# NEW ENTROPIES, BLACK HOLES, AND HOLOGRAPHIC DARK ENERGY

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The Bekenstein-Hawking entropy is a cornerstone of horizon thermodynamics but quantum effects correct it, while inequivalent entropies arise also in non-extensive thermodynamics. Reviewing our previous work, we advocate for a new entropy construct that comprises recent and older proposals and satisfies four minimal key properties. The new proposal is then applied to black holes and to holographic dark energy and shown to have the potential to cause early universe inflation or to alleviate the current Hubble tension. We then analyze black hole temperatures and masses consistent with alternative entropies.

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1. *Introduction*. Einstein said of thermodynamics that "A theory is the more impressive the greater the simplicity of its premises is, the more different kinds of things it relates, and the more extended is its area of applicability. Therefore the deep impression which classical thermodynamics made upon me. It is the only physical theory of universal content concerning which I am convinced that within the framework of the applicability of its basic concepts, it will never be overthrown" [1]. Indeed, thermodynamics is applied to a large variety of physical theories and situations, but its application to systems with long range interactions (where the partition function commonly diverges), and to gravity in particular, constitutes a challenge. A major discovery in the 1970s was the formulation of black hole thermodynamics [2,3]. It started with the discovery of the Bekenstein-Hawking entropy of black holes [4] $S = A/4G$, where $A$ is the area of the event horizon and we use geometrized units in which the speed of light $c$, the reduced Planck constant $\hbar$, and the Boltzmann constant $K_B$ are unity. The pieces of the puzzle fell into place when Hawking discovered that the Schwarzschild black hole radiates scalar quanta at the Hawking temperature

$$T_H = \frac{1}{8\pi M}, \qquad (1)$$

where $M$ is the black hole mass [5]. While, in classical thermodynamics, entropy is universal and defined uniquely, quantum effects correct the Bekenstein-Hawking entropy pointing to its modification in full quantum gravity, while inequivalent

entropies arise also in non-extensive thermodynamics. New entropy proposals come from classical gravity as well. Here we summarize recent work and ideas on the application of alternative entropies to cosmology and black holes [6,7].

New entropy proposals in the literature include non-extensive entropies that reduce to $S$ in some limit, such as the Tsallis entropy ([8], see also [9,10]) for systems with long range interactions

$$S_T = \frac{A_0}{4G}\left(\frac{A}{A_0}\right)^\delta,$$ (2)

where $A_0$ is a constant area and the dimensionless parameter $\delta$ quantifies non-extensivity. Other proposals are the Rényi entropy [11-13]

$$S_R = \frac{1}{\alpha}\ln(1 + \alpha S)$$ (3)

related to information theory, the Sharma-Mittal entropy [14]

$$S_{SM} = \frac{1}{R}\left[(1 + \delta S_T)^{R/\delta} - 1\right]$$ (4)

(with $R$ and $\delta$ free parameters), the Barrow entropy [15]

$$S_B = \left(\frac{A}{A_{Pl}}\right)^{1+\Delta/2}$$ (5)

proposed as a toy model for quantum spacetime foam (where $A_{Pl}$ is the Planck area), the Kaniadakis entropy [16]

$$S_K = \frac{1}{K}\sinh(KS)$$ (6)

generalizing the Boltzmann-Gibbs entropy in relativistic statistical systems [16], and the non-extensive Loop Quantum Gravity proposal [12,17]

$$S_q = \frac{1}{1-q}\left[e^{(1-q)\Lambda(\gamma_0)S} - 1\right],$$ (7)

where the entropic index $q$ quantifies how the probability of frequent events is enhanced relatively to infrequent ones, $\Lambda(\gamma_0) = \ln2/\sqrt{3}\pi\gamma_0$, and $\gamma_0$ is the Barbero-Immirzi parameter, usually taking one of the two values $\ln2/\pi\sqrt{3}$ or $\ln3/2\pi\sqrt{2}$, depending on the gauge group used.

These new entropies share four properties, which we promote to minimal requirements for any alternative entropy proposal:

1. Generalized third law: The entropy vanishes when the Bekenstein-Hawking entropy $S$ does. While, in the standard thermodynamics of closed systems in equilibrium, $e^S$ expresses the number of states and the entropy $S$ vanishes at zero temperature because the ground (vacuum) state should be unique, the Bekenstein-

Hawking entropy diverges when $T_H \to 0$ and vanishes as $T_H \to \infty$. We require generalized entropies to vanish when the Bekenstein-Hawking entropy $S$ does.

2. Monotonicity: The entropy is a monotonically increasing function of the Bekenstein-Hawking entropy $S$.

3. Positivity: The entropy is positive, as the number of states $e^S$ is greater than unity.

4. Bekenstein-Hawking limit: The entropy reduces to the Bekenstein-Hawking prescription $S$ in an appropriate limit.

A new and very general entropy with the above properties and incorporating all the above-mentioned entropy proposals as special limits is [6]

$$S_G\left(\alpha_\pm, \beta_\pm, \gamma_\pm\right) = \frac{1}{\alpha_+ + \alpha_-}\left[\left(1 + \frac{\alpha_+}{\beta_+} S^{\gamma_+}\right)^{\beta_+} - \left(1 + \frac{\alpha_-}{\beta_-} S^{\gamma_-}\right)^{-\beta_-}\right], \qquad (8)$$

where we assume all the parameters ($\alpha_\pm, \beta_\pm, \gamma_\pm$) to be non-negative. This proposal reproduces (2)-(7) for appropriate parameter values [6].

