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A SYNOPSIS OF 'POISSON FLATS IN EUCLIDEAN SPACES'

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Abstract. Homogeneous isotropic Poisson s-flats in Euclidean d-dimensional space E^d $(0 \leqslant s < d)$ are defined as a natural generalisation of the standard linear Poisson process, for which $s=0, d=1$. Invariant densities of integral geometry enter naturally into the ergodic theory of n-subsets of s — flats in such systems. A wide class of ergodic gamma-type probability distributions, in a sense dual to the Poisson distributions for numbers of hits, is derived. Extensions of the theory to more general systems incorporating mixtures, anisotropy, and associated cylinder sets are discussed, and mention is made of the fundamental role of the anisotropic case as a local limit when random s — dimensional varieties are superposed in E^d. Finally, the ergodic probability distributions of several associated random tessellations of convex polytopes are investigated.

Introduction. The present paper is based on two lectures given by the author at the Symposium on Integral Geometry and Geometrical Probability held at Oberwolfach in June, 1969. The main object then was to present, within the limits of a paper of reasonable length, a wide-ranging account of Poisson flats in Euculidean spaces. Emphasis was placed on the underlying connection with integral geometry, and on the wide range of possible random geometric models encompassed by such a structure. As inevitable costs of such an ambitious project, the material was somewhat selective with few examples being included, full proofs were for the most part omitted, and questions of rigour were glossed over or even ignored. This paper has much the same defects, but on the other hand it is hoped it offers a relatively effortless and illuminating introduction to this area of random geometry.

Much of the material relating to the random tessellations determined by Poisson hyperplanes is taken from the author's unpublished Ph. D. thesis [13], while Theorem 2 was announced in [14]. Specialised accounts of the planar cases $s=0$, $d=2$ and $s=1$, $d=2$ are to be found in Miles [15, 17]. The author is preparing a series of papers in which the contents of the present paper are developed in full detail. The first of these is [16].

Preliminaries. First we introduce the Poisson and gamma probability distributions which are of fundamental importance in this work.

The Poisson distribution with parameter (= its expectation = its variance) λ has p.m.f.

$$p_i = e^{-\lambda}\lambda^i/i! \qquad (i=0,1,\cdots). \tag{1}$$

The family of gamma distributions $\Gamma_\theta(\nu, \lambda)$ (θ, λ, ν all >0) has p.d.f.

$$f(x) = \theta\lambda^{\nu/\theta}x^{\nu-1}e^{-\lambda x^\theta}/\Gamma\left(\frac{\nu}{\theta}\right) \quad (x>0). \tag{2}$$

and kth order moment

$$\mu_k = \left\{\Gamma\left(\frac{\nu+k}{\theta}\right)/\Gamma\left(\frac{\nu}{\theta}\right)\right\}\lambda^{-k/\theta} \quad (k=1, 2,\cdots). \tag{3}$$

If ν/θ is a positive integer, then the d.f. of $\Gamma_\theta(\nu, \lambda)$ is

$$F(x) = q\left\{\frac{\nu}{\theta}-1, \lambda x^\theta\right\} \quad (x\geqslant 0), \tag{4}$$

where the tails of the Poisson (λ) distribution given by (1) are defined by

$$p\{i, \lambda\} = 1 - q\{i, \lambda\} = e^{-\lambda}\{1+\cdots+(\lambda^i/i!)\}. \tag{5}$$

The cases $\theta=1$ and $\theta=1$, $\nu=1$ yield the standard gamma distribution $\Gamma(\nu, \lambda)$ of index ν and parameter λ, and the exponential distribution of parameter λ, respectively.

The random systems we consider lie in E^d, in which x, $y,\cdots$ are points and o the origin ($d=1, 2,\cdots$). The notation $|\cdots|$ is used for the modulus of a real number, the length of a vector, full dimensional Lebesgue measure, and for determinants; the interpretation will be apparent from the context. The *ball* $|x|\leqslant q$ of centre 0 and radius q is denoted by Q_q; then its boundary ∂Q_q is the *sphere* $|x|=q$. The volume d-cotent $|Q_q|$ of Q_q is $\upsilon_d q^d$, where $\upsilon_d \equiv \pi^{d/2}/\Gamma\left(\frac{d}{2}+1\right)$; the surface $(d-1)$—content $|\partial Q_q|$ of Q_q is $\sigma_d q^{d-1}$, where $\sigma_d \equiv 2\pi^{d/2}/\Gamma\left(\frac{d}{2}\right)$. The weighted linear sum of the subsets $X_i \subset E^d$ with weights λ_i $(1\leqslant i\leqslant n)$ is

$$\sum_i^n \lambda_i X_i = \left\{\sum_i^n \lambda_i x_i : x_i \in X_i,\ 1\leqslant i\leqslant n\right\}. \tag{6}$$

We abbreviate '$X\cap Y\neq\varnothing$' to '$X\uparrow Y$', in words '*X hits Y*'. The range space of a variable$\cdots$ is often denoted by $[\cdots]$, while the subset comprising the single point$\cdots$ is $\{\cdots\}$. Finally, $\{x_*\}$ is shorthand for $\{x_i\}$ over the full range of i.

§ 1. Theory

An *s-flat* is the translate $J_s = \{x\} + J_{s(0)}$ of an s-dimensional linear subspace, or *s-subspace*, $J_{s(0)}$ in E^d $(0\leqslant s< d)$. Examples:

—s: 0 ~ point, 1 ~ line, 2 ~ plane, 3 ~ (ordinary) space, $\cdots$, $d-1$ ~ ~ hyperplane. Thus, for a system of random s — flats in E^d, there are six cases of practical importance, viz. $0 < s < d \leqslant 3$; and ten if 'time' is included. Since an s-subspace in E^d has $s(d-s)$ degrees of freedom, an s-flat has $(s+1)(d-s)$ degrees of freedom and hence may be parametrized by a point $b \in [b] \subset E^{(s+1)(d-s)}$. This parametrization is trivial when $s=0$. An s-flat hitting $X \subset E^d$ is termed an *s-secant* of X. Thus an arbitrary probability distribution concentrated on

$$B_x \equiv \{b : J_s(b) \uparrow X \subset E^d\} \tag{7}$$

(supposed measurable) 'is' the distribution of a randoms-secant of X.

Suppose, for each measurable subset B of $[b]$, $\int_B f(b)\, db$ is invariant with respecf to the group of Euclidean motions in E^d; that is, $\int_B f(b)\, db = \int_{B'} f(b')\, db'$ for all such motions. The existence and uniqueness, up to a contsant factor, of f is ensured by the general theory. See, for example, Santaló [21; Part III] or Nachbin [18; Chapter III]. According to Nachbin (Example 6, pp. 143—4), the s-flats of E^d form a locally compact homogeneous space under the Euclidean group, on which there is an invariant measure, unique up to a constant factor. We describe $f(b)$ as the *invariant density* in B with respect to the parametrization b, and $E(B) = \int_B f(b)\, db$ as the corresponding *invariant measure*. An explicit form of the nivariant density was first given by Blaschke, in a short monograph [2] which marked the birth of integral geometry as such; see also Petkantschin [19] and Santaló [21; § 24]. Suppose $u_1, \cdots, u_d$ are orthonormal vectors in E^d; $J_{s(0)}$ is spanned by $u_1, \cdots, u_s$; dO^s represents an $(s-1)$-dimensional volume element of the s-dimensional unit sphere with centre o throngh $u_1, \cdots, u_s$ (dO^d, which may be equated with 'du', is often written below simply as dO); and dx^{d-s} represents a $(d-s)$-dimensional volume element of the orthogonal complement of $J_{s(0)}$. In terms of these quantities, we have the intuitively apparent and convenient exterior differential relation

$$f(b)\, db = dJ_s / \int dJ_{s(0)} = dJ_{s(0)}\, dx^{d-s} / \int dJ_{s(0)} \tag{8}$$

where

$$dJ_{s(0)}\, dO^s \cdots dO^1 = dO^d \cdots dO^{d-s+1} \tag{9}$$

(dO^1 represents the 'volume element' of a measure concentrating unit mass on both ± 1), which implies

$$\int dJ_{s(0)} = \sigma_d \cdots \sigma_{d-s+1} / \sigma_s \cdots \sigma_1. \tag{10}$$

