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INVARIANCE PROPERTIES OF THE CORRELATION MEASURE OF LINE PROCESSES

Given a group L which acts continuously on a space Y, and the space Γ of the resulting equivalence classes of Y, the representation of an L — invariant measure ν on Y in the form $\nu = \int_\Gamma \nu_\gamma x\,(d\gamma)$ with L — invariant measures ν_γ concentrated on the respective equivalence classes is used to study the invariance of ν under various other transformations V of Y not belonging to L. The main application concerns the case $Y = X \times X$ there X is the space of all oriented lines in the plane R^2, L the group of all transformations of Y induced by Euclidean motions of R^2, ν the covariance measure of a second order stationary line process z in R^2, i. e. $\nu(A_1 \times A_2) = E(z(A_1)\,z(A_2))$ for all Borel subsets A_1 and A_2 of R^2, and V a map of one of the following types: the transformation of Y induced by a reflexion in R^2, or by the change of the orientation of all lines, or by a "translation" within X. It is assumed that z has almost surely no antipallel lines.

The invariance of ν under reflexions had been proved by R. Davidson (unpublished) by a different method under the assumption that z has almost surely no parallel or antiparallel lines, and ν is absolutely continuous in $Y - \{(w, w): w \in X\}$ relative to the product méasure of the invariant measure in X. This latter assumption was eliminated in his paper appearing in this same volume.

Introduction

The invariance under reflexions of the correlation measure of a second order stationary line process in the plane with no parallel or antiparallel lines was proved by R. Davidson in his unpublished thesis under the assumption that the correllation measure has a density relative to the relevant integral geometric measure outside the diagonal [5; see also 6]. The purpose of this note if to show how this result can be obtained in full generality from the familiar theory of the disintegration of measures. The method obtained here works in similar cases as well.

I am very much indebted to R. Davidson for making available to me his thesis and for many stimulating discussions, to D. Kendall and the British Council for arranging our first encounter, and to U. Krause for a remark which influenced the organization of the paper.

§ 1. Images of disintegrated measures

Let Y be a locally compact space which has a countable base, and ν a positive Radon measure on Y. Consider an equivalence relation $\sim$ in Y and assume that $\sim$ is ν—measurable. We can then select a locally compact space Γ with a countable base and a ν—measurable map r of Y onto Γ such that $x \sim y$ if and only if $r(x) = r(y)$; in fact, the existence of such a pair Γ, r is necessary and sufficient for the ν—measurability of $\sim$ [3, § 3, n° 4]. For any $\gamma \in \Gamma$ we denote the equivalence class in Y described by γ, i. e. $r^{-1}\{\gamma\}$, by Y_γ. We know [3, § 3, n° 2] that there is a pseudo-image of ν under r; by definition this is a positive Radon measure $\varkappa$ on Γ such that a subset Δ of Γ is a $\varkappa$—null set if and only if $r^{-1}\Delta$ is a ν—null set. For fixed $\varkappa$ there exists by [3, § 3, n° 3] a scalarly $\varkappa$—integrable family of positive measures $(\nu_\gamma)_{\gamma\in\Gamma}$ on Y with the property that ν_γ is carried by Y_γ for any $\gamma \in \Gamma$, and

$$\nu = \int\limits_\Gamma \nu_\gamma \varkappa(d\gamma). \tag{1.1}$$

If $\varkappa'$ stands for any other pseudo-image ot ν under r and $(\nu'_\gamma)_{\gamma\in\Gamma}$ for any scalarly $\varkappa'$—integrable family of positive Radon measures on Y such that ν'_γ is also carried by Y_γ and $\nu = \int\limits_\Gamma \nu_\gamma \varkappa(d\gamma)$, then we have $\nu_\gamma = \rho(\gamma)\,\nu'_\gamma$ for $\varkappa$—almost all γ where $\rho = d\varkappa'/d\varkappa$ denotes the Radon-Nikodym derivative of $\varkappa'$ with respact to $\varkappa$.

Next let V be a map of Y into itself which is measurable and proper with respect to ν. This amounts to the fact that $V^{-1}K$ is ν—integrable for every compact subset K of Y, and it suffices to require this for all elements K of a base of Y consisting of compact sets [2, § 6, n°1]. For fixed K the set $V^{-1}K$ will be ν_γ—integrable for $\varkappa$—almost all γ [2, § 3, n° 3], hence for $\varkappa$—almost all γ the map V will be measurable and proper with respect to ν_γ, too. We now assume that there is a $\varkappa$—measurable and $\varkappa$—proper bijective map v of Γ onto itself with the property that $VY_\gamma \subseteq Y_{v(\gamma)}$ for $\varkappa$—almost all γ, and that the image $\varkappa'$ of $\varkappa$ under v has the same null sets as $\varkappa$. Thus $\varkappa'$ is a pseudo-image of ν under r.