A simpler alternative proposal is the 3-parameter entropy [6]

$$S_G\left(\alpha, \beta, \gamma\right) = \frac{1}{\gamma}\left[\left(1 + \frac{\alpha}{\beta} S\right)^{\beta} - 1\right], \qquad (9)$$

where ($\alpha, \beta, \gamma$) are non-negative. This quantity reduces to the Sharma-Mittal entropy (4) with $S_T = S$ (or $\delta = 1$) when $\gamma = \alpha$. If $\gamma = (\alpha/\beta)^\beta$, the limit $\alpha \to \infty$ yields

$$\lim_{\alpha \to \infty} S_G\left(\alpha, \beta, \gamma = \left(\frac{\alpha}{\beta}\right)^\beta\right) = S^\beta \qquad (10)$$

and the choices $\beta = \delta$ and $\beta = 1 + \Delta/2$ reproduce the Tsallis and Barrow entropies (2) and (5), respectively. The limit $\alpha \to 0$, $\beta \to 0$ with $\alpha/\beta$ finite gives instead the Rényi entropy (3) (where $\alpha/\beta$ is replaced by $\alpha$ and $\gamma = \alpha$). Finally, $\beta \to \infty$ and $\gamma = \alpha$ gives the new quantity $S_G(\alpha, \beta \to \infty, \alpha) \to \left(e^{\alpha S} - 1\right)/\gamma$ satisfying our four entropy requirements.

2. *Black holes and the holographic Universe*. Let us apply the generalized entropy to the Schwarzschild geometry

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \frac{dr^2}{1 - 2GM/r} + r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right), \qquad (11)$$

where $M$ is the black hole mass. One can attempt to identify the Tsallis or the Rényi entropies (2) or (3) with the black hole entropy [18]; then, if we assume that the mass $M$ coincides with the thermodynamical energy $E$ [12,13], $dS_G = dE/T_G$ requires the temperature $T_G$ to be defined by

$$\frac{1}{T_G} \equiv \frac{d\,S_G}{dM} \neq \frac{1}{T_H}.$$ (12)

Alternatively, assuming the Hawking temperature (1) as the thermodynamic temperature, the first law

$$dE_G = T_H d\,S_G = \frac{\alpha}{\gamma\sqrt{16\pi G}}\left[ S^{-1/2} + \frac{\alpha(\beta-1)}{\beta}S^{1/2} + O\!\left(S^{3/2}\right) \right]$$ (13)

gives

$$E_G = \frac{\alpha}{\gamma}\left[ M + \frac{4\pi G\,\alpha(\beta-1)}{3\beta}M^3 + O\!\left(M^5\right) \right] \neq M \,.$$ (14)

The search for new entropies must deal with this problem, which requires a deeper reexamination of the broader thermodynamical formalism [6]. Black hole thermodynamics is expected to change drastically when quantum gravity becomes important. Eventually, the latter should change classical and black hole thermodynamics, not only redefining entropy but also correcting well-established quantities such as temperature and energy.

Thermodynamics has been applied fruitfully to another area of gravitational physics, that is, cosmology, to which we now turn our attention. In the holographic dark energy (HDE) scenario [19], thermodynamics plays a primary role since it is applied successfully to explain dark energy with the entropy of the cosmological horizon. In this context, the density of the HDE is proportional to the square of the inverse holographic cutoff $L_{IR}$, $\rho_{hol} = 3C^2/\kappa^2\,L_{IR}^2$, where $C$ is a free parameter. This holographic cutoff $L_{IR}$ is usually taken to be the size of the particle horizon $L_p$ or of the future horizon $L_f$, but there is no compelling argument for choosing this quantity. Following [19], the cutoff is assumed to depend on $L_{IR}$ ($L_p, \dot{L}_p, \ddot{L}_p, ..., L_f, \dot{L}_f, ..., a$), which gives the "generalized HDE" [19]. In the spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) universe described by the line element

$$ds^2 = -dt^2 + a^2(t)\!\left(dx^2 + dy^2 + dz^2\right)$$ (15)

in comoving coordinates, one speculates that the generalized HDE originates from the entropy of the cosmological horizon. The physical radii of the particle and event horizons of the FLRW universe (27) are $L_p \equiv a(t)\int_0^t dt'/a(t')$ and $L_f \equiv a(t)\int_t^\infty dt'/a(t')$ (when these integrals converge). Differentiating gives

$$H\!\left(L_p, \dot{L}_p\right) = \frac{\dot{L}_p}{L_p} - \frac{1}{L_p}, \quad H\!\left(L_f, \dot{L}_f\right) = \frac{\dot{L}_f}{L_f} + \frac{1}{L_f},$$ (16)

where $H \equiv \dot{a}/a$, an overdot denoting differentiation with respect to $t$. In the thermodynamical approach to gravity (e.g., [20]), the Einstein-Friedmann equa-

tions are derived from the Bekenstein-Hawking entropy $S$: the apparent horizon of the universe (27) has radius $r_H = H^{-1}$, area $A = 4\pi H^{-2}$, and Bekenstein-Hawking entropy $S = \pi H^{-2}/G$. We have

$$dQ = -dE = -\frac{4\pi}{3} r_H^3 \dot{\rho}\, dt = -\frac{4\pi}{3H^3} \dot{\rho}\, dt = \frac{4\pi}{H^2}(\rho + P)\, dt \tag{17}$$

