.2}$$

Therefore the expected value of $\chi\,(\partial K \cap g\partial K)$ is

$$E\,(\chi\,(\partial K \cap g\partial K)) = \frac{8\,F\,M_2}{4\pi^2 V + 2FM_2 + \frac{3}{2}\, M_1^2}\,. \tag{2.3}$$

Notice that, being K convex, the intersections $\partial K \cap g\partial K$ are closed orientable surfaces. Thus the possible values of χ are, either $\chi = 2, 4, 6, \cdots$ or $\chi = 0, -2, -4, -5, \cdots$ If K is an euclidean sphere, obvionsly we have $E\,(\chi) = 2$.

Conjecture. *For all convex sets K of E_4 the inequality*

$$E\,(\chi\,(\partial K \cap g\partial K)) \leqslant 2 \tag{2.4}$$

holds good, equality for the euclidean sphere.

Putting

$$\Delta = 8\pi^2 V + 3M_1^2 - 4FM_2 \tag{2.5}$$

the conjecture is equivalent to prove that $\Delta > 0$. For the euclidean sphere, accrding to (1.3) we have $\Delta = 0$.

In support of this conjecture we will prove it for rectangular parallelepipeds. Let a, b, c, d be the sides of a rectangular parallelepiped in E_4 and assume

$$a \leqslant b \leqslant c \leqslant d \tag{2.6}$$

It is known that (Hadwiger [6])

$$V = abcd,\ F = 2\,(abc + abd + acd + bcd),$$

$$M_1 = \frac{2}{3}\pi\,(ab+ac+ad+bc+bd+cd),\ M_2 = \frac{4}{3}\pi\,(a+b+c+d).$$

With these values we verify the identity

$$\frac{3}{4\pi}\Delta = (4-\pi)[a^2c^2 + a^2(c-b)^2 + b^2(c-a)^2 + a^2(d-b)^2$$
$$+ c^2(d-a)^2 + b^2(b-c)^2 + c^2(d-b)^2] + (18\pi - 56)\,abcd$$
$$+ (4\pi - 12)(a^2b^2 + a^2c^2 + b^2c^2) + (8-2\pi)[(b-a)\,acd +$$
$$+ (c-b)\,abd + (d-c)\,acb] + (4-\pi)\,d^2\,[(2A^2 - B^2)$$
$$(a^2 + b^2) + (Ac - Ba)^2 + (Ac - Bb)^2],$$

where $A^2 = (3\pi - 8)/(8-2\pi)$, $B^2 = (8-2\pi)/(3\pi - 8)$.

Since all terms are positive, we have $\Delta > 0$.

For an ellipsoid of revolution whose semiaxes are a, a, a, λa we have (Hadwiger [6])

$$V = \frac{\pi}{2}\lambda a^4,\ F = 2\pi^2\lambda^2 a^3\,F\left(\frac{5}{2}, \frac{1}{2}, 2;\ 1-\lambda^2\right),$$

$$M_1 = 2\pi^2\lambda^3 a^2\ F\left(\frac{5}{2}, 1, 2;\ 1-\lambda^2\right), \tag{2.7}$$

$$M_2 = 2\pi^2\lambda^4 a\ F\left(\frac{5}{2}, \frac{3}{2}, 2;\ 1-\lambda^2\right)$$

where F denotes the hypergeometric function. In this case the conjecture writes

$$1 + 3\lambda^5 F_1^2 - 4\lambda^5 F_{1/2} F_{3/2} > 0 \tag{2.8}$$

where

$$F_{1/2} = F\left(\frac{5}{2}, \frac{1}{2}, 2;\ 1-\lambda^2\right),$$

$$F_1 = F\left(\frac{5}{2}, 1, 2; 1-\lambda^2\right),$$

$$F_{3/2} = F\left(\frac{5}{2}, \frac{3}{2}, 2; 1-\lambda^2\right).$$

I do not know if (2.8) holds for all values of λ.