Apart from the constant factor $\int dJ_{s(0)}$, this is the form given by Santaló [22]. Examples: —$s=0$: $f(x)=1$; $s=d-1$: if, in polar coordinates, (p, u) is the foot of the perpendicular from 0 to the hyperplane, then $f(p, u)=2/\sigma_d$, the corresponding element being $(2/\sigma_d)\,dpd0$. Integrating (8) over B_X, and defining X_{d-s} to be the orthogonal projection of X onto the orthogonal complement of $J_{s(0)}$, we obtain

$$F(B_X) = \int |X_{d-s}|\, dJ_{s(0)} \Big/ \int dJ_{s(0)} = M_{d-s}\{X\}, \tag{11}$$

the *mean* $(d-s)$*-projection* of X. Examples (see [16; §2, 3]): — $-M_d\{X\}=|X|$; for convex X, $M_{d-1}\{X\}=\left\{\Gamma\left(\frac{d}{2}\right)/2\pi^{1/2}\,\Gamma\left(\frac{d+1}{2}\right)\right\}|\partial X|$; $M_0\{X\}=1$. For $0<M_{d-s}\{X\}<\infty$, the p.d.f.

$$f_X(b)=\begin{cases} f(b)/M_{d-s}\{X\} & (b\in B_X) \\ 0 & (b\notin B_X) \end{cases} \tag{12}$$

determines a *uniform isotropic* (random) *s-secant of* X. Such secants possess rather natural properties. Thus, if J_s is a uniform isotropic s-secant of X, then

(i) $P(J_s \uparrow Y \subset X) = M_{d-s}\{Y\}/M_{d-s}\{X\}$, *independent of the 'position' of* Y *within* X;

(ii) given $J_s \uparrow Y \subset X$, then J_s is uniform isotropic in Y; further,

(iii) given that the flat of intersection of independent uniform isotropic secants in X hits X, this flat is a uniform isotropic secant of X of appropriate dimension.

Consider the random system comprising N independent uniform isotropic s-secants of X. The distribution of $\#_Y$, the number of secants hitting $Y\subset X$, is binomial $(N, M_{d-s}\{Y\}/M_{d-s}\{X\})$. As N, $M_{d-s}\{X\}$ both $\to\infty$ in such a way that $N/M_{d-s}\{X\}\to\rho$, the distribution of $\#_Y$ tends to Poisson $(\rho M_{d-s}\{Y\})$, in the usual way. *This and the relation* (II) *suggest the consideration of the stochastic flat process* $\mathfrak{M}(\rho; s, d)$ *in* E^d *corrseponding to the (inhomogeneous) Poisson point process in* $[b]$ *of intensity* $\rho f(b)$. This definition ensures that $\mathfrak{M}(\rho; s, d)$ is stochastically invariant with respect to Euclidean motions. Thus it is both *homogeneous* (sometimes described as 'strict stationarity'), i. e. stochastically invariant under translations; and *isotropic*, i. e. stochastically invariant under rotations. Accordingly, we describe $\mathfrak{M}(\rho; s, d)$ as *the isotropic homogeneous Poisson s-flat process of intensity* ρ *in* E^d. Immediate from (II) and the Poisson structure is

Theorem 1. *The number* $\#_X$ *of s-flats os* $\mathfrak{M}(\rho; s, d)$ *hitting* $X\subset E^d$ *has a Poisson* $(\rho M_{d-s}\{X\})$ *distribution. Further, given that* $\#_X=N$, *these* N *s-secants of* X *are independent uniform random s-secants of* X. It is left to the reader to derive the p.g.f. of the multivariate Poisson distribution of $(\# X(1),\cdots,\# X(m))$ for arbitrary $X(i)\subset E^d$. Examples: $\mathfrak{M}(\rho;\,0,1)$ is the standard linear Poisson point process, and $\mathfrak{M}(\rho;\,0,d)$ the corresponding d-dimensional point process. For clarity, the 0-flats of $\mathfrak{M}(\rho;\,0,d)$ are termed particles. Since a hyperplane partitions E^d into two separated half-spaces, $\mathfrak{M}(\rho;\,d-1,d)$ (which includes $\mathfrak{M}(\rho;\,0,1)$) has the effect of partitioning E^d into a tessellation P of random convex polytopes (see § 3). (Thus a 1-dimensional convex polytope is an interval). The intensity ρ is also characterized as the mean s-content of s-flat of $\mathfrak{M}(\rho;\,s,d)$ per unit volume in E^d. We now state two of the fundamental properties of isotropic Poisson flat systems.

Independent Superposition. If $\mathfrak{M}(\rho_1;\,s,d),\cdots,\mathfrak{M}(\rho_m;\,s,d)$ are independent, then $\bigcup_{i=1}^{m}\mathfrak{M}(\rho_i;\,s,d)$ is a $\mathfrak{M}\left(\sum_{i=1}^{m}\rho_i;\,s,d\right)$.

Arbitrary Section. If J_t is an arbitrary t-flat in E^d $(d-s\leqslant t<d)$, then $J_t\cap\mathfrak{M}(\rho;\,s,d)$ is a $\mathfrak{M}(\rho';\,s+t-d,t)$, where

$$\rho'=\left\{\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{t+1}{2}\right)/\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{s+t-d+1}{2}\right)\right\}\rho. \tag{13}$$

Henceforth, since ρ is mostly fixed, 'ρ' is usually omitted from '$\mathfrak{M}(\rho;s,d)$'.

In this paragraph we forget $\mathfrak{M}(s,d)$ for the moment, and return to the integral geometry of s-flats in E^d. An *n-figure* of s-flats is defined to be an ordered set of n distinct s-flats (an un-ordered such set is termed an *n-set*). Parametrically,

$$c=(b_1,\cdots,b_n)\in[c]\subset E^{n(s+1)(d-s)}. \tag{14}$$

The corresponding invariant density in $[c]$ is clearly

$$f(c)=\prod_i^n f(b_i)\qquad (c\in[c]). \tag{15}$$

However, we shall use instead a special 'structural' parametrization $\bar{c}$ of an n-figure, best illustrated by an example. Taking $s=0$, $d=2$, and $b=(x,y)$, set

$$\left|\begin{array}{l}(x_1,y_1)=(x,y)\\(x_2,y_2)=(x+l\cos\theta,\;y+l\sin\theta)\\(x_i,y_i)=(x+\lambda_i l\cos\overline{\theta+\Phi_i},\;y+\lambda_i l\sin\overline{\theta+\Phi_i})\qquad(3\leqslant i\leqslant n).\end{array}\right. \tag{16}$$

Then

$$\bar{c}=(\underbrace{x,y}_{\text{location}};\quad\underbrace{\theta}_{\text{orientation}};\quad\underbrace{l}_{\text{scale}};\quad\underbrace{\lambda_3,\Phi_3,\cdots,\lambda_n,\Phi_n}_{\text{shape}}) \tag{17}$$

is structural, the components contributing to the four elements of structure of the n-figure being indicated. Since in this case $f(c)=1$,

$$\tilde{f}(\tilde{c}) = \left|\frac{\partial c}{\partial \tilde{c}}\right| f(c) = l^{2n-3} \prod_{3}^{n} \lambda_i. \tag{18}$$

For general (s, d), the intersection of the n component s-flats of an n-figure is in general an $\{ns-(n-1)d\}$—flat. Attention is here restricted to the case in which this flat is either void or at most a point, i. e. $n > d/(d-s)$. Then a *centre* $z \in E^d$ specifying the location of the n-figure can be defined, and we may write $\tilde{c} = (z, \alpha)$. Thus $z = (x, y)$ in (17). Clearly the invariant density factorises to give $\tilde{f}(\tilde{c}) = 1 \cdot f(\alpha)$. For $n > d/(d-s)$, there is in general a unique non-degenerate sphere hitting all n component s-flats an of minimal radius—the *minimal sphere* of the n-figure. In the minimal sphere has radius l, then we may write $\alpha = (l, \beta)$, in which the $\{n(s+1)(d-s)-(d+1)\}$-tuple β specifies *shape* and *orientation*. The 'characteristic length' l determines the *scale* of the n-figure. The reader should envisage the sequential construction of an n-figure by (i) z; (ii) β; (iii) l. Finally, integral geometric argument, which for reasons of space must be omitted, serves to generalise (18) to

$$\tilde{f}(\tilde{c}) = 1 \cdot f(\alpha) = 1 \cdot f(l, \beta) = l^{n(d-s)-(d+1)} f(1, \beta), \tag{19}$$

where $f(l, \beta) \equiv [f(1, \beta)]_{l=1}$.