using covariant conservation $\dot{\rho} + 3H(\rho + P) = 0$; from the Gibbons-Hawking temperature $T = H/2\pi$ and the first law of thermodynamics $TdS = dQ$, it follows that $\dot{H} = -4\pi G(\rho + P)$ and integration gives the Friedmann equation

$$H^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}, \tag{18}$$

where the cosmological constant $\Lambda$ appears as an integration constant. If the Bekenstein-Hawking entropy $S$ is replaced by a different (non-extensive) concept of entropy, the Friedmann equation (18) is modified, which is attributed to the HDE. For example, the Tsallis entropy (2) yields the modified Friedmann equation

$$H^2 = \frac{8\pi G}{3}(\rho + \rho_T) + \frac{\Lambda}{3}, \quad \rho_T = \frac{3}{8\pi G}\left[H^2 - \frac{\delta}{2-\delta} H_1^2 \left(\frac{H}{H_1}\right)^{2(2-\delta)}\right]. \tag{19}$$

Interpreting $\rho_T = 3C^2/\kappa^2 L_{IR,T^2}$ as the HDE due to the infrared holographic cutoff $L_{IR,T}$ leads to

$$L_{IR,T} = \frac{1}{C}\left[\left(\frac{\dot{L}_f}{L_f} + \frac{1}{L_f}\right)^2 - \frac{\delta}{2-\delta} H_1^2 \left(\frac{\frac{\dot{L}_f}{L_f} + \frac{1}{L_f}}{H_1}\right)^{2(2-\delta)}\right]^{-1/2}. \tag{20}$$

The Barrow entropy (5) describing the spacetime quantum foam phenomenologically gives

$$\rho_B = \frac{3}{8\pi G}\left[H^2 - \left(\frac{1+\Delta/2}{1-\Delta/2}\right)\frac{16\pi G}{A_{Pl}^2}\left(\frac{H^2}{4\pi A_{Pl}}\right)^{1-\Delta/2}\right] \tag{21}$$

while, with the new three-parameter entropy proposal (74), one obtains

$$\rho_G = \frac{3}{8\pi G}\left[H^2 - \frac{\pi\alpha}{G\beta\gamma(1-\beta)}\left(\frac{G\beta H^2}{\pi\alpha}\right)^{2-\beta} {}_2F_1\left(1-\beta, 2-\beta, 3-\beta; -\frac{G\beta H^2}{\pi\alpha}\right)\right], \tag{22}$$

where ${}_2F_1(a, b, c; z)$ is the hypergeometric function.

The formal conservation law $\dot{\rho}_G + 3H(\rho_G + P_G) = 0$ defines the pressure $P_G$ of the HDE and its equation of state parameter

$$w_G \equiv \frac{P_G}{\rho_G} = -1 - \frac{2\dot{H}}{3}\left[H^2 - \frac{\pi\alpha}{G\beta\gamma(1-\beta)}\left(\frac{G\beta H^2}{\pi\alpha}\right)^{2-\beta} {}_2F_1\left(1-\beta, 2-\beta, 3-\beta; -\frac{G\beta H^2}{\pi\alpha}\right)\right]^{-1}$$

$$\times\left[1 - \frac{2-\beta}{\gamma(1-\beta)}\left(\frac{G\beta H^2}{\pi\alpha}\right)^{1-\beta} {}_2F_1\left(1-\beta, 2-\beta, 3-\beta; -\frac{G\beta H^2}{\pi\alpha}\right)\right. \tag{23}$$

$$\left. + \frac{2-\beta}{\gamma(3-\beta)}\left(\frac{G\beta H^2}{\pi\alpha}\right)^{2-\beta} {}_2F_1\left(2-\beta, 3-\beta, 4-\beta; -\frac{G\beta H^2}{\pi\alpha}\right)\right].$$

When matter can be neglected and $\Lambda = 0$, the Friedmann equation $H^2 = (8\pi G/3)\rho_G$ and Eq. (22) yield ${}_2F_1 = 0$. The zeros $Z_i$ of this hypergeometric function are de Sitter universes with Hubble functions given by $Z_i = -G\beta H^2/\pi\alpha$ and effective cosmological constants $\Lambda_{eff} = 3\pi\alpha Z_i/G\beta$. If $\Lambda_{eff}$ is large, it can cause early universe inflation; if it is very small, it may describe the present accelerated expansion; if it is slightly larger than the present dark energy, it could potentially solve the Hubble tension problem [21,22].

Consider the case of a small $Z_1$: the hypergeometric function is approximated as

$$\begin{aligned}{}_2F_1\left(1-\beta, 2-\beta, 3-\beta; -\frac{G\beta H^2}{\pi\alpha}\right) &\simeq 1 - \frac{(1-\beta)(2-\beta)}{3-\beta}\frac{G\beta H^2}{\pi\alpha} \\ &+ \frac{(1-\beta)(2-\beta)^2}{4-\beta}\left(\frac{G\beta H^2}{\pi\alpha}\right)^2,\end{aligned} \tag{24}$$

then $Z_1 = -G\beta H^2/\pi\alpha \simeq -(3-\beta)/(1-\beta)(2-\beta)$ and $H^2 \sim (3-\beta)\pi\alpha/(1-\beta)(2-\beta)G\beta$. If $3-\beta \sim O(10^{-2n})$, $\alpha \sim O(10^{-2m})$, it is $H^2 \sim (10^{-n-m+28}\,\text{eV})^2$ and $n+m=61$ gives the current dark energy scale $H \sim 10^{-33}$ eV. If another zero $Z_2$ exists with $|Z_2|$ slightly smaller than $Z_1$, the effective cosmological constant can potentially solve, or alleviate, the Hubble tension problem [21,22].

