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CONSTRUCTION OF LINE PROCESSES: SECOND ORDER PROPERTIES

The second-order properties of stationary processes of lines in the plane are thoroughly treated, and the problem of construction of such processes is discussed.

1. Integral geometry

Let w be an oriented line in the Euclidean plane R^2; then the standard coordinates of w are (p, θ), where $-\infty < p < \infty$ and θ is an angle. Here p is the perpendicular (signed) distance of W from some fixed origin O and θ is the angle made by this perpendicular with some fixed direction Ox. Thus:

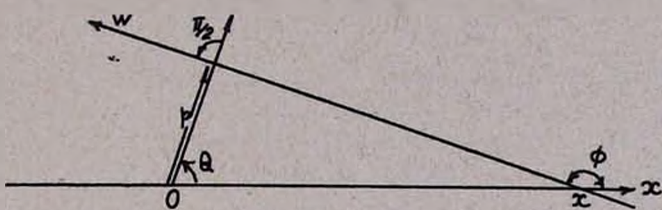

Fig. 1.

In fig. 1 $p>0$; but if w' were parallel with w, the same distance from, and the other side of O, we would have $p(w') = -p(w)$.

Let C be the cylinder $\{(p, \theta): -\infty < p < \infty, 0 \leqslant \theta \leqslant 2\pi\}$. Then there is a biunique correspondence between the lines w in R^2 and the points w of C. C is to have the ordinary Euclidean topology, and all sets in C that we shall consider will be Borel.

Let M^* be the group of rigid motions (translations and rotations) of R^2. Then each $T^* \in M^*$ induces a motion T of C. Let us write, for each positive q, $B_q = \{(p, \theta): |p| < q\}$.

Proposition 1. *Let M be the set of motions T of C as T^* runs through M^*. Then M is the group generated by the motions*

$$R_\alpha: p \to p, \ \theta \to \theta + \alpha \quad (0 \leqslant \alpha < 2\pi)$$

$$S_d: p \to p + d\cos\theta, \ \theta \to \theta \ (-\infty < d < \infty).$$

Proof. Immediate from the following observations:
1. R_α corresponds to rotating Ox clockwise through the angle α;
2. S_d corresponds to tanslating O a distance $-d$ along Ox;
3. The two classes of motion above generate M^*. Q. e. d.

It is to be noted that R_2 is a rotation of C, and that S_d is a *parallel shear* of C. That is, if C is slit along a generator and S_d is applied to the strip so obtained, the image of a line perpendicular to the axis of the strip is a sine curve.

Proposition 2. *No translation of* C *is a member of* M.

Proof. Let the translation be $T = \{p \to p + q;\ \theta \to \theta\}$ $(q \neq 0)$. Without loss of generality q is positive. Then consider the band $B_{q/2} = B$ say. It is clear that $B \cap T(B) = \varnothing$. On the other hand, B is the set of lines lying within the non-zero distance $q/2$ of O; so, under any $T' \in M$, $T'(B)$ is the set of lines lying within distance $q/2$ ot some point $O' \in R^2$ Therefore, $B \cap T'(B) \neq \varnothing$, so that $T \neq T'$. Q. e. d.

Proposition 3. (Crofton [4], Santalo [13]). There is, up to positive factors, a unique positive Borel measure on C invariant under M, and this measure (which we shall denote by m) can be taken to have the density $dp d\theta$.

Returning to fig. 1, we now have, for almost all (w. r. t. m) w, the representation $w = (p, \theta) = (x, \Phi)$, where $x = p \sec\theta$, $\Phi = \theta + \pi/2$; x is the distance of the meet of w with Ox from O, and Φ is the angle of intersection. We now have also $m(dw) = |\sin\Phi|\, dx\, d\Phi$.

2. Line-process

By a line-process Z we mean a non-negative integer-valued random. Borel measure on C, which satisfies

(I) For all positive q, $Z(B_q)$ is a. s. finite;

(II) Z has a. s. no atoms of mass greater than 1.

Thus Z corresponds to a random aggegate of lines in R^2, only finitely many of which cut any circle. We define further the conditions

(III) $E(Z^2(B_q))$ is finite for all finite positive q, and

$$\text{for all } T \in M,\ E(Z(A)) = E(Z(TA));$$
$$E(Z(A)\,Z(B)) = E(Z(TA)\,Z(TB)),$$

provided there is a q such that $A \cup B \subset B_q$.

(IV) Z has a. s. no parallel (or antiparallel) lines.

(V) The finite-dimensional distributions of Z are stationary under M,

$$\text{that is, } p\Big(\bigcap_{i=1}^{k}(Z(A_i) = n_i)\Big) = p\Big(\bigcap_{i=1}^{k}(Z(TA_i) = n_i)\Big) \text{ for all } k,$$
$$n_1, \cdots, n_k,\ A_1, \cdots, A_k, \text{ and } T \in M.$$

We shall study only those line-processes satisfying (I) — (IV) — these will form the class LP4; those members of LP4 satisfying also (V) will form the class LP5.

We pause to look at (IV). It will be clear throughout the paper that parallel lines are a pathology, essentially because we have a distance between them, so that we have all the complications of the theory of ordinary point processes on the line. For LP4 the theory is quite different.