Now re-consider $\mathfrak{M}(s, d)$. Define $\mathfrak{M}_n(s, d)$ to be the aggeregate of n-figures generated by $\mathfrak{M}(s, d)$ $(n=1, 2, \cdots)$. Thus each n-set consisting of members of $\mathfrak{M}(s, d)$ gives rise to $n!$ members of $\mathfrak{M}_n(s, d)$, and $\mathfrak{M}_n(s, d)$ may be regarded as a stochastic point process in $[c]$ or $[\tilde{c}]$. The a.s. (almost sure) countability of the members of $\mathfrak{M}(s, d)$ implies the same property for $\mathfrak{M}_n(s, d)$. Define $H_n\{X, \delta\alpha\}$ to be the number of members of $\mathfrak{M}_n(s, d)$ with centre $z \in X \subset E^d$ and with α—value lying in the notionally small $\{n(s+1)(d-s)-d\}$-dimensional interval '$\delta\alpha$' in $[\alpha]$ with opposite vertices α and $\alpha + \delta\alpha$. In addition to its normal meaning as an increment in α, $\delta\alpha$ is also used for the above-mentioned interval and the $\{n(s+1)(d-s)-d\}$-content of this interval; the interpretation intended will be apparent from the context. The corresponding normalized 'empiric average' is

$$h_q = H_n\{Q_q, \delta\alpha\}/|Q_q|. \tag{20}$$

Then

$$E(h_q) = |Q_q|^{-1} E \int_{Q_q} H_n\{dz, \delta\alpha\} = |Q_q|^{-1} \int_{Q_q} EH_n\{dz, \delta\alpha\}. \tag{21}$$

Appealing to the extreme 'Poisson' independence, and the relation $f(c)\,dc = \tilde{f}(\tilde{c})\,d\tilde{c}$, we have

$$E[H_n\{dz, \delta\alpha\}^k] = \rho^n f(\alpha)\, dz\, \delta\alpha\, \{1+0(\delta\alpha)\} \tag{22}$$

for an arbitrary positive integer k. Combining (21) and (22) with $k=1$ and applying the homogeneity of $\mathfrak{M}(s, d)$, we obtain

$$E(h_q) = \rho^n f(\alpha)\,\delta\alpha\,\{1 + 0(\delta\alpha)\}. \tag{23}$$

Henceforth, for brevity, factors like $\{1+0(\delta\alpha)\}$ are usually omitted. Similarly,

$$\operatorname{Var} h_q = |Q_q|^{-1} \int_{Q_q}\int_{Q_q} [E\{H_n(dx, \delta\alpha)\, H_n(dy, \delta\alpha)\} - EH_n(dx, \delta\alpha)\, EH_n(dy, \delta\alpha)]$$

$$\equiv |Q_q|^{-1} \int_{Q_q}\int_{Q_q} g(x-y, \delta\alpha)\, dx\, dy, \tag{24}$$

say. Now g is zero except on a subset S of $Q_q \times Q_q$, where $|S|/|Q_q| \to \to a(\delta\alpha) < \infty$ as $q \to \infty$. Further, application of Schwartz's Inequality and (22) with $k=2$ shows that $g \leqslant b(\delta\alpha) < \infty$ on S. It follows that $\operatorname{Var} h_q \to 0$ as $q \to \infty$, and so $h_q \underset{m.s.}{\to} \rho^n f(\alpha)\,\delta\alpha$, a constant, as $q \to \infty$. But, since $\mathfrak{M}(s, d)$ is homogeneous, Wiener's d-parameter ergodic theorem [30; Theorem II″] applies, giving $h_q \underset{a.s.}{\to}$ a random variable, h say, as $q \to \infty$. Since m.s. and a.s. limits coincide (=in probability limit), we have $h_q \underset{a.s.}{\to} \rho^n f(\alpha)\,\delta\alpha$ as $q \underset{a.s.}{\to} \infty$. In fact, more generally

$$H_n\{X, \delta\alpha\}/|X| \underset{a.s.}{\to} \rho^n f(\alpha)\,\delta\alpha\,\{1+0(\delta\alpha)\} \tag{25}$$

as $X \to \infty$, where $X = X_q = qX_1$, X_1 is a bounded region of E^d containing o with $|X_1| > 0$, and $X \to \infty$ is equivalent to $q \to \infty$. The significance of (25) is emphasised by

$$H_n\{X\delta\alpha_1\}/H_n\{X, \delta\alpha_2\} \underset{a.s.}{\to} f(\alpha_1)\,\delta\alpha_1\,\{1+0(\delta\alpha_1)\}/f(\alpha_2)\,\delta\alpha_2\,\{1+0(\delta\alpha_2)\} \tag{26}$$

as $|X| \to \infty$. *That is, $f(\alpha)$ is moreover the (a.s.) ergodic density of α for $\mathfrak{M}_n(s, d)$.* Ergodic densities, like invariant densities, are only defined up to a constant factor. It may be said that the ergodic density $f(\alpha)$ is the 'quotient' of the invariant density by the uniform density of the centre: $f(\alpha) = \bar{f}(c)/f(z)$. Factorization of the ergodic density for disjoint sets of components of α means that these sets are ergodically independent (providing also, of course, that the joint range is the corresponding product range, as is usually the case). Thus l and β are ergodically independent. The ergodic density of l is not normalizable over its full range $[0, \infty)$, although it may of course be normalized over a truncated range. Generally speaking, as in (18), the orientation components of β are normalizable, whereas the shape components are not.

We now show the ergodic density $l^{n(d-s)-(d+1)}$ may be normalized into an ergodic probability density in a rather natural manner. Consider a mapping

$$\alpha \to Y(\alpha) \subset E^d \tag{27}$$

where $Y(\alpha)$ may possibly be $\varnothing$, and it is supposed (writing, for brevity, $M(\alpha)$ for $M_{d-s}\{Y(\alpha)\}$):

(i) $M(\alpha)$ exists and is finite on $[\alpha]$;

(ii) $M(\alpha)$ and, in a suitable sense, $Y(\alpha)$ itself are continuous in $[\alpha]$, except possibly on sub-varieties of $[\alpha]$;

(iii) $A \equiv \{\alpha\colon M(\alpha)$ is positive and continuous$\}$ has positive Lebesgue measure in $[\alpha]$.

The corresponding mapping

$$\bar{c} = (z, \alpha) \to \{z\} + Y(\alpha) \tag{28}$$

is *translation invariant*. We shall be concerned with the aggregate

$$\mathbf{Y}_n = \{\{z\} + Y(\alpha);\ (z, \alpha)\colon \in \mathfrak{M}_n(s, d)\} \tag{29}$$

of random 'associated sets' generated by $\mathfrak{M}_n(s, d)$. Now define, for $\alpha \in A$, $H_n^{(m)}\{X, \delta\alpha\}$ to be the number of the $H_n(X, \delta\alpha)$ members of $\mathfrak{M}_n(s, d)$ with centre in X and α-value in $\delta\alpha$, whose associated set is hit by exactly m s-flats of $\mathfrak{M}(s, d)$, *excluding the m component s-flats*. Finally, write $\mathfrak{M}_n^{(m)}(s, d)$, $\mathbf{Y}_n^{(m)}$ for the corresponding sub-aggregates of $\mathfrak{M}(s, d)$, $\mathbf{Y}_n$.

The preceding ergodic theory may be repeated, utilising the extreme Poisson independence properties of $\mathfrak{M}(s, d)$, the assumed continuity of $Y(\alpha)$, and Theorem 1, to show that

$$H_n^{(m)}\{X, \delta\alpha\}/H_n\{X, \delta\alpha\} \xrightarrow[a.s.]{} (\rho M(\alpha))^m e^{-\rho M(\alpha)}/m! \quad (\alpha \in A) \tag{30}$$

as $|X| \to \infty$. Thus the ergodic density of $\mathfrak{M}_n^{(m)}(s, d)$ is

$$f^{(m)}(\alpha) = f(\alpha)\, M(\alpha)\, e^{-\rho M(\alpha)} \qquad (\alpha \in A). \tag{31}$$

The exponential factor clearly has a powerful effect rendering normalizable hitherto non-normalizable ergodic densities.

In applications, $Y(\alpha)$ is usually *homothetically invariant* in A, i. e.

$$Y(\alpha) = Y(l, \beta) = l\,Y(1, \beta), \tag{32}$$

which implies

$$M(\alpha) = l^{d-s} M(1, \beta). \tag{33}$$

In this case, we may substitute $M(\alpha)$ for l in (19) and (31), i. e. $(l, \beta) = \alpha \to \alpha' = (M, \beta)$. There results.

$$\frac{H_n^{(m)}\{X, \delta(M, \beta)\}}{|X|} \xrightarrow[a.s.]{} \frac{\rho^{m+n}}{m!(d-s)} \underbrace{\underbrace{\frac{f(1, \beta)}{M(1, \beta)^{n-d/(d-s)}}}_{f^{(m)}(\beta)} \underbrace{M^{m+n-1-\frac{d}{d-s}}\, e^{-\rho M}}_{f^{(m)}(M)}}_{f^{(m)}(M, \beta)} \delta M \delta\beta \tag{34}$$

Thus M and β are ergodically independent, with densities as indicated.