II. Absolute total curvatures of compact manifolds immersed in euclidean space

1. I n t r o d u c t i o n. In this section we extend and complete the contents of [10]. We shall first state some known formulas which will be used in the sequel.

Let L_h be a h-dimensional linear subspace in the $(n+N)$-dimensional euclidean space E_{n+N}. We will call it, simply, a h-space. Let $L_h(0)$ be a h-space in E_{n+N} through a fixed point 0. The set of all oriented $L_h(0)$ constitute the Grassman manifold $G_{h,n+N-h}$. We shall represent by $dL_h(0)$ the element of volume of $G_{h,n+N-h}$, which is the same thing as the density for oriented h-spaces through 0. The expression of $dL_h(0)$ is well known, but we will recall it briefly for completeness (see [9], [2]).

Let $(O; e_1, e_2, \cdots, e_{n+N})$ be an orthonormal frame in E_{n+N} of origin O. In the space of all orthonormal frames of origin O we define the differential forms

$$\omega_{im} = -\omega_{mi} = e_m de_i. \tag{1.1}$$

Assuming $L_h(O)$ spanned by the unit vectors $e_1, e_2, \cdots, e_h$, then

$$dL_h(O) = \Lambda\omega_{im} \tag{1.2}$$

where the right side is the exterior product of the forms ω_{im} over the range of indices

$$i = 1, 2, \cdots, h; \; m = h+1, h+2, \cdots, n+N.$$

The $(n+N-h)$ – space $L_{n+N-h}(O)$ orthogonal to $L_h(O)$ is spanned by $e_{h+1}, \cdots, e_{n+N}$ and (1.2) gives the duality

$$dL_h(O) = dL_{n+N-h}(O) \tag{1.3}$$

The measure of the set of all oriented $L_h(0)$ (= volume of the Grassman manifold $G_{h,n+N-h}$) may be computed directly from (1.2) (see 9]). or applying that it is the quotient space $SO(n+N)/SO(h) \times \times SO(n+N-h)$ (see [2]). The result is

$$\int_{G_{h,n+N-h}} dL_h(O) = \frac{O_{n+N-1}\, O_{n+N-2} \cdots O_{n+N-h}}{O_1 O_2 \cdots O_{h-1}} \tag{1.4}$$

$$= \frac{O_h O_{h+1} \cdots O_{n+N-1}}{O_1 O_2 \cdots O_{n+N-h-1}}$$

where O_i is the area of the i-dimensional unit sphere (1, (1.6)).

Another known integral formula which we will use is the following.

Consider the unit sphere $\sum_{n+N-1}$ of dimension $n+N-1$ of center O. Let V^s be a s-dimensional variety in $\sum_{n+N-1}$. Let $\mu_{s+h-n-N}(V^s \cap L_h)$ be the $(s+h-n-N)$—dimensional measure of the variety $V^s \cap L_h(O)$ of dimension $s+h-(n+N)$ and let $\mu_s(V^s)$ be the s-dimensional measure of V^s (all these measures considered as measures of subvarieties of the euclidean space E_{n+N}). Then

$$\int_{O_{h,\ n+N-h}} \mu_{s+h-n-N}(V^s \cap L_h(O))\, dL_h(O) =$$

$$= \frac{O_{n+N-h}\, O_{n+N-h+1} \cdots O_{n+N-1}\, O_{h+s-n-N}}{O_1 O_2 \cdots O_{h-1}\, O_s} \mu_s(V^s) \tag{1.5}$$

Note that this formula assumes the h-spaces L_h oriented (see [8]). In particular, if $s=1$ an $h=n+N-1$ that is, for a curve V^1 of ength U we have

$$\int_{O_{n+N-1,\ 1}} \nu\, dL_{n+N-1}(O) = \frac{2\, O_{n+N-1}}{O_1} U \tag{1.6}$$

where ν is the number of points of the intersection $V^1 \cap L_{n+N-1}(O)$.