P r o p o s i t i o n 4. *Any strictly stationary line-process Z (i. e., one that satisfies (V)) has a. s. no, or infinitely many, pairs of parallel lines.*

P r o o f. Let the event $F=$ (There exists at least one pair of parallel lines) have positive probability. Let N be the process of points on the whole line Ox given by its intersections with those lines of Z which have at least one other line of Z parallel to them. Then N is strictly stationary (this may be verified by elementary calculations) and has at least two point in it. By Cauchy's inequality, $E_F \exp - N([-2r, 2r[) \leqslant E_F \exp - 2N([-r, r[)$ for any positive r. Using this doubling repeatedly, by bounded convergence we get $E_F \exp - N(R) \leqslant p(N([-r, r[) = 0 \mid F)$, where R is the whole real line. Since $p(\cdot \mid F)$ is a probability measure, we obtain by continuity.

$$E_F \exp - N(R) \leqslant p(N(R) = 0 \mid F) = 0,$$

by definition of F. Thus $N(R)$ is a. s. infinite given F. Q. e. d.

N o t e. Ryll-Nardzewski [12] states effectively this result, but the method used here appears to be useful later.

The obvious example of an LP5 is the Poisson line-process, studied e. g. by Miles [8[, [9], which has, for disjoint $A_1, \cdots, A_k$ on C,

$$p\left(\bigcap_{i=1}^{k} (Z(A_i) = n_i)\right) = \prod_{i=1}^{k} \{e^{-\lambda m(A_i)} (\lambda m(A_i))^{n_i} / n_i!\},$$

where λ is a non-negative constant.

We may generalize this somewhat as follows. Let Λ be any random Borel measure on C, satisfying (I) and (III) (with therein Λ for Z), and (V) if we desire to construct an LP5 rather than an LP4. Then, having sampled Λ, we put on an inhomogoneous Poisson process with local rate $\Lambda(dw)$, so that

$$p\left(\bigcap_{i=1}^{k} (Z(A_i) = n_i)\right) = E_{\Lambda} \prod_{i=1}^{k} \{e^{-\Lambda(A_i)} (\Lambda(A_i))^{n_i} / n_i!\}.$$

We shall, of course, require the satisfaction of (II) and (IV) for the resulting line-process Z; and it turns out that this inposes heavy conditions on Λ. Processes of the type just described are called *doubly stochastic Poisson* processes, and we say that they form the class dsP. The fundamental, and as yet unsolved, question is, Do there exist members of LP5 —'dsP? We shall return to this question in section 5; meanwhile we shall investigate the second-order properties of the members of LP4 with this question in mind.

3. Second-order properties

In this section, Z is arbirary in LP4. Define, for arbitrary bounded Borel sets A, B of C, $\mu(A \times B) = EZ(A)Z(B)$.

P r o p o s i t i o n 5 (Krickeberg). μ has a unique extension to a Borel measure on $C \times C$.

Proof by countable additivity and square-summability of Z.

Theorem 1. *Every $Z \in LP4$ is second-order stationary under reflections of R^2 (equivalently, under translations of C).*

Note. This theorem, for the special case where Z possesses a second-order product-moment density, appears in my Cambridge Ph. D. thesis (1967). It was proved in full generality by Krickeberg (1969). His proof (which proceeds by disintegration of μ) is much more sophisticated and elegant than ours here, but ours is thematic.

Proof of the theorem. It is clearly only necessary to consider the particular reflection in Ox. This, T_0 say, is given by $T_0(p, \theta) = (-p, \pi - \theta)$. It is required to prove that if A, B are bounded Borel sets, then $EZ(A)Z(B) = EZ(TA)Z(TB)$; for it is easy to see that the pseudo-Haar measure on C, and hence the first-order moment measure of Z, are invariant under T_0 (and translation of the cylinder, of course). By Proposition 5, it suffices only to consider the case where A and B are congruent rectangular shields on C. That is, A has the form $[p, p+1[\times [\gamma, \delta[$ and B has a congruent form. It will be noticed that we are taking A and B to be half-open. By a rotation we may take

$$A = [p, p+1[\times [-\alpha-\beta, -\alpha[; \; B = [q, q+1[\times [\alpha, \alpha+\beta[,$$

where $0 < \beta < \pi$.

We suppose at first that there is no generator of C common to $\bar{A}$ and $\overline{B+\pi}$; equivalently, that $\max(|\alpha|, |\alpha+\beta|) < \pi/2$.

Let Q_d be the translation of C through d: $(p, \theta) \rightarrow (p+d, \theta)$. Then we at once verify that if we put $d = -(p+q+1)$, then $T_0A^\circ = Q_dB^\circ$ and $T_0B^\circ = Q_dA^\circ$, where the superscript $^\circ$ denotes that the interior is taken. Now what we shall demonstrate is stationarity under Q_d. Since the open rectangles generate the Borel sets, second-order stationarity under T_0 will follow from Proposition 5. Also it is clear that we could deduce stationarity under Q_d from that under T_0.

The idea of the proof is to approximate the effect of translation on A and B by splitting them up lengthwise and applying suitable shears to the pairs of split pieces. Define the distortion of the shear $(p, \theta) \rightarrow (p + d\cos(\theta-\alpha), \theta)$ as $|d|$; this is the maximum displacement under the shear. For any shield C (that does not encircle C), let $P(C)$ be its point furthest clockwise and with least (algebraic) p-value. For convenience, put $T = Q_d$.