In general, the hypergeometric function can have several or infinitely many zeros. If there are a root of order unity or a large and negative root $Z_i$, then one can obtain the large Hubble rate corresponding to the inflationary epoch. Retaining, for illustration, the first three terms in Eq. (24),

$$\frac{G\beta H^2}{\pi\alpha} = -\frac{4-\beta}{2(2-\beta)(3-\beta)}\left(1 \pm \sqrt{1 - \frac{4(3-\beta)^2}{(4-\beta)(1-\beta)}}\right), \tag{25}$$

and assuming $\beta \lesssim 3$, we have

$$Z_+ = -\frac{4-\beta}{2(2-\beta)(3-\beta)}, \quad Z_- = -\frac{3-\beta}{2(1-\beta)(2-\beta)}. \tag{26}$$

For $n+m=61$ one finds again the present Hubble scale. If, instead, $G\beta H^2/\pi\alpha = Z_+$, then $H^2 \sim \left(10^{-n-m+28}\,\mathrm{eV}\right)^2$ and, for $n+m=61$, it is $H^2 \sim \left(10^{-2m+89}\,\mathrm{eV}\right)^2$. At the GUT scale $\sim 10^{16}\,\mathrm{GeV} = 10^{25}\,\mathrm{eV}$ and inflation with $H \sim 10^{22}\,\mathrm{eV}$, we obtain $m \sim 33$ or 34, so $Z_+$ may explain early universe inflation.

One can also study generalized HDE from the full six-parameter entropy (72) instead of using the simpler proposal (73), as we did here. Correspondingly, there are many more possibilities to realize realistic cosmic histories.

3. *Alternative entropies and corresponding energies*. We describe spherical, static, and asymptotically flat spacetimes with the geometry

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2 d\Omega_{(2)}^2 . \tag{27}$$

where $\lambda(r) \to 0$ and $\nu(r) \to 0$ as $r \to +\infty$. Let us consider Einstein gravity and interior solutions. The $t$-$t$ Einstein equation is

$$-\kappa^2\rho = \frac{1}{r^2}\left(re^{-2\lambda} - r\right)' , \tag{28}$$

where $\rho$ is the energy density, $f' \equiv df/dr$, and the mass is given by

$$e^{-2\lambda} \equiv 1 - \frac{\kappa^2 m(r)}{4\pi r} , \tag{29}$$

$$4\pi r^2 \rho = m'(r). \tag{30}$$

Here

$$m(r) = 4\pi\int_0^r r'^2 \rho(r')dr' + m_0 , \tag{31}$$

with $m_0$ an integration constant. The metric inside a matter ball must be regular at $r=0$, hence

$$\lambda \to 0, \quad \lambda'(r) = \frac{m - rm'}{r(r - 2m)} \to 0 \tag{32}$$

as $r \to 0$ or else there is a conical singularity. Then, $m_0 = 0$ and

$$m(r) = 4\pi\int_0^r r'^2 \rho(r')dr' . \tag{33}$$

For asymptotically Schwarzschild geometries

$$M = m(r \to \infty) = 4\pi\int_0^\infty dr\, r^2 \rho(r) \tag{34}$$

while, in the presence of a central singularity, the integration constant $m_0$ remains and

$$M = m(r = \infty) = 4\pi\int_0^\infty dr\, r^2 \rho(r) + m_0 . \tag{35}$$

The total mass is not $m(r = \infty)$ but [7]

$$\overline{M} = \int d^3x \sqrt{\gamma}\rho(r) = 4\pi \int_0^\infty \rho(r)r^2 \left[1 - \frac{2Gm(r)}{r}\right]^{-1/2} dr$$

$$= 4\pi \int_0^\infty dr\,\rho(r)r^2 \left[1 + \frac{Gm(r)}{r} - \frac{3G^2m^2(r)}{r^2} + O\!\left(G^3\right)\right],$$

(36)

where $\gamma$ is the determinant of the 3D Riemannian metric

$$\gamma_{\ell m}\,dx^\ell dx^m = e^{2\lambda}dr^2 + r^2 d\Omega^2_{(2)}.$$

(37)

The gravitational binding energy of the ball is $E_B = M - \overline{M}$. The second term in the last line of Eq. (36) is interpreted as the Newtonian gravitational potential energy

$$-4\pi G\int_0^\infty dr\,\rho(r)r^2\,\frac{m(r)}{r} = -\frac{G}{2}\int dV \int dV'\frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|},$$

(38)

where the general-relativistic corrections are of order $G^2$ and higher.

If a black hole geometry is asymptotically Schwarzschild we can impose $m(r \to \infty) = M$, fixing the integration constant $m_0$ (one obtains $m_0 = M$ for the Schwarzschild black hole, for which $\rho$ can be seen as proportional to a Dirac delta centered at $r = 0$). The mass $M$ coincides with the Arnowitt-Deser-Misner mass.