2. Defintions. Let X^n be a compact n-dimensional differentiable manifold (without boundary) of class C^∞ in E_{n+N}. To each point $\in X^n$ we attach the ρ-space $T^{(q)}(p)$ spanned by the vectors

$$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}; \frac{\partial^2}{\partial x_1^2}, \ldots, \frac{\partial^2}{\partial x_n^2}; \ldots; \frac{\partial^q}{\partial x_1^q}, \ldots, \frac{\partial^q}{\partial x_n^q} \tag{21}$$

which we will call the q—th tangent fibre over p. Its dimension is

$$\rho(n, q) = \sum_{i=1}^{q} \binom{n+i-1}{i} \tag{2.2}$$

Assuming

$$1 \leqslant r \leqslant n+N-1, \quad \rho \leqslant n+N-1 \tag{2.3}$$

we define the r-th total absolute curvature of order q of X^n as follows:

a) *Case* $1 \leqslant r \leqslant \rho$. Let O be a fixed point of E_{n+N} and consider a $(n+N-r)$-space $L_{n+N-r}(O)$. Let Γ_r be the set of all r-spaces L_r of E_{n+N} which are contained in some of the fibres $T^{(q)}(p)$, $p \in X^n$, pass through p, and are orthogonal to $L_{n+N-r}(O)$. The intersection $\Gamma_r \cap L_{n+N-r}(O)$ will be a compact variety in $L_{n+N-r}(O)$ whose dimension δ we shall compute in the next section. Let $\mu(\Gamma_r \cap L_{n+N-r}(O))$ be the measure of this variety as subvariety of the euclidean space

$L_{n+N-r}(O)$; if $\delta = 0$, then μ means the number of intersection points of Γ_r and $L_{n+N-r}(O)$.

Then we define the r-th total absolute curvature of order q of $X^n \subset E_{n+N}$ as the mean value of the measures μ for all $L_{n+N-r}(0)$, that is, according to (1.4)

$$K_r^{(q)}(X^n) = \frac{O_1 O_2 \cdots O_{n+N-r-1}}{O_r O_{r+1} \cdots O_{n+N-1}} \int_{G_{n+N-r,\, r}} \mu(\Gamma_r \cap L_{n+N-r}(O))\, dL_{n+N-r}(O). \tag{2.4}$$

The coefficient of the right side may be substituted by

$$O_1 O_2 \cdots O_{r-1} / O_{n+N-r} \cdots O_{n+N-1}.$$

b) *Case* $\rho \leqslant r \leqslant n+N-1$. Instead of the set of L_r which *are contained* in some $T^{(q)}(p)$ we consider now the set of L_r which *contain* some $T^{(q)}(p)$, $p \in X^n$, and are orthogonal to $L_{n+N-r}(O)$. As before we represent this set by Γ_r and the r-th total absolute curvature of order q of $X^n \subset E_{n+N}$ is defined by the same mean value (2.4).

3. Properties. We proceed now to compute the dimension of $\Gamma_r \cap L_{n+N-r}(0)$.

a) *Case* $1 \leqslant r \leqslant \rho$. The set of all $L_r \subset E_{n+N}$ is the Grassman manifold $G_{r+1,\, n+N-r}$ whose dimension is $(r+1)(n+N-r)$. The set of all L_r which are contained in $T^{(q)}(p)$ and pass through p is the Grassman manifold $G_{r,\, \rho-r}$ of dimension $r(\rho-r)$; therefore the set of all L_r which are contained in some $T^{(q)}(p)$, $p \in X^n$, has dimension $r(\rho-r)+n$. On the other side, the set of all $L_r \subset E_{n+N}$ which are orthogonal to $L_{n+N-r}(O)$ has dimension $n+N-r$. Consequently, the intersection of both sets, as sets of points of $G_{r+1,\, n+N-r}$, has dimension

$$r(\rho-r)+n+n+N-r-(r+1)(n+N-r) = r\rho+n-r(n+N).$$

Since to each L_r orthogonal to $L_{n+N-r}(0)$ corresponds one and only one intersection point with this linear space, the preceding dimension coincide with the dimension δ of $\Gamma_r \cap L_{n+N-r}$, that is,