Divide A and B each into $n = 2^m$ $(m \rightarrow \infty)$ congruent shields A_i, B_j, each of which has the same length, and $1/n$'th the angular width, of the original shield. Let S_{ij} be the (unique) shear such that $P(S_{ij}TA_i) = P(B_j)$ and $P(S_{ij}TB_j) = P(A_i)$. This shear exists and is unique because the angular interval subtended by $\overline{A \cup B}$ is less than π. For the same reason, there is m so large that we could adjoin shields congruent to A_i, B_j on each side of A and B, and the angular interval subtended by the union of the augmented A and B would still be less than π. Let the augmenting, extreme shields be, without loss of generality, A_0 and B_{n+1}; and let $S_{0,\, n+1}$ (which again exists by the small-angular-interval argument) be defined similarly to S_{ij}. Let the distortion of $S_{0,\, n+1}$ be δ; it is easy to see that this is at least as large as the distortion of each S_{ij}.

Now we have to consider the sets $S_{ij}TA_i\Delta B_j\,(=B_{ij}$ say) and $S_{ij}TB_j\Delta A_i$ ($=A_{ij}$, say). For we have

$$|\mu(TA\times TB)-\mu(A\times B)|=|EZ(A)Z(B)-\sum_i\sum_j EZ(TA_i)Z(TB_j)|$$

$$=|\sum_i\sum_j E\{Z(A_i')Z(B_j)-Z(S_{ij}TB_j)Z(S_{ij}TA_i)\}|,\text{ by stationarity}$$

$$\leqslant\sum_i\sum_j\mu(A_{ij}\times B_{ij}).$$

Now $A_{ij}\subset A_i'$, the disjoint union of two shields with least p-values those of the ends of A_i, comprising the same generators as A_i, half-open similarly to A_i (and so to A itself), and of length (each) $\beta\delta/2^m$ So

$$|\mu(TA\times TB)-\mu(A\times B)|\leqslant\sum_i\sum_j\mu(A_i'\times B_j'),$$

where the B_j', are similarly defined for B. Let $A'=\bigcup_{i=1}^{n}A_i'$, $B'=\bigcup_{j=1}^{n}B_j'$. Then $|\mu(TA\times TB)-\mu(A\times B)|\leqslant\mu(A'\times B')$. But each of A' and B' decreases to the empty set as $m\to\infty$, so the right-hand side decreases to zero. Therefore the left-hand side must vanish, and we have stationarity. in the special case where $\bar{A}$ and $\overline{B+\pi}$ have no generator in common.

In the general case, we again divide the shields A and B into n shields congruent to each other and of the same length as A and B. Then, in the previous notation, $EZ(A)Z(B)=E\sum_i\sum_j Z(A_i)Z(B_j)=$

$=E(\sum+\sum')$, say, where $\sum$ is taken over those i and j such that $\bar{A}_i$ and $\overline{B_j+\pi}$ have a generator in common, and $\sum'$ is the remainder. Now consider $\sum$. Since the sets $\bigcup_{i,j}(A_i\times B_j)$ involved in $\sum$ decrease to the line segment

$\{(w,\, w')\colon \theta'=\theta+\pi;\ p,\ p',\ \theta,\ \theta'$ lie within bounds fixed by A and $B\}$

in the product subspace $A\times B$ of $C\times C$; and since this line segment is a. s. not charged by Z (because it has a. s. no parallel lines), $\sum\to 0$ a. s.. Therefore, $\sum_i\sum_j Z(A_i)Z(B_j)-\sum'$ decreases to zero a. s. as $n=2^m\to\infty$. Taking expectations and using the previous result, we have the theorem. Q. e. d.

Example. If Z may charge the line $\{(w,\, w')\colon \theta'=\theta+\pi\}$, that is, Z admits antiparallel lines, Z needsnot be stationary under reflections.

For let Z_1 be a Poisson process of oriented lines. For each $w \in Z_1$ introduce w' antiparallel to, and to the right of, w, at a distance d (fixed, positive) from w. Let Z be the whole process so obtained — the *rail-way-line* process. Then Z satisfies all of (I) — (V) except (IV). But if we reflect Z we get pairs of antiparallel lines which lie always on each other's left, instead of on the rigt; and it follows from this that Z is not stationary under reflections, either strictly or to the second order.

We now associate to every $Z \in$ LP4 a Y-process, which will turn out to be a nonatomic random measure on the circle, second-order stationary under its rotations, and strictly stationary if Z is.

Definition: Let K^* be the class of binary-rational endpointed, clockwise half-open intervals I of the circle K. For $I \in K^*$, let $I^* \subset C$ be the shield of base I and height 1; we suppose that I^* is closed below and open above. Let T be the translation of the cylinder through height 1. Define

$$Y(I) = \underset{n\to\infty}{\mathrm{l.\,i.\,m.}} \sum_{r=0}^{n} Z(T^r I^*)/(n+1).$$

This is a reasonable definition because, by Theorem 1, the sequence $Z(T^r I^*)$ is second-order stationary. By the assumption (III) on Z we have that $EY^2(K)$ is finite; and it is clear, by taking suittable subsequences of $\{n \to \infty\}$, that $Y(\cdot)$ is a monotone square — summable non-negative random set function on K^*. It is then a trivial matter to deduce, via the Riesz-Markov theorem, that we can extend Y in a unique manner to a random Borel measure on K. It is also trivial that Y inherits the stationarity (under rotations) of Z.

Proposition 6. *Y has a. s. no atoms.*

Proof. There exists a sequence $\{n_k \to \infty\}$ such that for all $I \in K^*$, Y is the a. s. limit of $\sum_{r=0}^{n} Z(T^r I^*)/(n+1)$; and so we can restrict attention to those realizations of Z for which all the a. s. limits exist (there are only countably many of them). Y can still be extended to a Borel measure on K. Suppose that Y has an atom at θ_0. It is clear that the atoms of Z not lying on the generator $\{\theta = \theta_0\}$ do not contribute, *via* the a. s. $(C, 1)$ limits, to our atom of Y. Therefore the only way there can arise an atom at θ_0 is that there be infinitely many atoms of Z on the generator $\{\theta = \theta_0\}$; but this would mean that there was in Z a whole sheet of parallel lines, which contradicts (IV). Q. e. d. We define the *intensity*, λ say, of Z, to be the mean number of points of Z in any set of m-measure 1; or equivalently, $\lambda = EZ(B_1)/4\pi$.