Let us consider now modified gravity, in which case we write the $t$-$t$ field equation as

$$-\kappa^2\rho_{eff} = \frac{1}{r^2}\left(re^{-2\lambda} - r\right)'.$$

(39)

Now the effective energy density $\rho_{eff}$ is defined by casting the field equations as effective Einstein equations with right-hand sides that contain effective stress-energy tensors built with the non-Einsteinian terms. Now the effective mass is

$$m_{eff}(r) = 4\pi \int_0^r dr' r'^2\,\rho_{eff}(r').$$

(40)

For example, for $F(R)$ gravity

$$S_{F(R)} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\,F(R) + S^{(matter)}$$

(41)

(where $S$ is the action, $R$ is the Ricci scalar, $F(R)$ is a nonlinear function, and $g$ is the determinant of the metric $g_{\mu\nu}$), we write $F(R) \equiv R + f(R)$ and $f_R(R) \equiv df(R)/dR$. The (0, 0) field equation defines the total energy density $\rho_{eff} = \rho + \rho_{F(R)}$, where

$$\rho_{F(R)} \equiv \frac{1}{\kappa^2} \left\{ -\frac{f}{2} - e^{-2\lambda} \left[ \nu'' + (\nu' - \lambda')\nu' + \frac{2\nu'}{r} \right] f_R \right.$$
$$\left. + e^{-2\lambda} \left[ f_R'' + \left( -\lambda' + \frac{2}{r} \right) f_R' \right] \right\}. \tag{42}$$

The resulting (effective) total mass

$$\overline{M}_{eff} = \int d^3x \sqrt{\gamma} \rho_{eff}(r) = \int d^3x \sqrt{\gamma} \left( \rho + \rho_{F(R)} \right). \tag{43}$$

receives contributions from both matter and gravity. The leading correction to the binding energy is

$$E_{B,eff} = -G \int dV \int dV' \frac{\left[ \rho(\mathbf{r}) + \rho_{F(R)}(\mathbf{r}) \right] \left[ \rho(\mathbf{r}') + \rho_{F(R)}(\mathbf{r}') \right]}{|\mathbf{r} - \mathbf{r}'|} + \dots, \tag{44}$$

$M_{eff} = m_{eff}(r \to \infty)$ is the total mass-energy of the system, while $m_{eff}(r)$ is the mass-energy of a 2-sphere of radius $r$.

A black hole in alternative theories of gravity may have horizon radius $r_h \neq 2GM_{eff} \equiv 2m_{eff}(r \to \infty)$. Now, if $M_{eff}$ is used as the internal energy and $S = 4\pi r_h^2/4$ as the black hole entropy, the new temperature given by

$$\frac{1}{T} = \frac{dS}{dM_{eff}} \tag{45}$$

differs from the usual Hawking temperature $T_H$. Alternatively, if the Hawking temperature is used, the entropy

$$dS = \frac{dM_{eff}}{T_H} \tag{46}$$

must replace the Bekenstein-Hawking entropy. The difference $M_{eff} - m_{eff}(r_h)$ could then be identified with the energy outside the horizon. For this black hole, $m_{eff}(r_h)$ would be the internal energy and Eq. (46) would become

$$dS_{bh} = \frac{dm_{eff}(r_h)}{T_H}. \tag{47}$$

4. *Temperatures corresponding to alternative entropies.* Denote the metric coefficients as

$$h(r) \equiv e^{2\nu(r)}, \quad h_1(r) \equiv e^{-2\lambda(r)}, \tag{48}$$

then the roots of $h(r) = 0$ locate the event horizon. If $h_1(r)$ does not vanish simultaneously with $h(r)$, the spacetime curvature diverges as $h(r) \to 0$. If $h_1(r)$ and $h(r)$ do vanish simultaneously, the surface $h_1(r) = h(r) = 0$ is an event horizon. In fact, consider the curvature invariants

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{1}{4h^4r^4}\left[4r^4h''^2h^2h_1^2 + 4r^4hh_1h'h''(h_1'h - h'h_1) + (h'^2h_1r^2)^2\right.$$
$$\left. -2r^4h'^3h_1h_1'h + (rhh')^2(h_1'^2r^2 + 8h_1)^2 + 8h^4(r^2h_1'^2 + 2(1-h_1)^2)\right], \tag{49}$$

$$R_{\mu\nu}R^{\mu\nu} = \frac{1}{8h^4r^4}\left[4r^4h''^2h^2h_1^2 + 4h\left[h(rh_1' + 2h_1)h' - rh'^2h_1 + 2h^2h_1'\right]r^3h_1h''\right.$$
$$+ r^4h'^4h_1^2 + r^2h^2(12h_1^2 + h_1'r^2)h'^2 - 2r^3hh_1(rh_1' + 2h_1)h'^3$$
$$+ 4rh^3(2h_1'rh_1 - 4h_1 + 4h_1^2 + h_1'^2r^2)h' + 4h^4(3h_1'^2r^2 + 4r(h_1-1)h_1' + 4(h_1-1)^2)\right], \tag{50}$$

$$R = \frac{2h''h_1r^2 - r^2h_1h'^2 + rh'h(rh_1' + 4h_1) + 4h^2(h_1 + rh_1' - 1)}{2h^2r^2}. \tag{51}$$

Their denominators contain positive powers of $h$ and these invariants diverge as $h \to 0$. If $h_1(r)$ and $h(r)$ vanish simultaneously, the invariants (49) remain finite where $h_1(r) = h(r) = 0$ since $h_1(r) = h_2(r)h(r)$ and $h_2 \neq 0$ and is regular where $h(r) = 0$. Then the substitution of $h_1 = h_2h$ in Eq. (49) yields