Let ν' be the image of ν and $\nu'_{v(\gamma)}$ the image of ν_γ under V. Taking any contunuous function f on Y with a compact carrier we have by (1.1):

$$\nu'(f) = \nu(f \circ V) = \int\limits_\Gamma \nu_\gamma(f \circ V)\,\varkappa(d\gamma) = \int\limits_\Gamma \nu'_{v(\gamma)}(f)\,\varkappa(d\gamma) = \int\limits_\Gamma \nu'_\gamma(f)\,\varkappa'(d\gamma),$$

hence

$$\nu' = \int\limits_\Gamma \nu'_\gamma \varkappa'(d\gamma). \tag{1.2}$$

Here, $(\nu'_\gamma)_{\gamma\in\Gamma}$ is a scalarly $\varkappa'$-integrable family of Radon measures on Y, and ν'_γ is carried by Y.

Consider an analogous ν-measurable and ν-proper map W of Y into itself and a $\varkappa$-measurable and $\varkappa$-proper bijective map w of Γ onto itself with the property that $WY_\gamma \subset Y_{w(\gamma)}$ for $\varkappa$-almost all γ, and that the image $\varkappa''$ of $\varkappa$ under w has the same null sets as $\varkappa$. Let ν'' be the image of ν and $\nu''_{w(\gamma)}$ the image of ν_γ under W, and set $\rho = d\varkappa/d\varkappa''$. Then (1. 2) implies immediately

Theorem 1. *The images ν' and ν'' of ν under V and W, respectively, coincide if and only if $\nu''_\gamma = \rho(\gamma)\nu'_\gamma$ for $\varkappa$-almost all γ.*

Recall that ν is said to be invariant under V if $\nu' = \nu$. Taking for W the identity we obtain

Corollary 1. *Under the assumptions made on V before, ν is invariant under V if and only if $\nu_\gamma = \rho(\gamma)\nu'_\gamma$ for $\varkappa$-almost all γ where $\rho = d\varkappa'/d\varkappa$. In particular, assuming one of the conditions "$\nu_\gamma = \nu'_\gamma$ for $\varkappa$-almost all γ" and "$\varkappa = \varkappa'$" to be satisfied, the other one is necessary and sufficient for the invariance of ν under V.*

Specializing once more to the case where v is the identical map of Γ we get

Corollary 2. *Suppose that $VY_\gamma \subset Y_\gamma$ for $\varkappa$-almost all γ. Then ν is invariant under V if and only if ν_γ is invariant under V for $\varkappa$-almost all γ.*

We now turn to the case where the equivalence relation is defined by a group L of transformations of Y, i. e., $x \sim y$ if and only if $Tx = y$ for some $T \in$ L. We are also going to change slightly our point of view by directing our attention not only to a single measure ν but rather to the set of all measures invariant under L. Thus let L be a locally compact group with a countable base which acts continuously on Y [1, § 2, n° 4].

We assume that there exists a locally compact space Γ with a countable base and a map r of Y onto Γ such that $x \sim y$ if and only if $r(x) = r(y)$, and such that a subset Δ of Γ is borelian if and only if $r^{-1}(\Delta)$ is borelian in Y. Note that if the quotient space $Y/\sim$ is separated in the quotient topology, we may take this space for Γ and the canonic mapping of Y onto Γ for r [1, § 2, n° 4 and § 4, n° 5]; however in the main application we have in mind, the quotient topology will not be separated. Under the preceding assumption, given any positive Radon measure ν on Y, the map r is ν-measurable, and every $T \in$ L, being a homeomorphism of Y, is ν-measurable and ν-proper. Each Y_γ is a Borel set in Y.

A positive Radon measure ν on Y is said to be L-invariant if it is invariant under every $T \in$ L. By corollary 2 of theorem 1, every L-invariant measure ν has a representation (1.1) with some positive Radon measure $\varkappa$ on Γ where, given $T \in$ L, the measure ν_γ is concentrated on

Y_γ and invariant under T for $\varkappa$-almost all γ. This amounts to $\nu_\gamma f = \nu_\gamma(f \circ T)$ for $\varkappa$-almost all γ and every continuous functions f on Y with a compact carrier. Since $\nu_\gamma(f \circ T)$ is, for fixed f of this kind, a continuous function of $T \in \mathbf{L}$ [4, § 1, n° 1] and $\mathbf{L}$ contains a countable dense set, we find that, for $\varkappa$-almost all γ, the measure ν_γ is invariant under $\mathbf{L}$. Conversely, having selected any positive Radon measure $\varkappa$ on Γ and a scalarly $\varkappa$-integrable family $(\nu_\gamma)_{\gamma \in \Gamma}$ of measures on Y such that, for $\varkappa$-almost all γ, ν_γ is concentrated on Y_γ and invariant under $\mathbf{L}$, the measure ν defined by (1.1) is invariant under $\mathbf{L}$.