The ergodic distribution of β is in general rather complex, but on the other hand we have

Theorem 2. *In the homothetically invariant case, the ergodic distribution of the mean projections M_{d-s} of the members of $\mathbf{Y}_n^{(m)}$, given any 'meaningful' condition on β, and thus in particular no condition, is* $\Gamma\left(m+n-\frac{d}{d-s}, \rho\right)$. Theorem 2 yields a wide class of distributions corresponding to varying choices of s, d, $Y(\alpha)$ and conditions on β. Note that Theorems 1 and 2 are in a sense dual, since they give respectively $P(\#|M_{d-s})$ and $P(M|_{d-s}|\#)$. In fact, Theorem 2 follows heuristically from the Bayes' relation

$$P(M_{d-s}|m, n) \propto P(M_{d-s}|n)\, P(m|M_{d-s}, n)$$

Examples of Theorem 2:- $(s, d)=$

(0,1): the total lengths of sets of n consecutive inter-particle intervals are $\Gamma(n, \rho)$;

(0,2): the areas of the 'empty' convex n-gons with particle vertices are $\Gamma(n-1, \rho)$;

(1,2): the in-radii of the polygons of the random tessellation **P** are exponential (2ρ);

(1,2): the perimeters of the n-gons of **P** are $\Gamma(n-2, \rho/\pi)$.

The reader is advised to verify these examples and perhaps construct others of his own. Note that, in applications of Theorem 2, $Y(\alpha)$ is usually *order invariant*, i. e. has the same value for all $n!$ n-figures corresponding to each n-set.

§ 2. Generalisations

We now consider three distinct generalisations of $\mathfrak{M}(s, d)$, which may all be simultaneously incorporated. As moreover s and d are arbitrary, a rather wide class of possible random models results.

1°. *Mixtures.* The *mixture*

$$\mathfrak{M}(\rho_0, \cdots, \rho_{d-1}; d) = \bigcup_{s=0}^{d-1} \mathfrak{M}(\rho_s; s, d),$$

where the $\mathfrak{M}(\rho_s; s, d)$ are supposed independent. Theorem 1 generalises trivially, as do the results in § 1 regarding independent superpositions and arbitrary sections. More interesting now is an n—figure, which comprises n_s s-flats $(0 \leq s < d;\ \Sigma_{s=0}^{d-1} n_s = n)$. Varga [27] gave invariant densities (which he termed 'Crofton formulae') for some such mixed n-figures in E^2 and E^3. The n component flats intersect in a $[\Sigma_s (sn_s) - (n-1)d]$-flat, and there is a characteristic length l when $\Sigma_s(d-s)\, n_s > d$. The invariant/ergodic density is

$$f(l, \beta) = l^{\Sigma_s (d-s) n_s - (d+1)} f(1, \beta). \tag{36}$$

Since in general $M_1\{Y(\alpha)\}, \cdots, M_d\{Y(\alpha)\}$ are not simply inter-related, there is no satisfactory extension of Theorem 2. However, an important exception is the choice of $Y(\alpha)$ as a ball, in which case $M_{d-s}=\varepsilon_{d-s}\times \times r^{d-s}$ $(0\leqslant s\leqslant d,\ r=\text{radius})$. For example, the locally maximal empty balls in the interstices between the members of $\mathfrak{M}(\rho_0,\cdots,\rho_{d-1};\ d)$ are 'tangential' to exactly $d+1$ flats; the radii of such balls touching n_s s-flats; $(0\leqslant s<d,\ \Sigma_s n_s=d+1)$ has ergodic p.d.f.

$$f(r)\,\alpha\, r^{\Sigma_s (d-s)\, n_s -(d+1)} \exp\,[-\Sigma_s\ \rho_s\, \varepsilon_{d-s}\, r^{d-s}], \tag{37}$$

a generalised gamma distribution. The radius distribution for the entire class of locally maximal empty balls is a weighted sum of distributions (37); the weights being

$$W(n_*)=\int\limits_{\substack{\text{locally}\\ \text{maximal}}} \lim_{|X|\to\infty} [H_{\{n_*\}}\{X,\ \delta r d\beta\}/|X|]. \tag{38}$$

Clearly $W(n_*)=\prod\limits_s \rho_s^{n_s}$.

2° *Anisotropy.* Since a point has no orientation, this extension is lonly possible for $1\leqslant s<d$. Parametrizing an s-subspace by the s $(d-s)$ tuple a, we have $b=(a,\ x^{d-s})$. Consider the anisotropic analogue

$$\hat{F}(db)=\Theta\,(da)\,dx^{d-s} \tag{39}$$

of (8), in which Θ is a general probability measure on [a]. The important property preserved from § 1 is that $\hat{F}(B)$ is invariant under transations. The analogue of (11) is

$$\hat{F}(B_X)=\int |X_{d-s}(a)|\ \Theta\,(da)\equiv \hat{M}_{n-s}\{X\}. \tag{40}$$

A *uniform* Θ (random) *s-secant of* X has probability element $\hat{F}(db)/\hat{M}_{d-s}\{X\}$ $(b\in B_X)$. $\mathfrak{M}(\rho;\ s,\ d;\ \Theta)$ is now defined in the natural way, $\#_X$ having a Poisson $(\rho\hat{M}_{d-s}\{X\})$ distribution (Theorem 1). The independent superposition

$$\bigcup_i \mathfrak{M}(\rho_i;\ s,\ d;\ \Theta_i)=\mathfrak{M}(\Sigma_i\,\rho_i;\ s,\ d;\ \Sigma_i\,\rho_i\,\Theta_i/\Sigma_i\,\rho_i), \tag{41}$$

and in general the section $J_t\cap\mathfrak{M}(\rho;\ s,\ d;\ \Theta)$ is a $\mathfrak{M}(\rho';\ s+t-d,\ t;\ \Theta')$. One difference from the isotropic case should be noted. In $\mathfrak{M}(s,\ d)$, almost surely no two s-flats are parallel. But here, if a_0 is an atom of Θ, then a 'fraction' (in an ergodic sense) $\Theta(\{a_0\})$ of the members of $\mathfrak{M}(s,\ d;\ \Theta)$ have orientation a_0. The translational invariance means that $\mathfrak{M}(s,\ d;\ \Theta)$ is homogeneous, and so admits an ergodic theory. Turning to n-figures, the 'nth product' of (39) admits a similar decomposition to that in (19), in the sense that the independent $l^{n(d-s)-(d+1)}$ density carries over. Consequently Theorem 2 extends, $\Gamma\left(m+n-\dfrac{d}{d-s},\ \rho\right)$ now being the ergodic distribution of $\hat{M}_{d-s}$ for $\mathbf{Y}_n^{(m)}$ in the homothetically invariant case. In particular, since $\hat{M}_i\{\zeta_q\} = M_i\{Q_q\}$, ergodic 'ball' distributions are unchanged. Thus (37) extends unchanged when 1° and 2° are combined.

It emerges from § 1 that the isotropic $\mathfrak{M}(s, d)$ is fundamental in the sense of integral geometry; on the other hand, $\mathfrak{M}(s, d; \Theta)$ is of fundamental importance as a *local limit*, as we now explain. Suppose V_0 is a movable smooth s-dimensional variety in E^d, whose position is determined by a centre $z \in E^d$ and a d-frame $\{u_1, \cdots, u_d\}$ of orthonormal vectors emanating from z, fixed with respect to V_0. Randomising by giving (z, u_*) thd distribution $D(z, u_*)$ furnishes the *random image* V of V_0. We suppse $D(z|u_*)$ is continuous for all $\{u_*\}$ for which it is defined, and set

$$Y = \{x \in E^d: 0 < \lim_{q \to 0} q^{s-d} P\{V \uparrow \{x\} + Q_q\} < \infty\}. \tag{42}$$

Consider n independent random images $V^1, \cdots, V^N$ of V_0. It may be shown that, under the above conditions, the local limit of the system in the neighbourhood of $x \in Y$ as $N \to \infty$, under the local dilation $(y-x)' = N(y-x)$ at x, is $\mathfrak{M}(\rho_x; s, d; \Theta_x)$. Both ρ_x and Θ_x depend on V_0 and D, as well as x.

Suppose that z is independent of $\{u_*\}$ and is uniform over some region R of E^d. Then, assuming edge effects near ∂R have been eliminated, the local limit is the same at all points of R. If, moreover, $\{u_*\}$ is uniform (i. e. normalized Haar measure on the d-dimensional rotation group) then the local limit is $\mathfrak{M}(s, d)$. The conditions for the local limit at x to be $\mathfrak{M}(\rho_x; s, d; \Theta_x)$ may be widened by, for instance, sampling the s-varieties from some distribution, and allowing them a certain degree of mutual dependence. Actually, $\mathfrak{M}(0, d)$ is equally fundamental as a local limit, having been discussed by Goldman [8].