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = h''^2h_2^2 + h'h_2h_2' + \left(\frac{h'h_2'}{2}\right)^2 + \left(\frac{2h_2h'}{r}\right)^2$$
$$+ \frac{2\left[r^2(hh_2' + h'h_2)^2 + 2(h_2^2h^2 - 1)^2\right]}{r^4}, \tag{52}$$

$$R_{\mu\nu}R^{\mu\nu} = \frac{h''^2h_2^2}{2} + \frac{h''h'h_2h_2'}{2} + \frac{h_2^2h'h''}{r} + \frac{h_2h^2(hh_2' + h_2h')}{r} + \frac{3h_2^2h'^2}{2r^2}$$
$$+ \frac{h'^2h_2^2}{8} + \frac{h_2h'(hh_2' + h_2h')}{r^2} - \frac{2h_2^2h'}{r^3} + \frac{2hh_2^2h'}{r^3} + \frac{hh_2^2h'}{2r} + \frac{h_2h'^2h_2'}{r}$$
$$\times \frac{3(hh_2' + h_2h')r^2 + 4r(hh_2 - 1)(hh_2' + h_2h') + 4(1 - hh_2)^2}{2r^4}, \tag{53}$$

and

$$R = 2h_2h'' + \frac{2rh_2h'}{r} + \frac{h'h_2'}{2} + \frac{2\left[h_1 + r(hh_2' + h_2h') - 1\right]}{r^2}, \tag{54}$$

and these invariants remain finite as $h(r) \to 0$.

Given that $h_1(r)$ and $h(r)$ vanish simultaneously on the event horizon, we can write $h_1(r) = e^{-2\lambda(r)}$ and the radius of the event horizon is

$$r_h = \frac{\kappa^2 m(r_h)}{4\pi} = 2Gm(r_h). \tag{55}$$

Close to the horizon, i.e., at $r \equiv r_h + \delta r$,

$$e^{-2\lambda} = h_1 = hh_2 = \frac{C(r_h)(r - r_h)}{r_h}, \tag{56}$$

$$e^{2\nu} = h = \frac{h_1}{h_2} = \frac{C(r_h)(r-r_h)}{h_2(r_h)r_h}, \tag{57}$$

where $C(r_h) \equiv 1 - m'(r_h)$. Wick-rotating the time $t \to i\tau$, the near-horizon geometry (27) becomes

$$ds^2 \simeq \frac{C(r_h)\delta r}{h_2(r_h)r_h} d\tau^2 + \frac{r_h}{C(r_h)\delta r} d(\delta r)^2 + r_h^2 d\Omega_{(2)}^2. \tag{58}$$

Introduce the new radial coordinate defined by $d\rho = d(\delta r)\sqrt{r_h/C(r_h)\delta r}$ and

$$\rho = 2\sqrt{\frac{r_h\,\delta r}{C(r_h)}}, \quad \delta r = \frac{C(r_h)\rho^2}{4r_h}, \tag{59}$$

then the near-horizon geometry (58) reads

$$ds^2 \simeq \frac{C(r_h)^2}{4h_2(r_h)r_h^2}\rho^2\,d\tau^2 + d\rho^2 + r_h^2 d\Omega_{(2)}^2 \tag{60}$$

To avoid conical singularities near $\rho = 0$, one imposes that the Euclidean time $\tau +$ is periodic of period $t_*$,

$$\frac{C(r_h)\tau}{2r_h\sqrt{h_2(r_h)}} \simeq \frac{C(r_h)\tau}{2r_h\sqrt{h_2(r_h)}} + 2\pi. \tag{61}$$

As a result, the temperature corresponds to $t_*^{-1}$. In the Euclidean path integral formulation of finite-temperature field theory

$$\int [D\phi] e^{\int_0^{t_*} L(\phi)} dt = \mathrm{Tr}\left(e^{-t_* H}\right) = \mathrm{Tr}\left(e^{-H/T}\right) \tag{62}$$

and the temperature of the Schwarzschild black hole

$$T = \frac{C(r_h)}{4\pi r_h\sqrt{h_2(r_h)}} = \frac{C(r_h)}{8\pi Gm_{eff}(r_h)\sqrt{h_2(r_h)}} = \frac{C(r_h)T_H}{\sqrt{h_2(r_h)}} \tag{63}$$

follows which, in general, differs from the Hawking temperature

$$T_H \equiv \frac{1}{8\pi Gm_{eff}(r_h)} \tag{64}$$

by the factor $C(r_h)/\sqrt{h_2(r_h)}$ that cannot be absorbed into a time rescaling because we have fixed the scale so that

$$h(r \to \infty) = h_2(r \to \infty)h_1(r \to \infty) = e^{2\nu(r\to\infty)} = 1. \tag{65}$$

Since Hawking radiation is a near-horizon phenomenon, thermal radiation can correspond to the temperature (63).

Identifying $m_{eff}(r_h)$ with the black hole internal energy, Eq. (47) yields

$$S_{bh} = \int \frac{dm_{eff}(r_h)}{T}. \tag{66}$$

The solutions of the gravitational field equations contain integration constants $c_i$ for $i = 1, ..., N$ (for example, in general relativity (GR) the mass $M$ of the Schwarzschild black hole appears as an integration constant in the metric coefficients $e^{2v} = e^{-2\lambda} = 1 - 2M/r$ when integrating the Einstein equations for spherical and asympotically flat vacuum). $N$ depends on the theory and $\lambda(r)$, $v(r)$, $m(r)$, $h(r)$, and $h_{1,2}(r)$ depend on the integration constants $c_i$. The solution $r_h(c_i)$ of Eq. (55) also depends on these integration constants (again, for the Schwarzschild black hole of GR, $r_h = 2M$). Eq. (55) yields

$$m(r_h) = m(r = r_h(c_i); c_i) = \frac{r_h(c_i)}{2G}, \tag{67}$$

then the integration constants $c_i$'s can be parametrized with a single parameter $\xi$, $c_i = c_i(\xi)$ (for example, the Reissner-Nordström black hole can be parametrized by the charge-to-mass ratio).