We have thus obtained a kind of survey on all $\mathbf{L}$-invariant measures on Y. This survey will turn out especially simple and useful if we make the following assumption: for every γ there is, up to a non-negative factor, one and only one $\mathbf{L}$-invariant positive Radon measure τ_γ concentrated on Y_γ. Then, unless $\tau_\gamma = 0$, every nonempty subset of Y_γ has a positive τ_γ-measure.

We assume that there exists a non-negative bounded borelian function h on Y with the following properties: for any $\gamma \in \Gamma$, the set $Y_\gamma \cap \{h > 0\}$ contains a non-empty subset which is open in Y_γ; for every compact subset Δ of Γ, the set $r^{-1}(\Delta) \cap$ carrier (h) is relatively compact. This assumption will obviously be satisfied in the later examples. Then $0 < \tau_\gamma(h) < \infty$ for every γ for which $\tau_\gamma \neq 0$, and by renormalizing τ_γ we can assume that for every γ we have $\tau_\gamma(h) = 1$ or $\tau_\gamma(h) = 0$. The set $\{x : x \in Y, \tau_{r(x)} = 0\}$ is null for every $\mathbf{L}$-invariant measure on Y, and could as well be discarged.

Now let ν be an $\mathbf{L}$-invariant positive Radon measure on Y, and $\varkappa$ a pseudo-image of ν under r. Starting with the decomposition (1.1) we find for $\varkappa$-almost all γ a number $\rho(\gamma) \geq 0$ such that $\nu_\gamma = \rho(\gamma)\tau_\gamma$. Given any compact subset Δ of Γ the function $g(x) = h(x)\, 1_\Delta(r(x))$ is ν-integrable with

$$\nu(g) = \int_\Gamma \nu_\gamma(g)\, \varkappa(d\gamma) = \int_\Gamma \tau_\gamma(h)\, \rho(\gamma)\, \varkappa d\gamma)$$

where $\tau_\gamma(h) = 1$ for $\varkappa$-almost all γ. Hence ρ is a locally $\varkappa$-integrable function on Γ, and the measure $\varkappa' = \rho\varkappa$ is a pseudoimage of ν under r. Writing again $\varkappa$ instead of $\varkappa'$ we obtain the decomposition

$$\nu = \int_\Gamma \tau_\gamma\, \varkappa(d\gamma) \tag{1.3}$$

where $\varkappa$ is unique. Conversely, given any positve Radon measure $\varkappa$ on Γ such that the family $(\tau_\gamma)_{\gamma \in \Gamma}$ is scalarly $\varkappa$-integrable, the measure ν defined by (1.3) is $\mathbf{L}$-invariant. Thus we have

T h e o r e m 2. *Under the assumptions made on* $\mathbf{L}$ *before, the formula (1.3) yields a one-to-one correspondence between* $\mathbf{L}$*-invariant measures* ν *on* Y *and measures* $\varkappa$ *on* Γ *such that* $(\tau_\gamma)_{\gamma \in \Gamma}$ *is scalarly* $\varkappa$*-integrable.*

Combining this with corollary 1 of theorem 1 we get the

Corollary. *Suppose that the preceding assumptions on* L *are satisfied. Let V be a homeomorphism of Y which induces a bijective transformation v of Γ such that $VY_{\gamma} = Y_{v(\gamma)}$ for every γ, or in other words $r \circ V = v \circ r$. Then v and v^{-1} are borelian. Suppose in addition that for every γ, the measure $\tau_{v(\gamma)}$ is the image of τ_γ under V, and let ν be an* L*-invariant measure represented in the form (1.3). Then an* L*-invariant measure ν on Y represented in the form (1.3) is invariant under V if and only if the corresponding measure $\varkappa$ on Γ is invariant under v.*

In fact, the assertion that v and v^{-1} are borelian follows immediately from our assumptions on r; hence v and v^{-1} are $\varkappa$-measurable for every Radon measure $\varkappa$ on Γ. To apply the corollary 1 of theorem 1 in the case where ν is invariart under L we need to know that v is $\varkappa$-proper and preserves $\varkappa$-null sets. The first statement results easily from the formula $\varkappa(\Delta) = \nu(h \cdot 1_{\gamma^{-1}(\Delta)})$ where Δ is any Borel set in Γ; this leads, by the way, directly to the ν-invariance of $\varkappa$. The second statement holds even if V is only assumed to preserve ν-null sets.

An important type of mappings to which the preceding corollary applies is the following one: let V be a homeomorphism of Y such that $VTV^{-1} \in$ L for every $T \in$ L. Then $y \sim y'$ entails indeed $Vy \sim Vy'$, and hence there is a bijective transformation v of Γ such that $r \circ V = v \circ r$.