$$\delta = \dim(\Gamma_r \cap L_{n+N-r}(0)) = r\rho + n - r(n+N).$$

Hence, in order that $K_r^{(q)}(X^n) \neq 0$, it is necessary and sufficient that

$$r\rho + n \geqslant r(n+N) \tag{3.1}$$

b) *Case* $\rho \leqslant r \leqslant n+N-1$. The set of all $L_r \subset E_{n+N}$ which contain a fixed L_ρ, constitute the Grassman manifold $G_{r+\rho,\, n+N-r}$ and therefore the dimension of the set of all L_r which contain some $T^{(q)}(p)$, $p \in X^n$, is $(r-\rho)(n+N-r)+n$. The remainder dimensions are the same as in the case a), so that the dimension of the set of all L_r which contain some $T^{(q)}(p)$, $p \in X^n$, and are orthogonal to $L_{n+N-r}(O)$ is

$$(r-\rho)(n+N-r)+n+n+N-r-(r+1)(n+N-r) = r\rho+n-\rho(n+N)$$

that is

$$\delta = \dim(\Gamma_r \cap L_{n+N-r}(O)) = \rho r + n - \rho(n+N)$$

In order that $K_r^{(q)}(X^n) \neq 0$, it is necessary and sufficient that

$$\rho r + n > \rho(n+N). \tag{3.2}$$

Of course, to (3.1) and (3.2) we must add the relations (2.3).

The most interesting cases correspond to $\delta = 0$, for which the measure μ in (2.4) is a positive integer and the total absolute curvature is invariant under similitudes. In this case the set of points $p \in X^n$ for which L_r contains or is contained in $T^{(q)}(p)$ can be divided according to the index of p, and we get different curvatures as those defined by Kuiper for the case $q=1$, $r=n+N-1$ [7]. We will not go into details here.

4. Examples.

4.1. Curves, $n=1$. For $n=1$ the condition (3.1) writes

$$1 > r + r(N-\rho)$$

and since $\rho \leqslant N$ the only possibility is $\rho = N$, $r=1$, which gives $\delta=0$. The corresponding curvature $K_1^{(N)}(X^1)$ is

$$K_1^{(N)}(X^1) = \frac{1}{O_N} \int_{G_{N,1}} \nu_1 \, dL_N(0) \tag{4.1}$$

where ν_1 is the number of lines in E_{n+N} orthogonal to $L_N(0)$ which are contained in some N—th tangent fiber of the curve X^1. Notice that $G_{N,1}$ is the unit sphere $\sum_N$ and $dL_N(0)$ is the element of area of this sphere in consequence of the duality (1.3). If $e_1, e_2, \cdots, e_{N+1}$ are the principal normals of X^1 then the formula (1.6) says that the right side of (4.1) is equal to the length of the spherical curve $e_{N+1}(s)$ (s=arc length of X^1) up to the factor $1/\pi$. That is, if $\varkappa_N$ is the N-th curvature of X^1 (see, for instance, Eisenhart [4], p. 107) we have

$$K_1^{(N)}(X^1) = \frac{1}{\pi} \int_{X^1} |\varkappa_N| \, ds. \tag{4.2}$$

For the case of curves in E_3, $N=2$, $\varkappa_N$ is the torsion of the curve and $K_1^{(2)}$ is up to the factor π^{-1}, the *absolute total torsion* of X^1.

