Մեխանիկորեն ազատ, թույլ մակերեսային անհարթությունների դեպքում առաջանում են ալիքի թույլատրելի հաճախականությունների գոտիներ (ինչպես նաև հաճախականությունների լռելյան գոտիները), և ալիքային էներգիայի տեղայնացումը տեղի է ունենում մերձմակերևութային գոտիներում:

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INSTABILITY OF A NORMAL SHEAR (SH) WAVE IN A WEAKLY INHOMOGENEOUS ELASTIC LAYER

The article studies the effects of longitudinal weak heterogeneity of the material, as well as the geometric weak heterogeneity of the elastic layer surfaces on the normal shear wave, under various mechanical boundary conditions. It is shown, that in case of hard-clamped smooth surfaces of an isotropic elastic layer, the asymmetric localization of the wave energy occurs near the middle plane of the layer. In case of mechanically free smooth surfaces, the localization of the wave energy occurs in the near-surface zones. But more intensively the localization appears again near the middle plane of the layer. In both cases, due to the material inhomogeneity influence on the normal wave, two new cramped frequencies appear. In case of weakly inhomogeneous mechanically free surfaces, the frequency transmission zones of the formed wave (as well as frequency default zones) and localization in the near-surface layers of heterogeneity appear.

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INITIATING STUDIES OF ZEROS OF SOLUTIONS OF SOME BASIC DIFFERENTIAL EQUATIONS

Abstract

Despite the fact that the zeros of solutions of ordinary differential equations were widely studied, we were not able to find publications concerning zeros of solutions of the basic equations $x' = f(x, t)$. A recent *principle of zeros* giving bounds for the number of zeros of real functions permits to initiate similar studies.

Also, we consider some large systems of equations; both autonomous and non-autonomous. Particularly we study the well-known system of equations $x' = P(x, y)$, $y' = G(x, y)$, where $P(x, y)$ and $G(x, y)$ are arbitrary polynomials. For an arbitrary solution $(x(t), y(t)), 0 \leq a \leq t \leq b < \infty$, of this system we give upper bounds for the number of zeros of $x(t)$ and/or $y(t)$ occurring on the segment $[a, b]$.

MSC2010: 34A26, 34A30, 34A40, 34A99, 35B05.

Keywords: *Zeros of solutions of equations.*

Introduction. An essential part of studies in ordinary differential equations $F(x, y, y', \dots) = 0$ (both real and complex) relates to oscillations (i.e. zeros) of their solutions. However, we were not able to find publications in mathematical journals related to zeros of the most basic equation $x' = f(x, t)$. Making use a recent principle of zeros of real functions we give easily (in section 1) upper bounds for the number of zeros of solutions $x(t)$.

As to the zeros of solutions of systems of equations $F_1(x, y, x', y') = 0$, $F_2(x, y, x', y') = 0$, they seemingly were not considered at all; at least we are not aware of corresponding publications in mathematical books and journals. This should be considered as an essential gap in differential equations since the solutions of the systems admit a lot of interpretations in different applied fields.

In section 2 we initiate similar studies by giving upper bounds for a number of zeros of solutions of some widely utilized autonomous and non-autonomous system of equations.

1. On zeros of solutions of equations $x' = f(x, t)$.

Consider the equation

$$x' = f(x, t) \qquad (1)$$

and its solution x(t) satisfying the following quite common restrictions: $x(t) \in C^2[a.b]$, $t \in [a, b]$, $M_x := max\_\{a \le t \le b\}|x(t)| < \infty$, $f(x, t) \in C^2(d)$, where d is the rectangle $-M_x \le x \le M_x, a \le t \le b$, and assume that $c_1 := sup_{(x,t)\in d}|f(x, t)| < \infty$, $c_2 := sup_{(x,t)\in d}|f_t'(x, t)| < \infty$ and $c_3 := sup_{(x,t)\in d}|f_x'(x, t)| < \infty$.

Denote by $N_{[a,b]}](\mathrm{x} = 0)$ the number of zeros of $x(t)$. If there are intervals in $[a, b]$, where $x(t)$ are identically equal to zero we will count each similar interval as one zero.

Theorem 1. *For any solution $x(t)$ satisfying the above assumptions we have*

$$N_{[a,b]}](\mathrm{x} = 0) < \frac{1}{\pi}[(c_1c_3 + c_2)M_x + 2{c_1}^2]|b - a| + 1. \qquad (2)$$

Comment 2.1. Notice that in the most considered case, when $f(x, t)$ is a polynomial $P(x, t)$, the constants c_1, c_2 and c_3 can be majorated by some easily determined magnitudes depending just on the coefficient and degrees of P and M_x.