Consider next the particular case where V commutes with every T of L, let ν_γ be any L-invariant positive Radon measure concentrated on Y_γ, and $\nu'_{v(\gamma)}$ its image under V. Then for every $T \in$ L and every continuous function f on Y with a compact carrier we have

$$\nu'_{v(\gamma)}(f \circ T) = \nu_\gamma(f \circ T \circ V) = \nu_\gamma(f \circ V \circ T) = \nu_\gamma(f \circ V) = \nu'_{v(\gamma)}(f),$$

i. e. $\nu'_{v(\gamma)}$ is also L-invariant. Thus, assuming that for every γ there exists, up to a factor, only one L-invariant positive Radon measure concentrated on Y_γ we find that $\nu'_{v(\gamma)}$ is a multiple of $\nu_{v(\gamma)}$. This shows that our hypothesis $\tau'_{v(\gamma)} = \tau_{v(\gamma)}$ is not an unrealistic one; it is, in fact, only a hypothesis regarding the normalization of the τ_γ's. If the normalizing function h is itself V-invariant, we get

$$\tau'_{v(\gamma)}(h) = \tau_\gamma(h \circ V) = \tau_\gamma(h) = 1$$

unless $\tau_\gamma = 0$, hence automatically $\tau'_{v(\gamma)} = \tau_{v(\gamma)}$. Given a finite group of homeomorphisms $V_1, \cdots, V_n$ of Y which commute with every $T \in$ L, we can always choose h so as to be invariant under every V_i by replacing it by $\sum_{i=1}^{n} h \circ V_i$. In particular, h can be chosen V-invariant if V is periodic.

§ 2. A particular case: a diagonal group in a product space

Let X be a locally compact space with a countable base, and $\mathbf{M}$ a locally compact group with a countable base which acts continuously on X. We fix a posiuive integer k and define Y to be the product space X^k. For $\mathbf{L}$ we take the group of all transformations of the form $(x_1, \cdots, x_k) \to (T_0 x_1, \cdots, T_0 x_k)$ where $T_0 \in \mathbf{M}$. Then $\mathbf{L}$ acts continuously on Y and has a countable base, the topology of $\mathbf{L}$ being given in the obvious way by that of $\mathbf{M}$.

As before we assume that we have a locally compact space Γ with a countable base and a map r of Y onto Γ which "represent" $Y/\sim$ in the sense previously defined.

An example of a homeomorphism V which commutes with every $T \in \mathbf{L}$ is given by

$$V(x_1, \cdots, x_k) = (x_{i_1}, \cdots, x_{i_k})$$

where $i_1, \cdots, i_k$ is a fixed permutation of the numbers $1, \cdots, k$. We can choose the normalizing function h to be invariant under every mapping of this kind.

Another type of mappings are the diagonal ones:

$$V(x_1, \cdots, x_k) = (V_0 x_1, \cdots, V_0 x_k)$$

where V_0 is a homeomorphism of X such that $T_0 \in \mathbf{M}$ implies $V_0 T_0 V_0^{-1} \in \mathbf{M}$. Obviously $T \in \mathbf{L}$ implies $VTV^{-1} \in \mathbf{L}$. If V_0 commutes with every $T_0 \in \mathbf{M}$, V commutes with every $T \in \mathbf{L}$, and if V_0 is periodic, V is periodic. Applying the corollary of theorem 2 we obtain

Proposition 1. *Let V be any transformation of Y of the kind described before, and v the corresponding tranformation of Γ. Suppose that for every γ there is, up to a factor, one and only one $\mathbf{L}$-invariant positive Radon measure τ_γ concentrated on Y_γ, ane that $\tau_{v(\gamma)}$ is the image of τ_γ under V. Then an $\mathbf{L}$-invariant measure ν on Y represented by (1.3) is invariant under V if and only if the corresponding measure $\varkappa$ on Γ is invariant under v.*

§ 3. Mixed moments of random measures

We start with a space X, a positive integer k, and a group $\mathbf{M}$ as in § 2. We denote by $\mathbf{B}(X)$ the sigma-ring of all Borel sets, and by $\mathbf{B}_0(X)$ the set of all relatively compact Borel sets of X. In addition we take a probability space $(\Omega, \mathbf{F}, P)$ to be kept fixed once for all, and let E stand for "expectation". As usual we mean by a positive random measure on X a map z of $\mathbf{B}(X)$ into the set of all non-negative extended real-valued random variables with the following property: if (A_n) is a sequence of dusjoint sets of $\mathbf{B}(X)$ and $A = \bigcup_n A_n$, then we have

$z(A)=\sum_n z(A_n)$ almost surely. Since $z(A_n)\geqslant 0$ it suffices to require only stochastic convergence.