The condition (3.2) gives $1 > \rho + \rho(N-r)$ and since $r \leqslant N$, this condition implies $\rho=1$, $r=N$. We have the curvature

$$K_N^{(1)}(X^1) = \frac{1}{O_N} \int_{G_{1,N}} \nu_N \, dL_1(O), \tag{4.3}$$

where ν_N is the number of hyperplanes L_N of E_{N+1} orthogonal to $L_1(O)$ which contain some tangent line of X^1. The same formula (1.6) gives now that the right side of (4.3) is equal to the length of the curve $e_1(s)$ (=spherical tangential image of X^1), up to the factor $1/\pi$. Therefore, if $\varkappa_1$ is the first curvature of X^1, (4.3) writes

$$K_N^{(1)}(X^1)=\frac{1}{\pi}\int_{X^1}|\varkappa_1|\,ds. \tag{4.4}$$

Notice that for each direction $L_1(O)$ there are at least two hyperplanes orthogonal to $L_1(O)$ which contain a tangent line of X^1 (the hyperplanes which separate the hyperplanes which have common point with X^1 of those which do not have such common point). Therefore the mean value $K_N^{(1)}$ is $\geqslant 2$ and (4.4) gives the classical Fenchel's inequality

$$\int_{X^1}|\varkappa_1|\,ds \geqslant 2\pi. \tag{4.5}$$

If the curve X^1 has at least 4 hyperplanes orthogonal to an arbitrary direction $L_1(0)$ which contain a tangent line of X^1 (as it happens for instance for knotted curves in E_3), the mean value $K_N^{(1)}(X)$ will be $\geqslant 4$, and we have the Fary's inequality

$$\int_{X^1}|\varkappa_1|\,ds \geqslant 4\pi \tag{4.6}$$

4.2 Surfaces, $n=2$.

1. *Total absolute curvatures of order* 1. We have $n=2$, $\rho=2$ and condition (3.1) writes $2\geqslant rN$. Therefore the possible cases are $r=1$, $N=1$; $r=2$, $N=1$ and $r=1$, $N=2$. For $2\leqslant r\leqslant N+1$, condition (3.2) gives $r\geqslant N+1$ and therefore the only possible case is $r=N+1$.

a) *Case* $r=1$, $N=1$. Surfaces in E_3. Having into account that $G_{2,1}$ is the unit sphere $\sum_2$, the curvature (2.4) writes

$$K_1^{(1)}(X^2)=\frac{1}{4\pi}\int_{\Sigma_2}\lambda\,dL_2(0) \tag{4,7}$$

where λ is the length of the curve in the plane $L_2(O)$ generated by the intersections of $L_2(O)$ with the lines of E_3 which are tangent to X^2 and are orthogonal to $L_2(O)$. If H denotes the mean curvature of X^2 and $d\sigma$ denotes the element of area of X^2, it is known that (4.7) is equivalent to the *total absolute mean curvature*

$$K_1^{(1)}(X^2)=\frac{1}{2}\int_{X^2}|H|\,d\sigma. \tag{4.8}$$

b) *Case* $r=2$, $N=1$. Surface $X^2\subset E_3$. The Grassman manifold $G_{1,2}$ is the unit sphere $\sum_2$ and (2.4) can be written

$$K_2^{(1)}(X^2)=\frac{1}{4\pi}\int_{\Sigma_2}\nu_2\,dL_1(0) \tag{4.9}$$

where ν_3 is the number of planes in E_3 which are tangent to X^2 and are orthogonal to the line $L_1(O)$. If K denotes the Gaussian curvature of X^2, since $dL_1(O)$ is the element of area on $\sum_2$, it is easy to see that (4.9) is equivalent, up to a constant factor, to the *total absolute Gaussian curvature* of X^2, that is

$$K_2^{(1)}(X^2)=\frac{1}{2\pi}\int_{X^2}|K|\,d\sigma. \tag{4.10}$$

c) *Case* $r=1$, $N=2$. Surfaces $X^2\subset E_4$. In this case, writting $\sum_3$ = unit 3-dimensional sphere, instead of $G_{3,1}$, we have

$$K_1^{(1)}(X^2)=\frac{1}{2\pi^2}\int_{\Sigma_3}\nu_1\,dL_3(0) \tag{4.11}$$

where ν_1 is the number of tangent lines to X^2 which are orthogonal to the hyperplane $L_3(0)$. Properties of this total absolute curvature it seems to be not known. A geometrical interpretation was given in [10].