2. On zeros of solutions of systems of equations

Consider first solutions $(x(t), y(t)), 0 \le a \le t \le b < \infty$, of the following autonomous system of equations

$$x' = P(x, y), \qquad y' = G(x, y), \qquad (3)$$

where $P(x, y)$ and $G(x, y)$ are arbitrary polynomials.

The case when P and Q are linear was studied in detail by Poincaré in his famous works, see for instance [3]. Particular cases of equation (1), where P and Q are polynomials of degree 2 and 3 arise in a huge number of applied problems (in biology, economics, physics, environmental sciences). The general case of equation (3) was touched much less.

Since the solution $(x(t), y(t))$ is a pair of functions we consider the number of zeros both for $x(t)$ and $y(t)$. Denote by $N_{[a,b]}(x = 0)$ (by $N_{[a,b]}(y = 0)$) the number of zeros of $x(t)$ (of $y(t)$). If there are intervals in $[a, b]$, where $x(t)$ (or $y(t)$) are identically equal to zero we will count each similar interval as one zero.

Denote $x(t), y(t) \in C^2[a, b]$, if $x(t), y(t)$ have continuos second derivatives in $[a, b]$, $-\infty < a \le t \le b < \infty$. Also denote $M_x = max_{a\le t\le b}|x(t)|$ and $M_y = max_{a\le t\le b}|y(t)|$.

Theorem 2. *For any solution* $(x(t), y(t)) \in C^2[a, b]$ *of (1) we have*

$$N_{[a,b]}(x = 0) \le K_x(b - a) + 1, \quad (4)$$

where K_x depends on (and easily determined by) P, Q, M_x, M_y.

Here

$$K_x = \frac{K_{1x}M_x + K_{2x}}{\pi},$$

$$K_{1x} = max_{|x|\le M_x\ |y|\le M_y}|P'_x(x, y)P(x, y) + P'_y(x, y)Q(x, y)|$$

and

$$K_{2x} = 2max_{|x|\le M_x\ |y|\le M_y}P^2(x, y).$$

Comment 1. Obviously, similar estimate holds also for the number of zeros $N_{[a,b]}(x = 0)$ of the solutions $y(t)$.

Comment 2. If we need to give upper bounds for the number $N_{[a,b]}(x = h)$ of solutions of $x(t) = h$ on $[a, b]$, where $h = const \ne 0$, we can (making use substitution $X(t) = x(t) - h$) consider the equation

$$X' = P(X, y)),\ y' = Q(X, y). \quad (5)$$

Then applying Theorem 2 to (5) we will get upper bounds for $N_{[a,b]}(x = h)$.

Comment 3: a practical problem. Assume $x(t) = h$ at the point $T_{x=h} \in [a, b]$ and $T^*_{x=h} \in [a, b]$ is the next point, where $x(t) = h$. The difference $T^*_{x=h}$- $T_{x=h}$ indicate the time we need to reach the same level $x(t) = h$. In other words, we consider the following question: how long does it take to recover a given level (t) =h?

A similar problem may arise in many applied fields.

If $h = 0$ we can apply Theorem 1 with a=$T_{x=0}$ and $b = T^*_{x=0}$ and taking into account that $N_{[T_{x=0},T^*_{x=0}]}(x = h = 0) = 2$ we get $1 \le K_x(b - a)$, i.e. we obtain the following

Corollary 1. *For the recovering time when* $h = 0$ *we have*

$$T^*_{x=0} - T_{x=0} \ge \frac{\pi}{K_x}. \quad (6)$$

If $h \ne 0$ we can consider the system of equations (4) and obtain quite similarly below bounds for $T^*_{x=h} - T_{x=h}$ when $h \ne 0$.

Comment 4: an illustration of the predator-prey model. The Lotka-Volterra's equation describes a predator-prey (or parasite-host) model which assumes that, for a set of fixed positive constants A (the growth rate of prey), B (the rate at which predators destroy prey), C (the death rate of predators), and D (the rate at which predators increase by consuming prey) the following equation holds:

$$x' = Ax - Bxy, y' = -Cy + Dxy. \quad (7)$$

Notice that usually, we know maximal possible upper bounds for the number of preys and predators in a given area so that we can assume that the quantities of M_x and M_y are known.

Consider the number $N_{[a,b]}(x=h)$ of solutions $x(t)=h$ on $[a,b]$; $N_{[a,b]}(x=h)$ shows how many time the quantity $x(t)$ of preys can be equal h for $t \in [a,b]$.