In this way, Eq. (63) turns Eq. (66) into

$$S_{bh} = \frac{1}{2G} \int d\xi \frac{\left[4\pi r_h\left(c_i(\xi)\sqrt{h_2(r = r_h(c_i(\xi)); c_i(\xi))}\right)\right]}{1 - \left.\dfrac{\partial m(r; c_i(\xi))}{\partial r}\right|_{r = r_h c_i(\xi)}} \sum_{i=1}^{N} \frac{\partial r_h(c_i)}{\partial c_i} \frac{\partial c_i}{\partial \xi} \tag{68}$$

and choosing $\xi = r_h$, Eq. (68) becomes

$$S_{bh} = \frac{1}{2G} \int_0^{r_h} d\xi \frac{\left(4\pi \xi \sqrt{h_2(r = \xi; c_i(\xi))}\right)}{1 - \left.\dfrac{\partial m(r; c_i(\xi))}{\partial r}\right|_{r = \xi}}, \tag{69}$$

where the integration constant is determined by the condition $S_{bh} = (r_h = 0) = 0$. In GR, the Schwarzschild black hole with $h_2(x) = 1$, $m = M = $ const is characterized by the Bekenstein-Hawking entropy. The different choice in which $h_2(r \to r_h)$ gives a contribution leads to an entropy $S_{bh}$ potentially different from the Bekenstein-Hawking one.

According to Eq. (69),

$$\frac{h_2(r = r_h; c_i(r_h))}{\left(1 - \left.\dfrac{\partial m(r; c_i(r_h))}{\partial r}\right|_{r = r_h}\right)^2} = 16G^2 [S'_{bh}(A)]^2 \tag{70}$$

and various entropy choices lead to corresponding forms of

$$\frac{h_2(r = r_h; c_i(r_h))}{\left(1 - \left.\dfrac{\partial m(r; c_i(r_h))}{\partial r}\right|_{r = r_h}\right)^2}. \tag{71}$$

In [6], we proposed two generalizations of entropy. We begin with the six-parameter entropy

$$S_G(\alpha_\pm, \beta_\pm, \gamma_\pm) = \frac{1}{\alpha_+ + \alpha_-} \left[ \left( 1 + \frac{\alpha_+}{\beta_+} S^{\gamma_+} \right)^{\beta_+} - \left( 1 + \frac{\alpha_-}{\beta_-} S^{\gamma_-} \right)^{-\beta_-} \right],$$

$$(72)$$

where we take all the parameters $(\alpha_\pm, \beta_\pm, \gamma_\pm)$ to be positive. Adjusting these parameters to suitable values, this entropy function reduces to the entropies (2)-(7). If we choose $\alpha_+ = \alpha_- = 0$ and $\gamma_- = \gamma_+ \equiv \gamma$, the values $\gamma = \delta$ or $\gamma = 1 + \Delta/2$ give back the Tsallis entropy (2) and the Barrow entropy (5). If we pick $\alpha_- = 0$ and we write $\alpha_+ = R$, $\beta_+ = R/\delta$, and $\gamma_+ = \delta$, then we recover the Sharma-Mittal entropy (4). Another possibility consists of the limit $\alpha_+ \to 0$ and $\beta_+ \to 0$ with $\alpha \equiv \alpha_+/\beta_+$ finite. Further setting $\gamma_+ = 1$, this procedure recovers the Rényi entropy (3). In the different limit $\beta_\pm \to 0$ of the entropy (72) with $\gamma_\pm = 1$ and $\alpha_\pm = K$ the latter is reduced to the Kaniadakis entropy (6). Finally, if we fix $\alpha$ and $\gamma$ to the values $\alpha_- = 0$ and $\gamma_+ = 1$ in Eq. (72), the limit $\beta_+ \to +\infty$ in conjunction with $\alpha = 1 - q$ reproduces the Loop Quantum Gravity entropy (7) with $\Lambda(\gamma_0) = 1$.

Another proposal in [6], containing only three parameters, consists of

$$S_G(\alpha, \beta, \gamma) = \gamma^{-1} \left[ \left( \frac{\alpha}{\beta} S + 1 \right)^\beta - 1 \right].$$

$$(73)$$

We choose again positive values of the parameters $\alpha$, $\beta$, and $\gamma$. When $\gamma = \alpha$, $S_G$ is the same as the Sharma-Mittal entropy (4) with $S_T = S$ and $\delta = 1$. If we fix $\gamma = (\alpha/\beta)^\beta$, then (73) becomes the Tsallis entropy proposal (2) if $\beta = \delta$ and the Barrow entropy (5) if $\alpha \to \infty$. To conclude, the limit $(\alpha, \beta) \to (0, 0)$ with $\alpha/\beta$ finite yields the Rényi entropy (3), provided that we substitute $\alpha$ in place of $\alpha/\beta$ and that $\gamma = \alpha$.