3° *Cylinders*. For each member J_s of $\mathfrak{M}(s, d)$, associate a random set W_{d-s} in the $(d-s)$-subspace orthogonal to J_s, the association being stochastically invariant with respect to translations of J_s. Consider the system of cylinder sets

$$\mathbf{C} = \{J_s + W_{d-s}: J_s \in \mathfrak{M}(s, d)\}. \tag{43}$$

The case $s = 0$ has been considered by Takács [26] and Giger and Hadwiger [5], amongst others. If the W_{d-s} are independently and identically distributed, then the number of cylinders containing $x \in E^d$ has a Poisson $(\rho E|W_{d-s}|)$ distribution. It is then an ergodic result that the 'fraction of E^d' which is i-covered (in the sense of the limiting fraction of Q_q as $q \to \infty$) is

$$p_i = (\rho E|W_{d-s}|)^i \exp[-\rho E|W_{d-s}|]/i! \qquad (i = 0, 1, \cdots). \tag{44}$$

If the random sets W_{d-s} are almost surely convex, then mutual intersection probabilities of the members of $\mathbf{C}$ hitting arbitrary fixed convex sets of E^d may be investigated by means of iteration of the complete system of kinematic formulae of integral geometry — see Streit [25]. This technique serves also to generalise Theorem 1 when, moreover, X itself is convex. Clearly independent superpositions and arbitrary sections yield corresponding Poisson cylinder systems, but equally clearly Theorem 2 does not extend. We conclude § 2 by exploring a special case of 3°.

Coverage and concentration. Consider N arbitrary subsets of a set X in a general space. Suppose $x \in X$ lies in $H(x)$ of these subsets, and define

$$\underline{H}_X = \min_{x \in X} H(x), \ \overline{H}_X = \max_{x \in X} H(x). \tag{45}$$

That is, the least—and most-covered regions of X are respectively $\underline{H}_X$ and $\overline{H}_X$— covered. Alternatively, $\underline{H}_X$ and $\overline{H}_X$ determine the overall *coverage* of X by the subsets, and the maximal *concentration* of the subsets in X, respectively. If the subsets are random, then we should like to know the jount p.m.f. of $(\underline{H}_X, \overline{H}_X)$.

Poisson discs. We now sketch the derivation of asymptotic probabilities of coverage and concentration for the special case of C in which discs of fixed radius r are centred at each particle of $\mathfrak{M}$ (0,2).

(i) *Coverage.* A disc in E^2 is a *loc. max.*$^{(j-1)}$ disc with respect to $\mathfrak{M}$ (0,2) if it contains $j+2$ particles, 3 of which lie in its perimeter circle in the form of an acute—angled triangle. Ignoring edge effects, which may be shown to be of negligible importance as $|X| \to \infty$, it is a geometrical identity that $\underline{H} > j$ iff every loc. max.$^{(j-1)}$ disc with centre in X has radius $< r$. Write $H^{(j-1)}\{X, \delta r\}$ for the number of loc. max.$^{(j-1)}$ discs with centre in X and radius in $(r, r+\delta r)$. By a specialisation of the theory of § 1, it may be shown [17; § 13] that, as $|X| \to \infty$,

$$H^{(j-1)}\{X, \delta r\}/|X| \underset{a.s.}{\to} 2\rho\ (\pi\rho)^{j+1} r^{2j+1} e^{-\pi\rho r^2}\ \delta r/(j-1)!\ \ (j=1, 2, \cdots). \tag{46}$$

Thus, asymptotically as $|X| \to \infty$, the aggregate of loc. max.$^{(j-1)}$ radii in X 'behaves like 'an independent sample of size $j\rho|X|$ from a $\Gamma_2(2j+2, \pi\rho)$ distribution. Falsely assuming such behaviour we should have, for arbitrary positive θ,

$$P(\underline{H}_X > j) = (1 - p\{j, \pi\rho r^2\})^{j\rho|X|}\, e^{-\theta} \tag{47}$$

as $|X| \to \infty$, provided $j\rho|X|\, p\{j, \pi\rho r^2\} = \theta$. This suggests

$$P(\underline{H}_X > j) \sim \exp\left[-j\rho|X|\, e^{-\pi\rho r^2}\left\{1 + \cdots + \frac{(\pi\rho r^2)^j}{j!}\right\}\right] \ \ (j=0,1,\cdots) \tag{48}$$

as $|X| \to \infty$. In fact, a rigorous *spatial* investigation of the homogeneous stochastic point process of loc. max.$^{(j-1)}$ centres and their associated radii shows that our 'dependent sampling' is 'asymptotically sufficiently independent' for $\sup_{r>0}|(\text{left side—right side})$ in (48)$| \to 0$ as $|X| \to \infty$. The main argument is similar to that of Watson [28]. The formula (48) with $j=1$ appears to be quite accurate even when $\rho|X|$ is as small as 100 — see Gilbert [7; p. 330] for the results of a computer simulation.

(ii) *Concentration.* Since the method applying here may be regarded as the dual of that of (i), we give even sketchier details. A disc in E^2 is a *loc. min.*$_{(k-1)}$ disc with respect to $\mathfrak{M}(0,2)$ if it contains $k+1$ particles, 2 of which lie in the perimeter circle at the ends of a diameter. Ignoring edge effects, $\overline{H}_X \leqslant k$ iff every loc. min.$_{(k-1)}$ and loc. max.$^{(k-2)}$ disc with centre in X has radius $> r$. Write $H_{(k-1)}\{X, \delta r\}$ for the total number of either of these types with radius in $(r, r+\delta r)$. Further specialisation of § 1 [17; § 13] shows that, as $|X| \to \infty$,

$$H_{(k-1)}\{X, \delta r\}/|X| \underset{a.s.}{\to} 2\,(k+1)\,\rho\,(\pi\rho)^k\, r^{2k-1}\, e^{-\pi\rho r^2}\,\delta r/(k-1)! \quad (k=1, 2, \cdots). \tag{49}$$

Thus $H_{(k-1)}\{X, \delta r\}$ approximates to a $(k+1)\rho|X|$ — sample from a $\Gamma_2(2k, \pi\rho)$ distribution. In analogy with (47),

$$P\,(\overline{H}_X \leqslant k) = (1 - q\,\{k-1,\ \pi\rho\})^{(k+1)\rho|X|}, \tag{50}$$

suggesting

$$P(\overline{H}_X \leqslant k) \sim \exp\left[-(k+1)\,\rho\,|X|\,e^{-\pi\rho r^2}\left\{\frac{(\pi\rho r^2)^k}{k!} + \cdots\right\}\right] \quad (k=1,2,\cdots). \tag{51}$$

This is in fact true in the same sense as (48), and with a similar justification.

These results generalise to higher dimensions. For Poisson spheres in E^3,

$$\left\{\begin{array}{l} P\,(\underline{H}_X \geqslant j) \sim \exp\left[-(3\pi^2/32)\, j\,(j+1)\,\rho\,|X|\,p\,\{j+1,\ 4\pi\rho r^3/3\}\right] \\ P\,(\overline{H}_X \leqslant k) \sim \exp\left[-\{4+(3/8)(\pi^2+16)(k-1)+(3\pi^2/32)(k-1)(k-2)\}\times\right. \\ \qquad\qquad \left.\times\,\rho\,|X|\,q\,\{k-1, 4\pi\rho r^3/3\}\right]. \end{array}\right. \tag{52}$$

It is hoped (48), (51) and (52) may prove useful in statistical applications; for instance, in testing the hypothesis of independent uniformity for point process data. A corresponding pair of formulae for general d may be derived. The interesting derivation utilises relations (66) and (67) below together with iteration of the general kinematic formulae for spaces of constant curvature, given by Santaló [23]. Incidentally, this iteration implies Wendel's [29] formula giving the probability that N independent isotropic random hemispheres on a sphere in E^d completely cover that sphere ($N=2, 3, \cdots$).

§ 3. Random Tessellations

A *tessellation* in E^d is defined to be an aggregate of convex polytopes which cover E^d without overlapping. (Henceforth we omit 'convex', since all polytopes considered are in fact convex). In this final section we investigate certain natural random tessellations generated by homogeneous Poisson flat systems.

But first consider a general random tessellation T in E^d for which a probability space with all necessary regularity properties has been established. Our examples are all of this type, since each depends in a simple way upon its underlying $\mathfrak{M}(s, d)$. A natural desirable property of T is homogeneity, i. e. stochastic invariance under translations. This is implied by isotropy, i. e. stochastic invariance under rotations. Note that arbitrary sections by a flat are random tessellations with corresponding properties. For a polytope T, let $Z=(Z_1,\cdots, Z_m)$ be a partial (or even complete) *description* of its Euclidean invariant properties. For us, important possible components of Z are $\{M_*\}$; $\{N_*\}$, where N_s is the number of s-facets (s-dimensional polytope facets) in ∂T; $\{L_*\}$, where L_s is the sum of the s-contents of the N_s s—facets; and I, the in-radius. It is convenient to write $V\equiv L_d\,(=M_d)$, $S\equiv L_{d-1}$ and $N\equiv N_0\,(=L_0)$. The polytope T is the convex hull of its set of N 0-facets or vertices. The standard reference on convex polytopes is Grünbaum [9].