d) *Case* $r=N+1$. Surfaces $X^2\subset E_{N+2}$. According to (2.4) we have the following curvature

$$K_{N+1}^{(1)}(X^2)=\frac{1}{O_{N+1}}\int_{\Sigma_N}\nu_{N+1}\,dL_1(O) \tag{4.12}$$

where ν_{N+1} is the number of hyperplanes of E_{N+2} which are tangent to X^2 and are orthogonal to the line $L_1(O)$ and $\sum_N$ denotes the N-dimensional unit sphere. Up to a constant factor this curvature coincides with the *curvature of Chern—Lashof* [3]. Since obviously $\nu_{N+1}\geq 2$ we have the inequality $K_{N+1}^{(1)}\geq 2$, with the equality sign only if X^2 is a convex surface contained in a linear subspace L_3 of E_4.

For $N=2$, X^2 is a surface imbedded in E_4 and the curvature (4.12) is a kind of dual af the curvature (4.11) (see [10].

2. *Total absolute curvatures of order* $q=2$. We have $n=2$, $p=5$ and the inequalities (3.1) and (3.2) say that the only possible cases are: a) $r=1$, $N=4$; b) $r=2$, $N=4$; c) $r=1$, $N=5$.

a) *Case* $r=1$, $N=4$. Surface X^2 in E_6. The Grassman manifold $G_{5,1}$ is the unit sphere $\sum_5$ and (2.4) can be written

$$K_1^{(2)}(X^2)=\frac{1}{O_5}\int_{\Sigma_5}\lambda\,dL_5(O) \tag{4.13}$$

where λ is the length of the curve in $L_5(O)$ generated by the intersections of $L_5(O)$ with the lines of E_6 which are orthogonal to $L_5(O)$ and belong to some of the 2-th tangent fibres of X^2.

b) *Case* $r=2$, $N=4$. Surface X^2 in E_6. We have

$$K_2^{(2)}(X^2)=\frac{O_1}{O_4O_5}\int_{G_{4,2}}\nu_2\,dL_4(O) \tag{4.14}$$

where ν_2 is the number of 2-spaces of E_6 which are orthogonal to $L_4(O)$ and are contained in some 2-th tangent fibre of X^2.

c) Case $r=1$, $N=5$. Surfaces X^2 in E_7.

We have

$$K_1^{(2)}(X^2)=\frac{1}{O_6}\int_{\Sigma}\nu_1\, dL_6\,(O) \tag{4.13}$$

where ν_1 is the number of lines of E_7 which are contained in some 2-th tangent fibre of X^2 and are orthogonal to $L_6\,(O)$.

The expression of these absolute total curvatures of order 2 by means of differential invariants of X^2 is not known.

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Հոդվածի առաջին մասում առաջ է քաշվում (2.4) անհավասարության իրավացիության հիպոթեզը $K\subset E_4$ և gK (g-ն E_4-ում իզոմետրիայի ձևափոխություն է) բազմությունների մակերևույթների հատման էյլերյան խարակտերիստիկի միջին արժեքի համար։ Այդ հիպոթեզը ստուգվում է, երբ K-ն զուգահեռանիստ է։

Երկրորդ մասը նվիրված է E_{n+N}-ում խորասուզված n չափանի կոմպակտ դիֆերենցելի բազմաձևության r-րդ q կարգի լրիվ բացարձակ կորության սահմանմանը։

Л. А. САНТАЛО. *Средние значения и кривизны* (резюме)

В первой части статьи выдвигается гипотеза о справедливости неравенства (2.4) для среднего значения эйлеровой характеристики пересечения поверхностей множеств $K\subset E_4$ и gK (g—преобразование изометрии в E_4). Эта гипотеза проверяется, когда K есть параллелепипед.

Вторая часть посвящена определению r-той полной абсолютной кривизны порядка q погруженного в E_{n+N} компактного дифференцируемого n-мерного многообразия.

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