Theorem 2. *Given λ, the covariance measure μ of Z is deuer mincd by that, ν say, of Y.*

Proof. It suffices to determine μ for congruent shields (as usual half-open) A and B of common length, a, in three cases:

(I) $\overline{I(A)} \cap \overline{I(B)} = \varnothing$; (II) $I(A) = I(B)$ or they are contiguous, but $A \cap B = \varnothing$; (III) $A = B$.

We may also, and do, demand that a be a binary rational.

In the first case we may, by shears and a translation and the limiting process used in the first part of Theorem 1, move A and B so that they coincide with the shields $a(I(A))^*$ and $a(I(B))^*$. In view of the definition of Y, and the fact that a is a binary rational, we have at once that $E\{Z(A)Z(B)\} = a^2E\{Y(I(A))\,Y(I(B))\}$. Using this, we may apply, in cases (II) and (III), the approximation method of the second part of the proof of Theorem 1, to obtain, in case (II), $E\{Z(A)Z(B)\} = a^2E\{Y(I(A))\,Y(I(B))\}$; while in case (III), $EZ^2(A) = = a^2EY^2(I(A)) + aEY(I(A)) = a^2EY^2(I(A)) + a\lambda|I(A)|$, where $|\cdot|$ is Lebesgue measure on the circle of unit radius. It follows that once we know λ, the covariance measure of Z is determined by that of its Y. Q. e. d.

C o r o l l a r y. *Given any $Z \in LP4$, there exists $Z^* \in LP4 \cap dsP$ with the same intensity and covariance measure.*

P r o o f. Given Z we have Y on K. Given Y on K construct Z^* on C as follows: the rate measure of Z^* is to be $\Lambda = Y \times l$, where is Lebesgue measure on the line; and Z^* is to be an inhomogeneous Poisson process with this rate measure. It is at once clear that Z^* satisfies (I) — (III). Further, Z^* gives rise to the same Y that we (and Z) started with, so the covariance measures of Z and Z^* are identical (by Theorem 2 and the fact that their intensities are obviously the same). Now since Y is continuous so is the rate measure of Z^*; which means that a. s. there will be no parallel lines in Z^* (contrariwise, there would be if Y did possess atoms, but it doesn't). So $Z^* \in$ LP4 (and LP5 if Z is). Q. e. d.

4. Existence of $Z \in$ LP4—dsP

P r o p o s i t i o n 7. *Let N be a pseudo-Poisson process on the linel Through each point of N put a line; the lines to have orientations independent of each other and N, and common density ∞ $\sin\Phi|$ (see fig. 1). Let Z be the line process so obtained. Then $Z \in LP4 - dsP$.*

P r o o f. By a pseudo-Poisson process we mean a strictly stationary point process on the line with stationary uncorrelated (but not independent) increments. Such processes have been constructed by Lee [6], Renyi [11] and Shepp [5]. Using the (x, Φ) representation of the lines of Z, it is easy to calculate the intensity and covariance measure of Z: on C, with the (p, θ) representation, these are $EZ(A) = \lambda m(A)$; $E\{Z(A)Z(B)\} = = \lambda m(A \cap B) + \lambda^2 m(A)m(B)$, where, if λ' is the expected number of N in interval of length 1, $\lambda' = 4\lambda$.

It is immediately clear that $Z \in$ LP4. Suppose $Z \in$ dsP. Then by Theorem 3 below, N is mixed Poisson. But a mixed Poisson process is ergodic if and only if it is the Poisson process; and the pseudo-Poison process N is ergodic but is, by construction, not Poisson. Q. e. d.

N o t e. For reasons which will appear in section 5, the Z constructed here does not belong to LP5.

Theorem 3. *Let* $Z \in LP4 \cap dsP$. (I) *Let* Z *have rate measure* Λ. *Then there is a version of* Λ *which is a product measure:* $\Lambda = Y \times l$, *where* l *is Lebesgue measure on the line.* (II) *If* w *is any line in the plane, then* $N = Z \cap w$ *is a mixed Poisson process; that is, there is a non-negative random variable* v *such that conditional on* v, N *is Poisson with rate* v.

Proof. We have from Theorem 2 that if A and B are congruent shields, then $E\{Z(A)\,Z(B)\} = a^2 E\{Y(I(A))\,Y(I(B))\} + EZ(A \cap B)$, where a is the common length of A and B. From this, and elementary calculations of the relation between the covariance measures of a doubly-stochastic Z and its rate measure Λ, we find that $E\{\Lambda(A)\Lambda(B)\} = a^2 E\{Y(I(A))\,Y(I(B))\}$. It follows at once that if A and B occupy the same generators, then $E\{\Lambda(A)\Lambda(B)\} = E\Lambda^2(A)$. Immediately, since their mean values are also the same, we have $\Lambda(A) = \Lambda(B)$ a. s. Since Λ is continuous im mean square, this implies that we may choose a version of Λ that is a product measure of the form described. This proves (I).