Let us come to discuss spherical spacetimes while using the Tsallis entropy (2). Equation (70) then becomes

$$\frac{h_2(r = r_h; c_i(r_h))}{\left( 1 - \frac{\partial m(r; c_i(r_h))}{\partial r} \bigg|_{r=r_h} \right)^2} = \delta^2 \left( \frac{4\pi r_h^2}{A_0} \right)^{2(\delta-1)}.$$

$$(74)$$

In the same geometry, the Rényi entropy construct (3) yields instead

$$\frac{h_2(r = r_h; c_i(r_h))}{\left( 1 - \frac{\partial m(r; c_i(r_h))}{\partial r} \bigg|_{r=r_h} \right)^2} = \frac{1}{\left( 1 + \frac{\pi\alpha r_h^2}{G} \right)^2}.$$

$$(75)$$

By contrast, the Kaniadakis entropy (6) yields

$$\frac{h_2\left(r=r_h\,;c_i\left(r_h\right)\right)}{\left(1-\left.\dfrac{\partial\,m\left(r;c_i\left(r_h\right)\right)}{\partial\,r}\right|_{r=r_h}\right)^2}=\cosh^2\left(\frac{\pi K r_h^2}{G}\right),$$

(76)

while our six-parameter entropy (72) produces

$$\frac{h_2\left(r=r_h\,;c_i\left(r_h\right)\right)}{\left(1-\left.\dfrac{\partial\,m\left(r;c_i\left(r_h\right)\right)}{\partial\,r}\right|_{r=r_h}\right)^2}=\frac{1}{\left(\alpha_++\alpha_-\right)^2}\left[\alpha_+\gamma_+\left(\frac{\pi r_h^2}{G}\right)^{\gamma_+-1}\left(1+\frac{\alpha_+}{\beta_+}\left(\frac{\pi r_h^2}{G}\right)^{\gamma_+}\right)^{\beta_+-1}\right.$$

$$\left.+\,\alpha_-\gamma_-\left(\frac{\pi r_h^2}{G}\right)^{\gamma_--1}\left(1+\frac{\alpha_-}{\beta_-}\left(\frac{\pi r_h^2}{G}\right)^{\gamma_-}\right)^{-\beta_--1}\right]^2.$$

(77)

We can also consider our simplified three-parameter entropy (73), which gives

$$\frac{h_2\left(r=r_h\,;c_i\left(r_h\right)\right)}{\left(1-\left.\dfrac{\partial\,m\left(r;c_i\left(r_h\right)\right)}{\partial\,r}\right|_{r=r_h}\right)^2}=\frac{\alpha^2}{\gamma^2}\left[1+\left(\frac{\pi\alpha r_h^2}{\beta G}\right)\right]^{2\beta-2}.$$

(78)

Specific models realizing these relations have been discussed in [7].

5. *Conclusions*. The Bekenstein-Hawking entropy is modified by quantum gravity phenomenology, as exemplified by the Barrow and the Loop Quantum Gravity proposals (5) and (7), or by non-extensive thermodynamics. While specific modifications abound in the literature and may be questionable, the general idea of departures from the simpler Bekenstein-Hawking prescription in the presence of phenomena such as spacetime foam, loops, or the unusual weighting of frequent/infrequent states, appears reasonable. Lacking knowledge of the "correct" entropy, we propose a phenomenological prescription which incorporates many recent and older entropies proposed in the literature and embodies four key properties that we identify as essential requirements for any physically reasonable entropy. Our most general construct contains six parameters, but a simplified version limited to three parameters seems to achieve the same goals, as shown in [6,7,18] and summarized here.

In addition to containing the previous Barrow, Loop Quantum Gravity, Rényi, Tsallis, Sharma-Mittal, and Kaniadakis entropies as special cases, and to reducing to the Bekenstein-Hawking entropy in an appropriate limit, our new proposal exhibits interesting phenomenology when applied to holographic dark energy in cosmology. In this context, there is the possibility of generating an effective

cosmological constant, which can either cause early universe inflation or alleviate the current Hubble tension afflicting the standard $\Lambda$CDM cosmological model [21,22]. Even though tiny, Planck-scale suppressed, infrared corrections to low-energy physics could at first sight seem unable to generate observable cosmological effects, this may not be the case. While the details of possible quantum gravity corrections remain obscure, one can follow Einstein's insight on the wide applicability of thermodynamics in physics and search for these corrections through their effects on entropy and thermodynamics. The new entropy proposals outlined here and in [6] seem to offer a practical implementation of this approach to cosmology and gravity.

Changing the notion of entropy jeopardizes the thermodynamics, unless the temperature and mass (i.e., internal energy) are also changed in a suitable way. We have proposed ways of making the entire thermodynamics consistent with alternative entropies, but we have not exhausted all possibilities. Alternatives will be explored in the future.

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# НОВЫЕ ЭНТРОПИИ, ЧЕРНЫЕ ДЫРЫ И ГОЛОГРАФИЧЕСКАЯ ТЕМНАЯ ЭНЕРГИЯ

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Энтропия Бекенстейна-Хокинга является краеугольным камнем термодинамики черных дыр, но квантовые эффекты дают поправки к ней. Не эквивалентные энтропии появляются также в не-экстенсивной термодинамике.

Мы делаем обзор предыдущих работ и предлагаем новую энтропию, которая обобщает другие варианты энтропий и удовлетворяет четырем ключевым условиям. Новая энтропия рассмотрена в теории черных дыр и голографической темной энергии. Показано, что она может объяснить инфляцию ранней Вселенной и указать на возможность решить недавнюю проблему с параметром Хаббла. Сделан также анализ температур и масс черных дыр совместимых с альтернативными энтропиями.

Ключевые слова: *энтропия: черные дыры: термодинамика*

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