The random measure z is said to be locally finite if $z(A)<\infty$ almost surely for any $A\in\mathbf{B}_0(X)$. It is called locally of k'th order if $E(z(A)^k)<\infty$ whenever $A\in\mathbf{B}_0(X)$. Assuming this the mixed moments of k'th order of the random variables $z(A_1),\cdots,z(A_k)$ exist for arbitrary sets $A_1,\cdots,A_k$ in $\mathbf{B}_0(X)$. The function

$$\nu(A_1\times\cdots\times A_k)=E\left(\prod_{l=2}^{k} z(A_l)\right) \tag{3.1}$$

defined on the semi-ring S of all sets of the form $A_1\times\cdots\times A_k$ where $A_l\in B_0(X)$, is additive, and it follows from Fatou's lemma that $\nu(B)=\lim_{n\to\infty}\nu(B_n)$ for any increasing or decreasing sequence (B_n) in S which converges to a set B in S. We can then show in the usual way [7, p. 56] that $\nu(B)=\sum_n\nu(B_n)$ if (B_n) is a sequence of disjoint sets in S, and $B=\cup_n B_n\in$S. Hence [7, § 2] ν can be uniquely extended to a positive Radon measure in $Y=X^k$ which will again be denoted by ν.

The measure ν is symmetric, i. e. invariant under any "permutation of the axes" of the type W studied in § 2. Of course, not every symmetric measure arises in this way as shown by the measure in the space $Y=\{0,1\}^2$ which consists of two unit masses at the points $(0, 1)$ and $(1, 0)$.

The random measure z is called stationary to the k'th order if it is locally of k'th order, and

$$E\left(\prod_{l=1}^{l} z(TA_l)\right)=E\left(\prod_{l=1}^{l} z(A_l)\right)$$

for any $l\leqslant k$, any sets $A_1,\cdots,A_l\in\mathbf{B}_0(X)$ and any $T\in\mathbf{M}$. In this case, ν is invariant under $\mathbf{L}$ as well as under any permutation of the axes, and is thus subject to proposition 1 of § 2.

Given a random measure z of k'th order and a set $B\in\mathbf{B}(Y)$ we agree to say that almost surely there are no k-tuples in B if $\nu(B)=0$. This can be justified as follows. Suppose we are given a version of z. By this we mean a non-negative function of $A\in\mathbf{B}(X)$ and $\omega\in\Omega$ which coincides, for each fixed A, almost surely with $z(A)$, and which is a measure on $\mathbf{B}(X)$ as a function of A for almost all fixed ω. To save letters we are going to denote a particular version again by z so that $z(A;\omega)$ is the value of that version for the set A and the outcome ω. For fixed ω, let $\nu(\cdot;\omega)$ be the product measure in Y defined by $\nu(A_1\times\cdots\times A_k;\omega)=z(A_1;\omega)\times\cdots\times z(A_k;\omega)$ if $A_1,\cdots,A_k\in\mathbf{B}_0(X)$. Then for every $B\in\mathbf{B}(Y)$ we have

$$\nu(B)=\int_\Omega \nu(B;\omega)P(d\omega). \tag{3.2}$$

In fact by (3.1) this formula is true if B has the form $A_1 \times \cdots \times A_k$ where $A_i \in \mathbf{B}_0(X)$, and hence is generally true. Loosely speaking, ν is the "mixture", with respect to P, of all measures $\nu(\cdot; \omega)$ where ω runs through Ω. It follows that $\nu(B) = 0$ if and only if $\nu(B; \omega) = 0$ for almost all ω.

By a point process in X we mean a positive random measure z with the property that, for every $A \in \mathbf{B}_0(X)$, $z(A)$ takes almost surely only integral values. In terms of a particular version, the phrase "There are almost surely no k-tuples in B" can then be described in the following way which makes it intuitively clear: for almost all ω, there is no $(x_1, \cdots, x_k) \in B$ such that $z(\{x_1\}; \omega) > 0, \cdots, z(\{x_k\}; \omega) < 0$.

§ 4. An example

Take for X the set of all oriented lines in the oriented Euclidean plane R^2. We parametrize X in the usual way [8]: having chosen a fixed origin o and a fixed axis w_0 through o, an oriented line w will cut w_0 at a point s and at an angle $\vartheta + \pi/2$. We then describe w by the pair (p, ϑ) where $p = s \cos \vartheta$ is the positive or negative distance of w from o. If w is parallel or antiparallel to w_0 we have $\vartheta = \frac{3}{2}\pi$ or $\vartheta = \frac{\pi}{2}$, respectively, and the sign of the distance p is determined by continuity, e. g. positive if w is antiparallel to w_0 and lies on the "left bank" of w_0. Thus $-\infty < p < \infty$ and $0 \leqslant \vartheta < 2\pi$; topologically, X is simply the infinite two-dimensional cylinder $R \times S$ where S is the circle.