Turning to (II), we first show that $Z \in$ dsP implies $N \in$ dsP (for the line). Now we may represent — with probability 1, by stationarity — Z on C', the (x, Φ) strip $\{-\infty < x < \infty,\ 0 < \Phi < \pi$ or $\pi < \Phi < 2\pi\}$. Further, Z is still constructed on C' by putting a random square-summable σ-finite Borel measure on C' and then putting on an inhomogeneous Poisson process with our random measure as rate. Then N is obtained from Z by mapping the point (x, Φ) down onto x. So we have the following diagram:

$$\begin{tikzcd} \Lambda \arrow[r, "\gamma"] \arrow[d, "\zeta"'] & Z \arrow[d, "\zeta"] \\ P \arrow[r, "\gamma'"] & N \end{tikzcd}$$

Fig. 2.

Here Λ is the rate measure of Z. γ and γ' are the operations of taking the inhomogeneous Poisson process, and ζ is the operation of integration over Φ. P is then a square-summable σ-finite Borel measure on the line; and, by considering rectangles on the strip, we see easily that the diagram is (not a. s., but in distribution) commutative. Thus N is indeed doubly-stochastic Poisson.

Clearly, N inherits the stationarity of Z; and the material motions of Z are

(A) $(x, \Phi) \to (x + d \tan \Phi, \Phi)$ (from translation of w perpendicular to its length);

(B) $(x, \Phi) \to (x + d, \Phi)$ (from translation of w along its length).

One shows, by methods similar to those of Theorem 1, that

1. If A and B are two rectangles of C' such that there is no generator common to A and B then for all real x we have $E\{Z(A)Z(B)\} = E\{Z(A)\,Z(B+x)\}$.

2. If A and B are two bands (sections transverse to its generators) of C', then for all real x we have

$$E\{Z(A)Z(B)\} - \lambda m(A \cap B) = E\{Z(A)Z(B+x)\} - \lambda m(A \cap (B+x)).$$

3. If I and J are any two intervals of the line, then for all real x we have $E\{N(I)N(J)\} - \lambda' |I \cap J| = E\{N(I)N(J+x)\} - \lambda' |I \cap (J+x)|$.

It is clear that 3. is an immediate deduction from 2., and that 2. will follow, under (IV), from 1. by the approximation method of the last part of the proof of Theorem 1. Again 1. may be proved using the motions (A) and (B) and the methods of the first part of the proof of Theorem 1.

Now that we have 3., we may apply the proof of (I) of the present Theorem to conclude that the measure P is such that for any two congruent intervals I and J on the line, $P(I) = P(J)$ a. s.. Consequently, P is a random multiple of Lebesgue measure, which is equivalent to saying that N is mixed Poisson. Q. e. d.

After this analysis there are two problems outstanding:

1. To find what restrictions are imposed on a random measure (second-order stationary on the line, say) by niceness of its covariance measure.

2. To characterize analytically the covariance measures of LP4's (see Theorem 2). The first problem is of considerable interest in its own right but is not on the theme of this paper; we treat the second. First we observe that this problem is equivalent, by Theorem 2, to characterizing covariance measures of square-summable second-order stationary nonatomic random measures Y on the circle K.

Theorem 4. (I). *μ (on K) is the kernel measure of the covariance measure of such a Y if and only if $\mu = \alpha\mu'$, $\alpha > 0$ a constant, where μ' has no atoms and lies in the convex set D' of probability measures generated by the set Sqp' of squared probability measures without atoms on K. (A probability measure is called squared if it is of the form $\nu * \nu^*$, where $\nu^*(I) = \nu(-I)$ and ν is itself a probability measure.)*

(II). *Let $Z \in LP4$. Then there exists $Z^* \in LP5 \cap dsP$ having the same intensity and covariance measure, and Z^* is got as follows:*

(a) *take a certain random non-atomic probability measure on K;*

(b) *put an independent uniform rotation on this measure—call the result Y_0;*

(c) *take a random variable Λ, independent of the previous choices — Λ may take values 0 and Λ_0, with probabilities q and $p = 1 - q$;*

(d) *set $Y = \Lambda Y_0$, and let Z^* be the doubly-stochastic Poisson process with rate measure $Y \times l$.*

Before going through the proof, we observe that this theorem raises the question, When is a probability measure on K in D'?; and we naturally push this question back one stage, by asking When is a probability measure squared? Neither of these questions has been answered (so, far as I know), even for the more famous (but possibly more difficult) case where the measures are on the line. For that case, we may make the following remarks. Let p be a probability measure on the line and let f be its characteristic function. That p lie in the convex hull of the set of squared probability measures it is necessary that f be real and non-negative; and if p possesses atoms, one of them must be at the origin. p will lie in the convex hull in question if f has the form $f=\frac{2}{3}+\frac{1}{3}$ (any real ch. f.), and p will be a squared measure if f is real and infinitely divisible, or if f is the square of a Polya characteristic function.

Proof of the theorem. We start with a Z and so also with its Y, which has a random Fourier sequence (F. S.) $\{b_n\}$ say: $b_n=\int e^{in\theta}\, Y(d\theta)$. We see at once that $E(b_0)=\lambda$, the intensity ot the process, whereas $E(b_n)=0$ for $n\neq 0$. Let M be the covariance measure of Y. Then by second-order stationarity of Y we may disintegrate M as $M=\mu\times\varkappa$, where $\varkappa$ is proportional to Lebesgue measure on K and μ is the kernel measure, also on K. Now we define the Fourier double sequence $\{a_{m,n}\}$ of M: $a_{m,n}=\int e^{in\theta-im\Phi}\, M(d(\theta,\Phi))$

$$=\int e^{in\theta-im\Phi}\, Y(d\theta)\, Y(d\Phi)=E(b_n\overline{b}_m).$$

Then, using the disintegration of M, it is clear that $a_{m,n}$ vanishes unless $m=n$, when we have $a_{n,n}=E(|b_n|^2)=a_n(\mu)$, say. For by the disintegration we have also $a_{n,n}=\int e^{in\psi}\,\mu(d\psi)$.