We now sketch the general procedure required to establish ergodic distributions for T. Suppose Z is an arbitrary description, and that Z' is a 'particular value' in $[Z]$. Define H_q to be the number of polytopes of T 'within' Q_q, and $H_q(Z')$ to be the number of these for which $Z_i\leqslant Z_i'$ $(1\leqslant i\leqslant m)$. We write "'within'" here on the supposition that edge effects due to polytopes which hit ∂Q_q may be shown by *ad hoc* means to be of negligible importance as $q\to\infty$. The homogeneity of T implies the homogeneity of stochastic processes ot the type $\{Y(x)\}$ $(x\in E^d)$, where $Y(x)$ is the value of the description Y for the polytope Tx of T containing x. The homogeneity allows the application of Wiener's d—parameter ergodic theorem [30; Theorem 11″] to the empiric averages $\int_{Q_q} Y(x)\,dx/|Q_q|$ of such processes. Such application to $Y_1(x)=1/V(x)$ and $Y_2(x)=K_{Z'}(x)/V(x)$, where the indicator random variable $K_{Z'}(x)$ indicates the event $[Z_i(x)\leqslant Z_i'\ (1\leqslant i\leqslant m)]$ ensures the existence of the almost sure limits H, $H(Z')$ of $H_q/|Q_q|$, $H_q(Z')/|Q_q|$, respectively, as $q\to\infty$. In the important metrically transitive case these limits, which are in general random, degenerate to constants. However, although in practice metrical transitivity is difficult to prove directly, the demonstration of the 'asymptotic independence of T in distant localities of E^d', a sort of d-dimensional mixing condition, suffices to ensure the constancy of H and $H(Z')$. Then

$$H_q(Z')/H_q \underset{a.s.}{\to} H(Z')/H\equiv F(Z'), \tag{53}$$

the value of the *ergodic d.f. of Z for* T at the particular value Z'.

The general problem is the determination of $F(Z)$ for the important and natural descriptions Z. Write $G(Z)$ for the usually well-defined d.f. of Z for the polytope containing an arbitrary point of E^d, e.g. T_0. A by-product of the above ergodic theory is the basic relation

$$F(dV,\ dZ) = G(dV,\ dZ)/V \int V^{-1}\, G(dV), \tag{54}$$

which may be derived heuristically by regarding, on account of homogeneity, the origin o to be a 'random point in E^d'. This relation is important, if only because G is usually more accessible than F. Another way of expressing (54) is to say that T_0 is a 'Vf' polytope, where f represents the 'density' (i.e. combined p.m.f./p.d.f.) of F. Then a 'random polytope of T' is an 'f' polytope. Another *re*-assuring by-product of the ergodic theory is that the empiric mean of scalar Z (the quotient of the empiric averages of $Y_3(x) = Z(x)/V(x)$ and $Y_1(x)$) converges to the ergodic mean:

$$E_q(Z) \underset{a.s.}{\rightarrow} E(Z) = \int Z F(dZ) \quad \text{as } q \rightarrow \infty. \tag{55}$$

Example: For $d=2$, $E(N) = 2\chi/(\chi-2)$ where χ is the mean number of sides meeting at each vertex of T (for further details, see Matschinski [11] and Miles [17; § 10]).

The random tessellation **P**. Let us combine the elements 2°, 3° in the case $s=d-1$. Suppose $(b,\ w)$ represents the *hyperslab* containing all points of E^d whose perpendicular distance from the hyperplane b is at most w; b, w are its *mid-hyperplane* and *semi-thickness*, respectively. Let Φ be a general probability distribution in $(a,\ w)$ for which $E(w) < \infty$ and the marginal distribution of a is Θ. Generate in the usual way the system $\mathfrak{M}(\rho;\ d-1,\ d;\ \Phi)$ of hyperslabs. Thus its members are $\{(p_i,\ u_i,\ w_i)\}(i=1,\ 2, \cdots)$, where

(i) the perpendiculars $\{p_i\}$ from o to the mid-hyperplanes constitute a $\mathfrak{M}(0,\ 1)$ of intensity 2ρ on $[0,\ \infty)$;

(ii) independently of (i), $\{(u_i,\ w_i)\}$ are independently sampled from Φ. Write $\mathbf{P}^+$ for the aggregate of polytope interstices between these hyperslabs. The system of mid—hyperplanes, which constitutes a $\mathfrak{M}(\rho;\ d-1,\ d;\ \Theta)$, has the effect of partitioning E^d into a random tessellation, **P** say. (Clearly, for $\mathbf{P}^+$ and **P** to exist, it is necessary that Θ be not 'too degenerate'). Using Grünbaum's [9] terminology, almost surely every polytope of $\mathbf{P}^+$ and **P** is simple, i. e. each s-facet lies in the intersection of $d-s$ (d—1)-facets $(0 \leqslant s < d)$. Further, each s-facet of **P** lies in the boundaries of 2^{d-s} members of **P** $(0 \leqslant \leqslant s < d)$. Now, given that o is not covered by any member of $\mathfrak{M}(d-1,\ d;\ \Phi)$, let T_0^+ be the member of $\mathbf{P}^+$ containing it. It may be shown that T_0^+ (under this condition) and T_0 (with no conditions imposed) have the same stochastic construction. Consequently, since (54) applies to both $\mathbf{P}^+$ and **P**, *the ergodic distributions of* $\mathbf{P}^+$ *and* **P** *are* *identical*! Thus, for example, Theorem 2 and 2° imply that, for P^+, the ergodic distribution of I is exponential (2ρ), and the conditional ergodic distribution of $\hat{M}_{d-s}$ given N_{d-s} is $\Gamma(N_{d-s}-d, \rho)$.

Although attention is restricted henceforth to **P** generated by $\mathfrak{M}(d-1, d)$, it should be borne in mind that the following results apply equally to associated $\mathbf{P}^+$. The invariant density element for a hyperplane in E^d is $f(h)\,db = (2/\sigma_d)\,dp\,d0$. Hence the invariant density element for a d-figure of hyperplanes is $(2/\sigma_d)^d \prod_1^d dp_i\, d0_i$. If z is their common point, then $p_i = z.u_i\,(1 \leqslant i \leqslant d)$, and so

$$\left|\frac{\partial p_*}{\partial z}\right| = \text{modulus of } |u_1 \cdots u_d| \equiv \Lambda_d(u_*) \tag{56}$$

say, the d-content of the parallelotope with edges $\{u_*\}$. Thus the invariant density element is alternatively $(2/\sigma_d)^d \Lambda_d(u_*)\, dz \prod_1^d d0_i$, from which it follows that the ergodic p.d.f. of $\{u_*\}$ at the vertices of **P** is

$$\Phi(u_*) = \Lambda_d(u_*) / \int \Lambda_d(u_*)\, d0_1 \cdots d0_d. \tag{57}$$

The invariant density for a $(d+1)$-figure of hyperplanes is $(2/\sigma_d)^{d+1} \prod_0^d dp_i\, d0_i$. Write y, I for the in-centre and in-radius of the simplex so formed. Define $u_i' = \pm u_i$ so that the feet of the perpendiculars from y to the hyperplanes are $\{y + Iu_*'\}$. Then $|p_i| = y.u_i' + I\ (0 \leqslant \leqslant i \leqslant d)$, and so

$$\left|\frac{\partial p_*}{\partial(y, I)}\right| = \text{modulus of} \begin{vmatrix} 1 \cdots 1 \\ u_0' \cdots u_d' \end{vmatrix} \equiv \nabla_d(u_*') \tag{58}$$

say, which is d! times the d-content $\nabla_d(u_*')$ of the simplex with vertices at the points $u_i' \in \partial Q_1$. Thus the invariant density element is alternatively $(2/\sigma_d)^{d+1} \nabla_d(u_*')\, dy\, dI \prod_0^d d0_i$, from which it follows that the in-simplices of **P** have ergodic p.d.f.

$$f(I, u_*) = 2\rho\, e^{-2\rho I} \cdot \nabla_d(u_*) / \int_K \nabla_d(u_*)\, d0_0 \cdots d0_d \tag{59}$$

restricted to $K = \{\{u_*\}$: there is no hemisphere of ∂Q_1 containing all the $u_i\}$. An 'f' polytope may be constructed about its in-centre by means of (59), the remaining $[N_{d-1} - (d+1)]$ $(d-1)$-facets being determined by a $\mathfrak{M}(d-1, d)$ restricted so that none of its members hit the already determined in-ball. Actually, the joint orientation densities of (57) and (59) extend to the anisotropic case upon weighting by $\prod \Theta(du_i)$.