Let $\mathbf{M}$ be the group acting on X induced by the Euclidean motions of the plane. Explicitly, if S is the motion

$$(\xi \cdot \eta) \to (\xi \cos \vartheta_0 - \eta \sin \vartheta_0 + \xi_0, \ \xi \sin \vartheta_0 + \eta \cos \vartheta_0 + \eta_0)$$

of R^2, taking for o the origin and for w_0 the first axis of coordinates the induced map of X becomes

$$(p, \vartheta) \to (p + \xi_0 \cos (\vartheta + \vartheta_0) + \eta_0 \sin (\vartheta + \vartheta_0), \ \vartheta + \vartheta_0),$$

and determines S uniquely. Under its natural topology obtained in this way, $\mathbf{M}$ is homeomorphic to the group of the Euclidean motions of R^2, and to $R \times R \times S$. Its action on X is continuous.

Denote by Y the product space X^2, and accordingly by $\mathbf{L}$ the group of all transformations of Y of the form $(w_1, w_2) \to (Tw_1, Tw_2)$ where $T \in \mathbf{M}$. There are then five types of equivalence classes of Y with respect to $\mathbf{L}$ which we are going to describe in terms of a partition of Γ into five sets $\{\delta\}$, $\{\alpha\}$, Γ_+, Γ_-, and Γ_0:

$\{\delta\}$: The set of all double lines (w, w), i. e. the diagonal in Y, forms one equivalence class which we will represent by a point δ of Γ.

$\{\alpha\}$: Likewise, the set of all pairs of coincident lines with opposite directions $(p, \vartheta), (-p, \vartheta + \pi))$ forms one equivalence class which we will represent by a point α of Γ.

Γ_+: Suppose that w_1 and w_2 are parallel and unequal, say $w_1 = (p, \vartheta)$ and $w_2 = (p + \gamma, \vartheta)$ where $\gamma \neq 0$. Note that „parallel" in this sense implies „equidirected", and the distance γ between w_1 and w_2 is taken as positive if w_2 lies on the right bank of w_1. Then $(w_1, w_2) \sim (w_1', w_2')$ if and only if w_1' and w_2' are also parallel with the same distance γ, i. e. $w_1' = (p', \vartheta')$ and $w_2' = (p' + \gamma, \vartheta')$. Hence we may take γ to represent the equivalence class which contains (w_1, w_2), and the part Γ_+ of Γ corresponding to the classes of pairs of unequal parallel lines is the real line minus the origin $R - \{0\}$. Γ_-: Suppose that w_1 and w_2 are antiparallel and non-coincident, say $w_1 = (p, \vartheta)$ and $w_2 = (\gamma - p, \vartheta + \pi)$ where $\gamma \neq 0$. Then $(w_1, w_2) \sim (w_2', w_2')$ if and only if w_1' and w_2' are also antiparallel with the same distance γ. Hence we may again take γ to represent the equivalence class which contains (w_1, w_2), and the part Γ_- of Γ corresponding to the classes of pairs of antiparallel and non-coincident lines is equal to $R - \{0\}$.

Γ_0: Suppose that w_1 and w_2 are neither parallel nor antiparallel and form the angle γ, say $w_1 = (p_1, \vartheta)$ and $w_2 = (p_2, \vartheta + \gamma)$, where γ is not an integral multiple of π. Then $(w_1, w_2) \sim (w_1', w_2')$ if and only if w_1' and w_2' form the same angle γ, i. e. $w_1' = (p_1', \vartheta')$ and $w_2' = (p_2', \vartheta' + \gamma)$. Hence we may take γ to represent the equivalence class which contains (w_1, w_2), and the part Γ_0 corresponding to the classes of pairs of neither parallel nor antiparallel lines is equal to the union of the two disjoint open intervals $0 < \gamma < \pi$ and $\pi < \gamma < 2\pi$.

The representation space $\Gamma = \{\delta\} \cup \{\alpha\} \cup \Gamma_+ \cup \Gamma_- \cup \Gamma_0$ is not separated in its quotient topology because the intersection of Γ_0 with any two neighbourhoods of any two points of Γ_+, or any two points of Γ_-, is never empty. We will, however, endow Γ with a finer topology, to be called the "natural" one, which makes it locally compact with a countable base. To this end we use the ordinary topologies of Γ_+, Γ_-, and Γ_0 given by their description in terms of $R - \{0\}$, $R - \{0\}$, and $]0, \pi[\cup]\pi, 2\pi[$, respectively, and let Γ_+, Γ_-, and Γ_0 be open in Γ. A basis of neighbourhoods of δ is to consist of the following sets: any union of δ with intervals $]-\varepsilon, 0[$ and $]0, \varepsilon[$ of Γ_+ and intervals $]0, \varepsilon[$ and $]2\pi - \varepsilon, 2\pi[$ of Γ_0 where $\varepsilon > 0$. Similary a typical neighbourhood of α consists of a union of α with intervals $]-\varepsilon, 0[$ and $]0, \varepsilon[$ of Γ_- and intervals $]\pi - \varepsilon, \pi[$ and $]\pi, \pi + \varepsilon[$ of Γ_0. Briefly, Γ is a circle with two lines brazed to it at apposite points.