We now have obvious conditions on λ, viz. that $\lambda^2\leqslant a_0$ and $\lambda>0$ if $a_0>0$. We shall see later that these are the only relations between λ and the F. S. of μ.

Let L be the set of all totally finite non-negative measures on K; let Lp be the set of those of L whose total mass is 1. Let $Sq\subset L$ be the set of squared measures in L (the 'square roots' also lying in L), and let $Sqp=Sq\cap Lp$. We give L the topology of ordinary convergence of the F. S. 's. To put a prime (') on any of these spaces is to restrict attention to the continuous measures in it. Let D be the closed convex hull of Sqp in L, and let D' be the closed convex hull of Sqp' in L' with the relative topology.

First we observe that $\mu\in L'$. For a. s. $Y\in L'$. Consequently, by the standard criterion for continuity of a measure on K, we have that $(C, 1)$ $\lim |b_n|^2=0$ a. s. Since Y is square-summable we may take expectations and interchange the limit and the expectation, to get $(C, 1)\lim a_n(\mu)=0$. Since all the a's are non-negative, we find that $(C, 1)\lim |a_n(\mu)|^2=0$ which means that μ is continuous.

We may suppose that μ is not the zero measure (which corresponds to a null process, so that the theorem is trivial). Let then $\mu' = \mu/a_0(\mu)$; then $\mu' \in Lp'$. Also, by the representation of the F. S. of μ, we have that $\mu' = \int\limits_{m \in Sq'} m p_0(dm)$, where p_0 is some probability measure on the Borel subsets of Sq' (which is itself a Borel subset of $[0, \infty[\times \times Sqp$, the latter being a compact metric spase); and $a_0(\mu') = = \int\limits_{m \in Sq'} a_0(m) p_0(dm) = 1$.

Now p_0 is not concentrated on the null measure z. Let its atom there then be a, $0 \leqslant a < 1$. Define p_1 on the Borel sets of Sq' by

$$p_1(A) = p_0((1-a)A - \{z\})/(1-a).$$

Then we have $\mu' = \int\limits_{m \in Sq' - \{z\}} m p_1(dm)$, and

$$\int\limits_{Sq' - \{z\}} a_0(m) p_1(dm) = (1-a)^{-1} \int\limits_{Sq' - \{z\}} a_0(m) p_0(d(1-a)m) = (1-a)^{-1}(1-a) = 1.$$

Now define p on Sqp' by, if $A \subset Sqp'$ (so that $A \times]0, \infty[\subset Sq'$),

$$p(A) = \int\limits_{A \times]0, \infty[} a_0(m) p_1(dm).$$

It is clear from the properties of p_1 just proved that p is a probability measure. We assert that $\mu' = \int\limits_{Sqp'} m^* p(dm^*)$. To show this we have to prove that for all n, $a_n(\mu') = \int\limits_{Sqp'} a_n(m^*) p(dm^*)$. But the right-hand side of this equals $\int\limits_{Sq'} a_n(m^*) a_0(m) p_1(dm)$, where m^* is m scaled to a probability; $= \int\limits_{Sq'} a_n(m) p_1(dm)$, by the definition of F. S. and the relations of m^* and m; $= a_n(\mu')$ by properties of p_1.
So we have exhibited μ' as a probability mixture of elements of Sqp'. It follows at once that $\mu' \in D'$.

We have now proved the first part of Theorem 4 (I). We shall now prove its (II) and the last part of (I) simultaneously. By the first part of (I), starting from any $Z \in$ LP4 we end up with a λ, an a_0 and a μ', such that λ and a_0 satisfy the conditions given earlier. So now let us start with λ and a_0 satisfying those conditions, and $\mu' \in D'$.

Since $\mu' \in D'$, μ' also lies in D, the closed convex hull of Sqp in Lp, so that D is compact. Therefore (see e. g. Phelps [10]) there exists a probability measure h on Sqp representing μ'. But if h puts positive mass on $Sqp - Sqp'$, μ' would have an atom at $\theta = 0$. Thus h must be concentrated on Sqp, that is, μ' is a probability mixture of elements $\nu * \nu^*$ of Sqp'. Thus we may sample, with respect to h, a probability measure $\nu \in Lp'$. We may then rotate it uniformly round K, obtaining a random strictly stationary probability measure Y_0 on K.

Now we have to deal with λ and a_0. Consider a random variable Λ taking the two values Λ_0 and 0 with probabilities p and $q = 1 - p$ respectivety. Then $E\Lambda = p\Lambda_0$; $E\Lambda^2 = p\Lambda_0^2$. So if we dilate Y_0 by Λ, we obtain $E(\Lambda Y_0(K)) = p\Lambda_0$; $E(\Lambda Y_0(K))^2 = p\Lambda_0^2$, since Y_0 has to be a probability measure and we assume that Λ and Y_0 are independent. Because of the conditions $\lambda^2 \leqslant a_0$, $\lambda > 0$ if $a_0 > 0$, we can solve the equations $p\Lambda_0 = \lambda$, $p\Lambda_0^2 = a_0$ for $\Lambda_0 > 0$ and the probability p, so that $\Lambda_0 = a_0/\lambda$ and $p = \lambda^2/a_0$. (If a_0 or λ vanishes they both do and we may take $\Lambda_0 = 0$, $p = 1$). Then if we put $Y = \Lambda Y_0$, we have $EY(K) = \lambda$, $EY^2(K) = a_0$; and in fact the kernel measure of the covariance measure of Y is, as desired, $a_0\mu' = \mu$. For the kernel measure of the covariance measure of Y_0, given ν, is computed as $\nu * \nu^*$; and then we take averages. Now if we put on Z^* as the doubly-stochastic Poisson process rate $Y \times 1$, $Z^* \in$ LP5 and has the same intensity and covariance measure as Z. Q. e. d.