Corollary 2. *For the equation (6) and any $h < M_x$ we have*

$$N_{[a,b]}(x=h) \le \frac{1}{\pi}[k_{1,x}+k_{2,x}](b-a)+1, \quad (8)$$

where $k_{1,x}$ and $k_{2,x}$ are determined (a bit lengthily, however) very simply:

$$k_{1,x} = B^2M_x^2M_y^2 + BDM_x^3M_y + [2AB(1+h)+BC]M_yM_x^2 + (1+h)A^2M_x^2,$$

$$k\_\{2,x\} = 2M\_\{x\}^2A^2 + 4ABM\_\{x\}^2M\_\{y\} + 2B^2M\_\{x\}^2M\_\{y\}^2.$$

Besides, arguing as in Comment 3, we get

$$T^*_{x=h} - T_{x=h} \ge \frac{\pi}{k_{1,x}+k_{2,x}}. \quad (9)$$

3. Proofs.

Proof of Theorem 1. Consider a real function $x(t)$, $t \in [a,b]$ with continuos x'' (i.e. $x(t) \in C^2[a,b]$) and denote by $N_{[a,b]}(x=0)$ the number of the zeros of $x(t)$; here we count in N each possible interval, where $x \equiv 0$ as one zero.

Our approach based on a general method (principle) permitting to give bounds for zeros of "enough smooth" real function, see the book [1], item 5.3.2. Below we utilize the simplest version of the principle (see paper [2] in this volume) which we state as the following

Lemma 1. *For an arbitrary function $x(t) \in C^2[a,b]$ we have*

$$N_{[a,b]}(x=0) < \frac{1}{\pi}\int_a^b [|x''(t)||x(t)| + 2[x'(t)]^2]dt + 1. \quad (10)$$

Theorem 1 immediately follows from inequality (10). Indeed, since for the solution of $x(t)$ of (1) we have

$$x''(t) = f'_x(x,t)x'(t) + f'_t(x,t) = f'_x(x,t)f(x,t) + f'_t(x,t),$$

consequently we have

$$|x''(t)| \le c_1c_3 + c_2,$$

so that taking into account that $|x(t)| \le M\_\{x\}$ and $[x'(t)]^2 \le {c_1}^2$ we get

$$N_{[a,b]}(x=0) <$$

$$\frac{1}{\pi}\int_a^b [|f'_x(x,t)f(x,t) + f'_t(x,t)||x(t)| + 2[x'(t)]^2]dt + 1 \le$$

$$\frac{1}{\pi}[(c_1c_3+c_2)M_x + 2{c_1}^2]|b-a| + 1,$$

and the inequality (1) is proved.

Proof of Theorem 2. Now we apply Lemma 1 to the solutions $(x(t), y(t)) \in C^2[a,b]$ of equation (2). Since $x'(t) = P(x,y)$ and $y'(t) = Q(x,y)$ we have

$$x''(t) = P'_t(x,y) = P'_x(x,y)x'(t) + P'_y(x,y)y'(t) =$$

$$P'_x(x,y)P(x,y) + P'_y(x,y)Q(x,y),$$

consequently we have

$$|x''(t)||x(t)| + 2[x'(t)]^2 \le K_{1x}|x(t)| + K_{2x}$$

and applying (10) we get

$$N_{[a,b]}(x = 0) <$$

$$\frac{1}{\pi}\int_a^b [K_{1x}|x(t)| + K_{2x}]dt + 1 \le \frac{1}{\pi}[K_{1x}M_x + K_{2x}](b - a) + 1.$$

i.e. we obtain Theorem 2.

Proof of Corollary 2. Making use substitution $X = x(t) - h$ we can rewrite the equation (5) as

$$X' = P(X,y)) = (X + h)A - B(X + h)y, \qquad y' = Q(X,y) = -Cy + D(X + h)y \quad (11)$$

Then the zeros X are the solutions of $x(t) - h = 0$; respectively

$N_{[a,b]}(x = h) = N_{[a,b]}(X = 0)$.

The corollary is trivial when we have only one solution of $x(t) - h = 0$.

Consider the case when we have two (or more) solutions of $x(t) - h = 0$. Then obviously $h < M_x$; otherwise, we can't have solutions of $x(t) - h = 0$.