Suppose v is an arbitrary unit vector. Almost surely the relation 'extreme point of a polytope in the direction v' sets up a (1,1) correspondence between the vertices and the members of **P**. Hence, since each vertex is almost surely a vertex of 2^d polytopes, it is clear that

$E(N)=2^d$. This essentially geometrical property extends to the anisotropic case. The stochastic construction of an 'f' polytope with respect to its 'extreme v point' is clear:

(i) construct $\mathfrak{M}(d-1, d)$;

(ii) independently construct d random hyperplanes through 0 with joint distribution (57) and, of the 2^d convex polyiopal cones into which E^d is thereby partitioned, let C_0 be the a.s. unique one having 0 as extreme v point;

(iii) $T_0 \cap C_0$ is then an 'f' polytope.

Given the 2^d cones *up to a random rotation* (normalised Haar measure), they do not have equal chances of being C_0. Define the polar angle of the convex cone C (apex o) to be the angle of the convex polar cone $C^p=\{x^p : x^p \cdot x > 0, \text{ all } x \in C\}$. Then the chance a given one of these cones is selected as C_0 in the random rotation is proportional to its polar angle. For example, when $d=2$, the angle at a vertex have common p.d.f. $\frac{1}{2}\sin\theta$, whereas the polygon angles at its extreme v point has p.d.f. $\{1-(\theta/\pi)\}\sin\theta$ $(0 \leqslant \theta \leqslant \pi)$.

Denote the t-facets of a polytope T by $T_{i,t}$, $(1 \leqslant i \leqslant N_t)$. Denote the mean s-projection of $T_{i,t}$, *with respect to the t-flat containing it*, by $M_{s,t}\{T_{i,t}\}$, and define

$$Y_{s,t}\{T\}=\sum_{i=1}^{N_t} M_{s,t}\{T_{i,t}\} \quad (0 \leqslant s < t \leqslant d). \tag{60}$$

Then the edge elements of the triangular array $\{\mathbf{Y}_{s,t}\}$ are

$$Y_{s,s}=L_s,\ Y_{s,d}=M_s,\ Y_{0,s}=N_s \quad (0 \leqslant s \leqslant d). \tag{61}$$

Miles [13] has shown that, for $\mathbf{P}$ with respect to $\mathfrak{M}(\rho;\, d-1, d)$,

$$E(Y_{s,t})=\left\{2^{d+s-t}\binom{d}{t}\Gamma\left(\frac{t}{2}+1\right)\Big/\Gamma\left(\frac{t-s}{2}+1\right)\right\}\left\{\Gamma\left(\frac{d+1}{2}\right)\Big/\Gamma\left(\frac{d}{2}\right)\rho\right\}^s$$
$$(0 \leqslant s \leqslant t < d), \tag{62}$$

and

$$E(L_r L_s)=\frac{2^d \pi^{1/2}}{\Gamma\left(\frac{r+1}{2}\right)\Gamma\left(\frac{s+1}{2}\right)}\left\{\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\rho}\right\}^{r+s}\sum_{i=\max(r,s)}^{d}\binom{d}{i}\left(\frac{\pi}{2}\right)^i \times$$
$$\times\frac{\Gamma\left(\frac{i+1}{2}\right)}{\Gamma\left(\frac{i}{2}+1\right)}(i)_r\,(i)_s \quad (0 \leqslant r \leqslant s \leqslant d). \tag{63}$$

The first and second order ergodic moments (62) and (63) allow the variance-covariance matrix of $(L_0, \cdots, L_d)$, and thus in particular (N, S, V), to be evaluated. The moments $E(N_*)$ in (62) may be obtained by a limiting process from a result of Cover and Efron [4;Theorem 1'].

Note that, for the mixture $\mathfrak{M}(\rho_0, \rho_{d-1}; d)$,

$$P\,(\text{a 'random polygon of } \mathbf{P}\text{' contains no particles}) = \int_0^\infty e^{-\rho_0 V}\, dF_{\rho_{d-1}}(V), \tag{64}$$

the Laplace—Stieltjes transform of the ergodic d.f. of V for $\mathbf{P}$. This offers a possible combinatorial method of investigating this perhaps most important ergodic distribution of $\mathbf{P}$.

See [15] for a more detailed discussion of the planar case, and [24] for the generalisation of this to the hyperbolic plane.

The random tessellations $\mathbf{V}$, $\mathbf{D}$ and $\mathbf{V}_n$, $\dot{\mathbf{V}}_n$ $(n=1, 2, \cdots)$ *generated by* $\mathfrak{M}(0, d)$. $\mathbf{V}$: Label the particles of $\mathfrak{M}(0, d)$ by y_*; for example, y_i might be the ith nearest particle to o $(i=1, 2, \cdots)$.

$$T_i = \{x \in E^d\colon |x-y_i| \leqslant |x-y_j|,\ j \neq i\} \tag{65}$$

is almost surely a simple polytope, and $\mathbf{V} = \{T_*\}$ is a random tessellation—the *Voronoi* tessellation generated by $\mathfrak{M}(0, d)$ (see Rogers [20; Chapter 7]). Each s-facet lies in the s-flat of points which are equidistant from a set of $d-s+1$ particles. Thus, unlike $\mathbf{P}$, each s-facet lies in the boundaries of $d-s+1$ members of $\mathbf{V}$. In the 'practical' cases $d=2, 3$, the first order ergodic moments of V, S and $\{N_*\}$ were determined by Meijering [12], while Gilbert [6] evaluated the second order moments of V by computer calculation of definite integrals.

Blaschke [3] and Petkantschin [19] independently obtained the form

$$dx_0 \cdots dx_s = \nabla_s(x_*)^{d-s}\, dJ_s\, dx_0^s \cdots dx_s^s \tag{66}$$

of the density of $(s+1)$-–figures of points in E^d. Here $\nabla_s(x_*)$ is $s!$ times the s-content $\Delta_s(x_*)$ of the s-simplex with vertices $\{x_*\}$, J_s is the s-flat containing $\{x_*\}$, and $\{x_*^s\}$ are the coordinates of these points with respect to J_s. This served as the basis of a study by Kingman [10] of the random s-flat containing $s+1$ independent uniform random points of a convex body in E^d; in particular, he solved Sylvester's classical problem for a d-ball. In fact, by means of (66), all the moments $E(\Delta_s^k)$ of the s-content Δ_s of the simplicial convex hull of an $(s+1)$-sample from certain spherically symmetric d-dimensional probability distributions (and distributions obtained by affine transformation from such distributions) may be determined; the $s+1$ sample points may even be dependent, but full spherical symmetry in E^d must be preserved. For example, if $x_1, \cdots, x_{r+s}$ $(r>0,\ s>0,\ 2\leqslant r+s\leqslant d+1)$ are independent, $x_1, \cdots, x_r$ and $x_{r+1}, \cdots, x_{s+r}$ being uniformly distributed in Q_q and ∂Q_q, respectively, then

$$E(\Delta^k_{r+s-1}) = \frac{q^{r+s-1}}{(r+s-1)!}\,\frac{\Gamma\left(\frac{(r+s)(d+k)}{2}-s+1\right)}{\Gamma\left(\frac{(r+s)(d+k)-k}{2}-s+1\right)}\times$$

$$\times\left\{\frac{\Gamma\left(\frac{d}{2}+1\right)}{\Gamma\left(\frac{d+k}{2}+1\right)}\right\}^r\left\{\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d+k}{2}\right)}\right\}^s\prod_{i=d-r-s+2}^{d}\left\{\frac{\Gamma\left(\frac{k+i}{2}\right)}{\Gamma\left(\frac{i}{2}\right)}\right\}. \tag{67}$$

Again, if $x_0,\cdots,x_s$ is an independent $(s+1)$-sample from the general d-dimensional normal distribution $N(\mu.\ \Sigma)$, then

$$E(\Delta^k_s) = \{(s+1)^{1/2}\,(2\,|\Sigma|^{1/d})^{s/2}/s!\}^k\prod_{i=d-s+1}^{d}\left\{\Gamma\left(\frac{k+i}{2}\right)/\Gamma\left(\frac{i}{2}\right)\right\},\ (1\leqslant s<d). \tag{68}$$

A useful variant of (66) is

$$dx_1\cdots dx_s = \Lambda_s(x_*)^{d-s}\,dJ_{s(0)}\,dx_1^s\cdots dx_s^s \tag{69}$$

in which $\Lambda_s(x_*)$ is the s-content of the parallelotope with sides $\{x_*\}$, $J_{s(0)}$ is the s-subspace spanning $\{x_*\}$, and $\{x^s_*\}$ are the coordinates of these points with respect to $J_{s(0)}$. Using (69), it may similarly be proved that, if $\{x_*\}$ is an s-sample from $N(\mu,\Sigma)$, $(s+1)^{1/2}$ times the s-content of the simplicial convex hull of o and $\{x_*\}$ also has the moments (68). Although this latter result is well-known, (68) itself seems new. Of course (see, for example, Anderson [1; § 7.5]) the distribution is also the distribution of the 'generalised sample variance' common of a $(d+1)$-sample from an s-dimensional normal distribution. The distributions of Δ_1 and Δ_2 in (68) are khown. They are $\Gamma_2(d,\ 1/4|\Sigma|^{1/d})$ and $\Gamma(d-1,\ 2/\sqrt{3}\,|\Sigma|^{1/d})$ respectively.