Although the canonical map r of Y onto Γ is no longer continuous with respect to the natural topology of Γ, it has a continuous restriction to each of the five Borel sets $r^{-1}\{\delta\}$, $r^{-1}\{\alpha\}$, $r^{-1}(\Gamma_+)$, $r^{-1}(\Gamma_-)$, and $r^{-1}(\Gamma_0)$. Hence r is borelian, i. e. $r^{-1}(\Delta)$ is a Borel set for every Borel set Δ in Γ. On the other hand, r is open in the natural topology since this topology is finer than the quotient topology. It follows that Δ is a Borel set if $r^{-1}(\Delta)$ is.

Next we look at the action of L in the various equivalence classes Y_γ. Each class Y_γ with $\gamma = \delta$, $\gamma = \alpha$, $\gamma \in \Gamma_+$ or $\gamma \in \Gamma_-$ can be mapped homeomorphically onto X by the projection $(w_1, w_2) \to w_1$. The action of L on Y_γ is hereby carried into the action of M on X. It is well known [8, § 2] that there exists one and, up to a positive factor, only one non-vanishing positive Radon measure on X which is invariant under M, namely, $|dp\,d\vartheta|$. The corresponding measure on Y concentrated on Y will be denoted by τ_γ.

Suppose that $\gamma \in \Gamma_0$. In this case the map $((p_1, \vartheta), (p_2, \vartheta + \gamma)) \to \to (\xi, \eta, \vartheta)$ where

$$\xi = (\sin\gamma)^{-1} (p_1 \sin(\vartheta + \gamma) - p_2 \sin\vartheta),$$

$$\eta = (\sin\gamma)^{-1} (p_2 \cos\vartheta - p_1 \cos(\vartheta + \gamma)),$$

is a homeomorphism of Y_γ onto the space $R \times R \times S$; the point (ξ, η) is the intersection of the lines (p_1, ϑ) and $(p_2, \vartheta + \gamma)$. Hence the action of L on Y_γ is carried by this map into the action on $R \times R \times S$ of the group of all transformations

$$(\xi, \eta, \vartheta) \to (\xi \cos\vartheta_0 - \eta \sin\vartheta_0 + \xi_0,\ \xi \sin\vartheta_0 + \eta \cos\vartheta_0 + \eta_0,\ \vartheta + \vartheta_0)$$

where $(\xi_0, \eta_0, \vartheta_0)$ runs through $R \times R \times S$. Again it is well known [8, § 5] that there exists one and, np to a positive factor, only one non-vanishing positive Radon measure on $R \times R \times S$ which is invariant under this group, namely, the so-called kinematic measure $|d\xi d\eta d\vartheta| = = |\sin\gamma|^{-1} |dp_1 dp_2 d\vartheta|$. The corresponding measure on Y concentrated on Y_γ will be denoted by τ_γ.

It follows now from theorem 2 that a positive Radon measure ν in Y is invariant under L if and only if it admits a disintegration (1.3) where $\varkappa$ is a positive Radon measure on Γ.

Next we consider various other transformations of Y. As before, let W be the permutation of the axes: $(w_1, w_2) \to (w_2, w_1)$. The subsequent maps have the form

$$V_i \colon (w_1, w_2) \to (V_{i0} w_1, V_{0i} w_2)$$

where the transformation V_{i0} of X for $i = 1, 2, 3$ does not belong to M.

V_{10}: The transformation of X induced by a reflexion at a fixed line in the plane R^2, e. g. by the reflexion at the axis w_0, i. e.

$$(p, \vartheta) \to (-p, \pi - \vartheta).$$

V_{20}: Change of the orientation of all lines, i. e. $(p, \vartheta) \to (-p, \vartheta + \pi)$.

V_{30}: Translation of the cylinder X, i. e. $(p, \vartheta) \to (p + p_0, \vartheta)$, p_0 fixed, $\neq 0$.

An elementary geometric reasoning shows that the group of transformations of X generated by M and any maps of two given types V_{i0} does not contain a map of the remaining type. An alternative proof will be given below.