5. The big problem

Do there exist elements of LP5 — dsP?

The relevance of the previous work to this is that it might have been possible to prove that the covariance measures of the class dsP did not exhaust those of $LP5$. Howerer, as we have shown, this is not the case (it is the case for point processes on the line: see, e. g.., Bartlott [2]).

How might one try to construct elements of LP5?

1. By taking a point process on a fixed line and putting lines through its points.

2. By taking a stationary point process in the plane and putting lines through its points.

3. By tinkering with a Poisson line process.

We first look at 1. Clearly this is the general method of construction; but how are we to put the lines through the points of the point-pro-cess (which we call N, ano which has, of course, to be strictly stationary under shifts of the line)?

P r o p o s i t i o n 7. *Let $Z \in LP5$ be constructed by method 1, where the lines are put through the points with orientations independent of each other and N, and with a common continuous density. Then N is mixed Poisson and $Z \in dsP$.*

P r o o f. That N is mixed Poisson may be deduced from the work of Breiman [3], as strengthened by Thedéen [14]. It is easy to modify their work to show the following: Let N be a spatially strictly stationary summable process of cars (points) on a road (line), with velocities independent of each other and the positions on the cars and having a common density which is a. e. continuous and bounded on compacta. Then letting N_t be the process of cars as observed at time $t > 0$ ($N_0 = N$), N_t converges in laws of finite-dimensional distributions to a mixed Poisson process as $t \to \infty$.

Applying this to our N, and observing that Z has to be stationary under translations of the fixed line through distances t perpendicular to its lengh, N (which is summable because Z is) must be mixed Poisson. By construction, then, $Z \in$ dsP. Q. e. d.

It is thus difficult to see how we should assign the orientations to the points of N to obtain a $Z \in$ — LP5; and dsP this is as far as I have been able to take method 1. But in any case, we have the following condition on N:

P r o p o s i t i o n 8. *N has a second-order product-moment density which is a constant.*

P r o o f. The second-order product-moment density is defined (see Bartlett [1] by $g(x, y) = \lim EN(I)\, N(J)/|I||J|$, where I and J are congruent intervals with centres, and shrinking down to, x and y respectively $(x \neq y)$. (Of course if g turns out to exist and be a constant it can be defined for $x = y$ by continuity.) But from Theorem 3 (II) we know that so soon as I and J are so small that they are disjoint, the numerator of the limit in the definition of $g(x, y)$ is a constant times the product of the lengths of I and J. It is thus clear that $g(x, y)$ exists for N and is a constant. Now admittedly Theorem 3 (II) only applies to $Z \in$ dsP, but by Theorem 2 the covariance behaviour of these Z exhausts that of all $Z \in$ LP4, and so also that of all $Z \in$ LP5. Q. e. d.

We now turn to method 2. Let $(P, T) = \{x_i, \theta_i\}$ be a marked point process (in the sense of Matthes [7]) in the plane, the x_i being the points of P and the θ_i being the orientations assigned to them. We identify $Z \in$ LP5 with (P, T) by constructing Z with lines going through the x_i with the orientations θ_i. We assume that (P, T) is strictly stationary under the rigid motions of the plane; in which case P is also stationary under these motions. Now P has to be locally square-summable, otherwise Z would certainly not be square-summable (consider those lines whose parent points lie in a convex compact region of non-square-summability of P). Consequently P is *well-distributed* in the sense of Goldman [5]. We assume throughout that P has a. s. infinitely many points.

P r o p o s i t i o n 9. *If T is a process of uniform orientations independent of each other and P, then Z puts a. s. infinitely many lines through each circle.*

Proof. Let the circle be of radius $r>0$ and centre the origin O. Let the points of P lie at distances $r_1 \leqslant r_2 \leqslant r_3 \leqslant \cdots$ from O. Then, by P's being well-distributed, there exist positive constants (conditional on P) h and $k<h$ such that for all n, $|n^{-\frac{1}{2}} r_n - h| < k$.

Now the probability that the line whose parent point lies at a distance $d>r$ from 0 will pass through our circle is $(1/\pi)\sin^{-1}(r/d) \sim \sim (r/\pi d)$ as $d \to \infty$. But the incidences of different lines on our circle are independent; so we may apply the divergence case of the Borel-Cantelli lemmas, with

$$\sum_{n=1}^{\infty} p_n = \sum_{n=1}^{\infty} (1/\pi)\sin^{-1}(r/r_n) > H.\sum_{n=1}^{\infty} n^{-\frac{1}{2}} \text{ for some } H = H(h, k) > 0$$

$$= \infty,$$

so that infinitely many of the lines of Z hit our circle. Q. e. d.