Applying (9) to the equation (11) we have

$$N_{[a,b]}(x = h) = N_{[a,b]}(X = 0) \le$$

$$\frac{1}{\pi}\int_a^b [|P'_X(X,y)P(X,y) + P'_y(X,y)Q(X,y)|M_X + 2P^2(X,y)|]dt + 1. \quad (12)$$

Further since $P(X,y)) = (X + h)A - B(X + h)y$ we have

$$|P'_X(X,y)P(X,y) + P'_y(X,y)Q(X,y)| =$$

$$|(1 + h)(A - By)[(X + h)A - B(X + h)y] - B(X + h)[-Cy + D(X + h)y]| =$$

$$|(1 + h)(A - By)^2(X + h) + BCy(X + h) - BD(X + h)^2y|$$

Noticing that due to definitions and meanings all values A, B, C, D, h are positive and also noticing that $M_X = M_x - h = max_{a\le t\le b}|x(t)| - h > 0$ (consequently $|X| \le M_x - h$) and that

$$(A - BM_y)^2 = A^2 + 2ABM_y + B^2M_y^2$$

we get

$$P'_X(X,y)P(X,y) + P'_y(X,y)Q(X,y)| =$$

$$(1 + h)(A^2 + 2ABM_y + B^2M_y^2)M_x + BCM_yM_x + BDM_x^2M_y =$$

$$B^2M_y^2M_x + BDM_x^2M_y + [2AB(1 + h) + BC]M_yM_x + (1 + h)A^{2M_x}; \quad (13)$$

here we take into account that $X + h \le M_X + h = M_x$.

On the other hand

$$2P^2(X,y) = 2[(X + h)A - B(X + h)y]^2 \le$$

$$2M_x^2A^2 + 4ABM_x^2y + 2B^2M_x^2y^2 \le$$

$$2M_x^2A^2 + 4ABM_x^2M_y + 2B^2M_x^2M_y^2 \quad (14)$$

Substituting (13) and (14) in (12) we obtain Corollary 2.

Comment 5. The reader can easily see that the method above is applicable also for some large classes of non-autonomous equations

$$x' = E(t,x,y), y' = R(t,x,y)$$

We just need to assume that functions $E(t,x,y)$ and $R(t,x,y)$ and their derivatives in t,x and y admit upper bounds. Then we will get quite a similar result for solutions $(x(t),y(t))$ of these equations.

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ԳՐԻԳՈՐ ԲԱՐՍԵՂՅԱՆ

ֆիզիկամաթեմատիկական գիտությունների դոկտոր

ՈՐՈՇ ՀԻՄՆԱԿԱՆ ԴԻՖԵՐԵՆՑԻԱԼ ՀԱՎԱՍԱՐՈՒՄՆԵՐԻ ԼՈՒԾՈՒՄՆԵՐԻ ԶՐՈՆԵՐԻ ՈՒՍՈՒՄՆԱՍԻՐՈՒԹՅՈՒՆՆԵՐԻ ՆԱԽԱՁԵՌՆՈՒՄ

Չնայած սովորական դիֆերենցիալ հավասարումների լուծումների զրոները լայնորեն ուսումնասիրված են, մենք չկարողացանք գտնել հրապարակումներ $x' = f(x,t)$ հիմնարար հավասարումների լուծումների զրոների վերաբերյալ: Իրական գործառույթների զրոների քանակին վերաբերող, վերջերս հայտնաբերված սկզբունքը թույլ է տալիս նախաձեռնել նմանատիպ ուսումնասիրություններ:

Սույն հոդվածում մենք ուսումնասիրել ենք նաև $x' = P(x,y), y' = G(x,y)$ հավասարումների հայտնի համակարգը, որտեղ $P(x,y)$ և $G(x,y)$ կամայական բազմանդամներ են: Այս համակարգի կամայական $(x(t),y(t)), 0 \le a \le t \le b < \infty$ լուծման համար մենք տալիս ենք վերին սահմաններ $x(t)$ և / կամ $y(t)$ –ի զրոների քանակի համար, որոնք ընկած են [a, b] հատվածում:

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ИНИЦИИРОВАНИЕ ИССЛЕДОВАНИЙ НУЛЕЙ РЕШЕНИЙ НЕКОТОРЫХ ОСНОВНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

Несмотря на то, что нули решений обыкновенных дифференциальных уравнений были широко изучены, мы не смогли найти публикаций, касающихся нулей решений основных уравнений $x' = f(x,t)$. Недавний принцип нулей, дающий границы для числа нулей реальных функций, позволяет инициировать аналогичные исследования.

Также мы рассматриваем некоторые системы уравнений: как автономные, так и неавтономные. В частности, мы изучаем известную систему уравнений $x' = P(x,y), y' = G(x,y)$, где $P(x,y)$ и $G(x,y)$ - произвольные многочлены. Для произвольного решения

$(x(t),y(t)), 0 \le a \le t \le b < \infty$ этой системы мы даем верхние оценки для числа нулей для $x(t)$ и / или $y(t)$ на отрезке $[a,b]$.

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