Obviously $T_0 \in$ M entails $V_{i0} T_0 V_{i0}^{-1} \in$ M. Accordingly, let w and v_i, $i = 1, 2, 3$, be the transformations of Γ with the property that

$r \circ W = w \circ r$ and $r \circ V_i = v_i \circ r$ for $i = 1, 2, 3$. It is easy to describe w and every v_i explicitly if we represent Γ_+, Γ_-, and Γ_0 as above by the spaces $R - \{0\}$, $R - \{0\}$, and $]0, \pi[\cup]\pi, 2\pi[$, respectively:

$$w(\delta) = \delta;\ w(\alpha) = \alpha;\ w(\gamma) = -\gamma \text{ if } \gamma \in \Gamma_+;$$

$$w(\gamma) = \gamma \text{ if } \gamma \in \Gamma_-;\ w(\gamma) = 2\pi - \gamma \text{ if } \gamma \in \Gamma.$$

$$v_1(\delta) = \delta;\ v_1(\alpha) = \alpha;\ v_1(\gamma) = -\gamma \text{ if } \gamma \in \Gamma_+;$$

$$v_1(\gamma) = -\gamma \text{ if } \gamma \in \Gamma_-;\ v_1(\gamma) = 2\pi - \gamma \text{ if } \gamma \in \Gamma_0.$$

$$v_2(\delta) = \delta;\ v_2(\alpha) = \alpha;\ v_2(\gamma) = -\gamma \text{ if } \gamma \in \Gamma_+;$$

$$v_2(\gamma) = -\gamma \text{ if } \gamma \in \Gamma_-;\ v_2(\gamma) = \gamma \text{ if } \gamma \in \Gamma_0.$$

$$v_3(\delta) = \delta;\ v_3(\alpha) = 2p_0 \text{ in } \Gamma_-;\ v_3(\gamma) = \gamma \text{ if } \gamma \in \Gamma_+;$$

$$v_3(\gamma) = \gamma + 2p_0 \text{ if } \gamma \in \Gamma_- \text{ and } \gamma + 2p_0 \neq 0,$$

$$v_3(\gamma) = \alpha \text{ if } \gamma = -2p_0 \in \Gamma_-;$$

$$v_3(\gamma) = \gamma \text{ if } \gamma \in \Gamma_0.$$

By looking at the changes of the orientation of $\Gamma_+ \cup \{\delta\}, \Gamma_- \cup \{\alpha\}$ and $\Gamma_0 \cup \cup \{\delta\} \cup \{\alpha\}$ under these maps we see in the first place that the group of transformations of Y generated by **L** and any maps of three given types out of the types W, V_1, V_2. and V_3 does not contain a map of the remaining type.

Moreover, by the definition of τ_γ the image of τ_γ under W or V_i is the measure $\tau_{w(\gamma)}$ or $\tau_{v_i(\gamma)}$, respectively. Hence by the corollary of theorem 2, an **L**—invariant measure ν on Y written in the form (1.3) is invariant under W or V_i if and only if the corresponding measure $\varkappa$ is invariant under w or v_i, respectively. In applying this we will make use of the following trivial remark: suppose we have two bijective borelian transformations v and v' of with a borelian inverse and a decomposition of Γ into two sets Γ_e and Γ_g which are invariant under both v and v' and such that v' is the identity on Γ_e and coincides $\varkappa$—almost everywhere with v on Γ_g. Then if $\varkappa$ is invariant under v, it is also invariant under v'.

Finally, let z be a positive random measure on X which is stationary to the second order, let ν be its covariance measure defined by $\nu(A_1 \times A_2) = \mathbf{E}(z(A_1)\, z(A_2))$, and let $\varkappa$ be the corresponding measure on Γ. Then ν is invariant under W, hence $\varkappa$ is invariant under w. Therefore, using the terminology introduced at the end of the preceding chapter and recalling that $\varkappa(\Delta) = 0$ is tantamount to $\nu(r^{-1}\Delta) = 0$ we find that:

ν is invariant under reflexions at fixed lines in the plane and under the change of the orientation of all lines if almost surely there are no pairs of antiparallel, non-coincident lines, i. e. $\varkappa(\Gamma_-) = 0$;

ν is invariant under translations of the cylinder X if almost surely there are no antiparallel (coincident or non-coincident) lines, i. e. $\varkappa(\Gamma_- \cup \{\alpha\})=0$.

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Կ. ԿՐԻԿԵԲԵՐԳ. Ուղիղների դաշտի կոռելյացիոն չափի ինվարիանտության հատկությունները *(ամփոփում)*

Հարթության վրա ուղիղների երկրորդ կարգի ստացիոնար դաշտերի կոռելյացիոն չափի ինվարիանտությունը անդրադարձումների նկատմամբ ապացուցված է Ռ. Դավիդսոնի կողմից կոռելյացիոն խտության գոյության դեպքում [5]—[6]։

Ներկա աշխատանքում այդ արդյունքը ապացուցված է ամենաընդհանուր դեպքում չափերի վերածման տեսության օգնությամբ։

К. КРИККЕБЕРГ. *Свойства инвариантности корреляционной меры полей прямых* (резюме)

Инвариатность относительно отражений корреляционной меры второго порядка стационарных полей прямых на плоскости была доказана Р. Давидсоном в предположении существования корреляционой плотности (см. [5], [6]). В настоящей работе этот результат доказан во всей общности с помощью теории разложения мер.

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