Now we look at the general case. It is clear, by simple addition, that the mean number of lines cutting any circle is infinite, so that method 2 cannot yield an LP4. But the problem of whether the actual number cutting any circle is infinite (even with positive probability) is of independent interest. We might proceed as follows:

The problem for the circle is clearly equivalent to that for a finite nonempty line segment, J say, forming part of some line w (say) in the plane. We divide the process Z up into sub-processes Z_n consisting of those lines of Z whose parent points lie between the distances n (inclusive) and $n+1$ (exclusive) from w. Those lines of Z whose parent points lie below (at negative distances from) w are from now on disregarded; if we can do without them so much the better. Then each Z_n is a marked point point process $\{x_i, y_i, \theta_i\}$ stationary under shifts of x, where x_i is the abscissa along w, y_i is the ordinate (lying in $[0, 1[$), and θ_i is the orientation assigned. Further, the Z_n have identical (but of course not independent) distributions. Let X, X_n be the Z-, Z_n-charges of J; let $Y_n = \sum_{r=0}^{n-1} X_r$. Then for all $t>0$ $E\exp -$

$$-tX = \lim_{n\to\infty} E\exp - tY_n = \lim_{n\to\infty} E\exp - t\sum_{r=0}^{n-1} X_r \leqslant \lim_{n\to\infty} \Big(\prod_{r=0}^{n-1} E\exp - ntX_r\Big)^{1/n},$$

by repeated use of Hölder's inequality.

Now since all the X's are non-negative integers, we have

$$p(X_r = 0) = p_r \text{ (say)} \leqslant E\exp - ntX_r \leqslant p_r + \exp - nt. \text{ So}$$

$$E\exp - tX \leqslant \lim_{n\to\infty} \Big(\prod_{r=0}^{n-1} (p_r + \exp - nt)\Big)^{1/n}.$$

Suppose that (and here is the gap) we have $p_r \leqslant 1-c$ $(c>0)$ uniformly in r; then the formula above gives

$$E \exp - tX \leqslant \lim_{n\to\infty} ((1-c+\exp - nt))^{1/n}$$

$$\to 1 - c, \text{ independent of } t > 0.$$

It would follow at once, from the discontinuity of the Laplace-Stieltjes transform at the origin, that X was infinite with the positive probability c.

Now observe the point where there was the gap. In fact we do not there need that p_r should be bounded away from unity; it is sufficient that $\{p_r\} \to 0$ as $r \to \infty$, that is, that the X_r do not converge to zero in probability. Of course in the case discussed in Proposition 7 we actually have convergence of the p_r to a non-zero limit; and there are other cases, e. g. when equal orientations are permitted, where we can bridge the gap. Even in the general case the result required appears likely enough; even more so, when we consider a very slightly modified version of the problem in different terms. We omit the ordinates y_i from the marked point processes Z_n. Then Z_0 may be regarded as an array of cars on a road at time zero, spatially stationary and with a stationary array of speeds which remain constant in time. Z_n is then the same process of cars observed at time n, at least in distribution; and X_n is the number of cars in a fixed interval J of road at time n. Or we may take X_n to be the number of cars that pass a fixed observer between times n and $n+1$; one does not thereby change the problem of whether X_n converges to zero in probability.

Turning now to method 3, an obvious way of constructing non-doubly-stochastic Poisson processes in R^1 or R^2 is to take a Poisson process and then modify it in some way. For example, one may attach new points to the old ones. or to selected clusters of old ones; or subtract points from clusters, or change the geometry of clusters. and so on. Now the first of these methods only works because of the compactness of the group of rotations about the 'old point', so that we can make a coordinate-free stationary assignment of the new points to the old. With lines that have to be skew, this cannot be done. For clusters, the situation is vaguer, but the general trouble is that any line will appear in infinitely many clusters, so that if we are deleting lines independently from each cluster they will all disappear. On the other hand, if we are adding lines, it seems that clusters may be associated with points of R^2, and then — since these points will form a stationary process — the troubles of method 2 arise.

Thus we think at the moment that dsP does exhaust LP5. If this is so, of course, we may use Miles results' on distributions associated with the Poisson process to get expressions for the same distributions associated with any $Z \in$ LP5. For example, let ρ be the experimental intensity of Z: $\rho = \lim_{r\to\infty} Z(B_r)/4\pi r$. Let δ be the diameter of the incircle of a random polygon; then $\delta\rho$ has an exponential distribution with mean 2 (see [8]).

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ՌՈԼԼՈ ԴԱՎԻԴՍՈՆ. Ուղիղների պատահական դաշտերի երկրորդ կարգի հատկությունները *(ամփոփում)*

Ապացուցված է մի շարք թեորեմներ հարթության վրա թույլ և ուժեղ իմաստով ստացիոնար ուղիղների դաշտերի մասին։ Հատուկ ուշադրություն է դարձվում ստացիոնար դաշտերի դասերի և այսպես կոչված կրկնակի ստոխաստիկ Պուասոնյան դաշտերի միջև եղած առնչությունները պարզաբանելու վրա։ Քննարկված է հետևյալ խնդիրը, որի լուծումը դեռևս գտնված չէ՝ գոյություն ունի արդյոք ուղիղների ուժեղ իմաստով ստացիոնար դաշտ, որը չի հանդիսանում կրկնակի ստոխաստիկ պուասոնյան դաշտ։

Р. ДАВИДСОН. *Построение полей случайных прямых и их свойства второго порядка* (резюме)

Доказан ряд теорем о стационарных в слабом и сильном смысле полях прямых на плоскости. Особое внимание уделяется выяснению соотношений между классами стационарных полей и классами так называемых дважды стохастических пуассоновских полей. Обсуждается задача, решение которой еще не найдено: существует ли стационарное в сильном смысле поле прямых, не являющееся дважды стохастическим пуассоновским полем.

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