We shall now show that

$$dx_0\cdots dx_d = R^{d^2-1}\nabla_d(u_*)\,dz\,dR\,d0_0\cdots d0_d, \tag{70}$$

where z, R are the circum-centre and circum-radius of $\{x_*\}$: $x_i = z + Ru_i$ $(0\leqslant i\leqslant d)$. If the foot of the perpendicular from 0 to the hyperplane $(p_i,\ u_i)$ is x_i, then

$$dx_0\cdots dx_d = \prod_{i=0}^{p} p_i^{d-1}\,dp_i\,d0_i \qquad \text{which, by (58)} \tag{71}$$

$$= \prod_{i=0}^{d} p_i(y,\ I,\ u_*)^{d-1}\,d0_i\cdot\Delta_d(u_*)\,dy\,dI$$

where y, I are the in-centre and in-radius, respectively, of the simplex formed by $\{(p_*,\ u_*)\}$,

$$\equiv \Phi(y,\ I,\ u_*)\,dy\,dI\prod_{i=0}^{d} d0_i,$$

say. But clearly

$$dx_0 \cdots dx_d = \psi(R, u_*)\, dz\, dR \prod_{i=0}^{d} d0_i, \tag{72}$$

for some function ψ, since z must be uniform and independent of (R, u_*). Now $y = 0$ iff $z = 0$, in which case $l = R$. Moreover, $dy = dz$ and $dl = dR$ at $y = z = 0$. (This may be demonstrated geometrically in the case in which an arbitrary p_i is adjusted by dp_i, leaving the remaining p_i and all the u_i fixed. It follows that it is also true when all the p_i are adjusted by dp_i together.) Hence, by (71) and (72),

$$\psi(R, u_*) = \Phi(0, R, u_*) = R^{d^2-1} \nabla_d(u_*), \tag{73}$$

which completes the derivation of (70)

Combining (66) and (70), we obtain

$$dx_0 \cdots dx_s = \nabla_s (u_*^s)^{d-s+1} R^{ds-1}\, dz dJ_{s(0)}\, dR\, d0_0^s \cdots d0_s^s, \tag{74}$$

where z, R are the circum-centre and circum-radius of $\{x_*\}$ in the s-flat $\{z\} + J_{s(0)}$ containing these points. This relation, assisted by (67), is tailor-made for the determination of the ergodic moments

$$E(L_s) = \frac{2^{d-s+1} \pi^{\frac{d-s}{2}} \Gamma\left(\frac{d^2 - ds + s + 1}{2}\right) \Gamma\left(\frac{d}{2}+1\right)^{d-s+\frac{s}{d}} \Gamma\left(d - s + \frac{s}{d}\right)}{(d-s)!\, d\, \Gamma\left(\frac{d^2 - ds + s}{2}\right) \Gamma\left(\frac{d+1}{2}\right)^{d-s} \Gamma\left(\frac{s+1}{2}\right) \rho^{s/d}}$$

$$(0 \leqslant s \leqslant d) \tag{75}$$

of V.

D: Each vertex of V is the circum-centre of a set of $d+1$ particles of $\mathfrak{M}(0, d)$, the convex hull of which is a simplex. The aggregate of such random simplices is a tessellation—the *Delaunay* tessellation D (for a verification, see Rogers [20; Chapter 8]). On account of (70), the ergodic p.d.f. of D

$$f(R; u_0, \cdots, u_d) = e^{-\rho \varepsilon_d R^d} R^{d^2-1} \Delta_d(u_*). \tag{76}$$

Thus R is independent of $\{u_*\}$ and has a $\Gamma_d(d^2, \rho\varepsilon_d)$ distribution. By means of (76), the formula $V = R^d \Delta_d(u_*)$ and the moments (67), we obtain all the ergodic moments

$$E(V^k) = $$

$$= \frac{(d+k-1)!\, \Gamma\left(\frac{d^2}{2}\right) \Gamma\left(\frac{d^2+dk+k+1}{2}\right) \Gamma\left(\frac{d+1}{2}\right)^{d-k+1} \prod_{i=2}^{d+1} \left\{\Gamma\left(\frac{k+i}{2}\right) / \Gamma\left(\frac{i}{2}\right)\right\}}{(d-1)!\, \Gamma\left(\frac{d^2+1}{2}\right) \Gamma\left(\frac{d^2+dk}{2}\right) \Gamma\left(\frac{d+k+1}{2}\right)^{d+1} \left(2^d \pi^{\frac{d-1}{2}} \rho\right)^k}$$

$$(k = 1, 2, \cdots) \tag{77}$$

of V for D. For $d = 1$, $\mathbf{D} \equiv \mathbf{P}$ and V is exponential (ρ).

$\mathbf{V}_n$, $\dot{\mathbf{V}}_n$ $(n=1, 2, \cdots)$: An arbitrary point of E^d almost surely possesses a set of n nearest particles; the points possessing the same set of n nearest particles form a simple polytope; the aggregate of such polytopes is defined to be $\mathbf{V}_n$. The random tessellation $\mathbf{V}_n$ is rather similar to $\mathbf{V}\equiv\mathbf{V}_1$, in that each s—facet is a facet of $d-s+1$ polytopes of $\mathbf{V}_n$. In fact, (74) implies that the joient orientation p.d.f. of the $d-s+1$ particles equidistant from an s—facet of $\mathbf{V}_n$ is given by

$$f(u_0^{d-s}, \cdots, u_{d-s}^{d-s})=\nabla_{d-s}\,(u_*^{d-s})^{s+1} \qquad (0\leqslant s\leqslant d-2) \qquad (78)$$

The joint orientation of the $\binom{d-s+1}{2}$ $(d-1)$-facets meeting at this s-facet may in theory be deduced from (78).

Taking into account the *order* of the n nearest particles, we obtain $\dot{\mathbf{V}}_n$, a 'refinement', of, and more complex than, $\mathbf{V}_n$. In fact, denoting the union of polytope boundaries of a tessellation $\mathbf{T}$ by $\partial\mathbf{T}$, we have

$$\partial\dot{V}_n=\partial\mathbf{V}_1\cup\cdots\cup\partial\mathbf{V}_n. \qquad (79)$$

It may be shown that the local limit (see § 2) as $n\to\infty$ of $\mathbf{V}_n$ at an arbitrary point of E^d is $\mathbf{P}$ with respect to $\mathfrak{M}(\rho_n;\, d-1,\, d)$, where

$$\rho_n=\left\{2^{3d-1}\,d!\,\Gamma\left(\frac{d}{2}+1\right)^{2-\frac{1}{d}}\Big/\pi^{1/2}\,(2d)!\right\}\rho^{1/d}\,n^{2-\frac{1}{d}}. \qquad (80)$$

The values of each of $E(V)$, $E(S)$ and $E(N)$ in the case $d=2$ are given in [17; § 10].

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Ռ. Ե. ՄԱՅԼՍ. Հարթությունների պուասոնյան դաշտեր էվկլիդյան տարածություններում *(ամփոփում)*

E^d-ում s-հարթությունների համասեռ իզոտրոպ պուասոնյան դաշտերը սահմանվում են որպես ուղղի վրա (s=0, d=1) ստանդարտ պուասոնյան պրոցեսի բնական ընդհանրացումներ։ Դիտարկվում է s-հարթությունների n-ենթաբազմությունների էրգոդիկ տեսությունը այդպիսի դաշտերում։ Դուրս է բերվում Γ տիպի բաշխումների մի լայն դաս։

Այդ տեսության այլ ընդհանրացումների թվում դիտարկվում են որոշ պատահական մոզաիկաներ ուռուցիկ պոլիտոպներից։

Р. Е. МАЙЛС. *Пуассоновские поля плоскостей в евклидовых пространствах*
(резюме)

Однородные изотропные пуассоновские поля s-плоскостей в E^d определяются как естественные обобщения стандартного пуассоновского процесса на прямой ($s=0$, $d=1$). Рассматривается эргодическая теория n-подмножеств s — плоскостей в таких полях. Выводится широкий класс распределений типа Г.

В числе других обобщений этой теории рассматриваются некоторые случайные мозаики из выпуклых политопов.

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