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BERGMAN-TYPE AND Q_k-TYPE SPACES OF p-HARMONIC FUNCTIONS

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Abstract. In this paper, we extend a Hardy-Littlewood type theorem to the exponentially p-harmonic Bergman space on the real unit ball $\mathbb{B}$ in $\mathbb{R}^n$. As an application, we characterize exponentially p-harmonic Bergman spaces in terms of Lipschitz type conditions. Furthermore, some derivative-free characterizations for n-harmonic Q_k spaces are established.

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1. Introduction and main results

For $n \geq 2$, let $\mathbb{R}^n$ denote the usual real vector space of dimension n. For two column vectors $x, y \in \mathbb{R}^n$, we use $\langle x, y\rangle$ to denote the inner product of x and y. The ball in $\mathbb{R}^n$ with center a and radius r is denoted by $\mathbb{B}(a, r)$. In particular, we write $\mathbb{B} = \mathbb{B}(0, 1)$ and $\mathbb{B}_r = \mathbb{B}(0, r)$. Let dv be the normalized volume measure on $\mathbb{B}$ and $d\sigma$ the normalized surface measure on the unit sphere $\mathbb{S} = \partial\mathbb{B}$.

The purpose of this paper is to investigate p-harmonic functions whose definition is as follows.

Definition 1.1. Let $p > 1$ and Ω be a domain in $\mathbb{R}^n$. A continuous function $u \in W^{1,p}_{loc}(\Omega)$ is *p-harmonic* if

$$\operatorname{div}\Big(|\nabla u|^{p-2}\nabla u\Big) = 0$$

in the weak sense, i.e.,

$$\int_\Omega \langle |\nabla u|^{p-2}\nabla u, \nabla\eta\rangle dv(x) = 0$$

for each $\eta \in C_0^\infty(\Omega)$.

p-harmonic functions are natural extensions of harmonic functions from a variational point of view. It has been extensively studied because of its various interesting

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features and applications. By a well-known regularity result due to Tolksdorf, p-harmonic functions are $C^1(\Omega)$. Moreover $u \in W^{2,2}_{loc}(\Omega)$ if $p \geq 2$ and $u \in W^{2,p}_{loc}(\Omega)$ if $1 < p < 2$ (cf. [12, 20]).

Let $p > 1$, we denote by $h_p(\mathbb{B})$ the set of all p-harmonic functions on the real unit ball $\mathbb{B}$ in $\mathbb{R}^n$. For $\alpha \in \mathbb{R}$ and $\beta > 0$, the so-called exponential weighted function $\omega_{\alpha,\beta}$, introduced by Aleman and Siskakis [2], is defined as

$$\omega_{\alpha,\beta}(x) = (1-|x|)^\alpha \exp\Big(\frac{-1}{(1-|x|)^\beta}\Big), \quad x \in \mathbb{B},$$

and the associated weighted volume measure is denoted by

$$dv_{\alpha,\beta}(x) = \omega_{\alpha,\beta}(x)dv(x).$$

For $1 < s < \infty$, $\alpha \in \mathbb{R}$ and $\beta > 0$, the *exponentially weighted p-harmonic Bergman space* $\mathcal{A}^s_{\alpha,\beta}(\mathbb{B})$ is defined as

$$\mathcal{A}^s_{\alpha,\beta}(\mathbb{B}) = \Big\{u \in h_p(\mathbb{B}) : \|u\|^s_{\mathcal{A}^s_{\alpha,\beta}} = \int_{\mathbb{B}} |u(x)|^s dv_{\alpha,\beta}(x) < \infty\Big\}.$$

In particular, if $\beta = 0$, then $\mathcal{A}^s_{\alpha,\beta}(\mathbb{B})$ becomes the weighted p-harmonic Bergman space, which is denoted by $\mathcal{A}^s_\alpha(\mathbb{B})$.

For $0 < s < \infty$, $\alpha > -1$, let f be a holomorphic function on the unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$. The famous Hardy-Littlewood theorem for holomorphic Bergman spaces asserts that

$$(1.1) \int_{\mathbb{D}} |f(z)|^s(1-|z|^2)^\alpha dA(z) \approx |f(0)|^s + \int_{\mathbb{D}} |f'(z)|^s(1-|z|^2)^{s+\alpha} dA(z),$$

where dA is the area measure on $\mathbb{C}$ normalized so that $A(\mathbb{D}) = 1$ (cf. [10]).

It is well-known that integral estimate (1.1) plays an important role in the theory of holomorphic functions. For the generalizations and applications of (1.1) to the spaces of holomorphic functions, harmonic functions, and solutions to certain PDEs, see [3, 4, 5, 9, 15, 11, 14, 21, 25] and the references therein. In [18], Siskakis extended (1.1) to the setting of exponentially weighted Bergman space of holomorphic functions for $1 \leq s < \infty$. For the further generalizations of (1.1) to holomorphic Bergman spaces with some general differential weights, see [15, 19]. By applying these results, Cho and Park characterized exponentially weighted Bergman space in terms of Lipschitz type conditions([5, Theorem A], [6, Theorem 3.1]).

In [11], Kinnunen et al. pointed out that (1.1) is also true for p-harmonic functions. More precisely, they obtained the following integral estimate.

Theorem A. *Let $\alpha > -1$, $1 < s < \infty$, then*

$$\int_{\mathbb{B}} |u(x)|^s (1-|x|)^\alpha dv(x) \approx |u(0)|^s + \int_{\mathbb{B}} |\nabla u(x)|^s (1-|x|)^{s+\alpha} dv(x) \tag{1.2}$$

for all $u \in h_p(\mathbb{B})$.

With developing of theory on the standard (weighted) Bergman space, more general spaces such as weighted Bergman spaces with exponential type weights have been extensively studied (see [2, 4, 5, 6, 8, 16]). As the first aim of this paper, we consider an analogue of (1.2) in the setting of exponentially weighted p-harmonic Bergman space $\mathcal{A}^s_{\alpha,\beta}(\mathbb{B})$. The following is our result in this line.

Theorem 1.1. *Let $1 < s < \infty$, $\alpha \in \mathbb{R}$ and $\beta \geq s-1$, then*

$$\int_{\mathbb{B}} |u(x)|^s dv_{\alpha,\beta}(x) \approx |u(0)|^s + \int_{\mathbb{B}} |\nabla u(x)|^s (1-|x|)^s dv_{\alpha,\beta}(x) \tag{1.3}$$

for all $u \in h_p(\mathbb{B})$.

To state our next results, let us recall the following notion.

The weighted hyperbolic distance d_λ, due to Dall'Ara [7], is induced by the metric $\lambda(x)^{-2} dx \otimes dx$, i.e,

$$d_\lambda(x,y) = \inf_\gamma \int_0^1 \frac{|\gamma'(t)|}{\lambda(\gamma(t))} dt, \quad x, y \in \mathbb{B},$$

where $\lambda(x) = (1-|x|^2)^2$ and $\gamma : [0,1] \to \mathbb{B}$ is a parametrization of a piecewise C^1 curve with $\gamma(0) = x$ and $\gamma(1) = y$. By [7], it was shown that $d_\lambda(x,y) \approx \frac{|x-y|}{[x,y]^2}$ when x, y are close sufficiently in $\mathbb{B}$, see Section 4 in [7] for details.

As an application of Theorem 1.1, we obtain a Lipschitz type characterization for exponentially weighted p-harmonic Bergman space $\mathcal{A}^s_{\alpha,\beta}(\mathbb{B})$.

Theorem 1.2. *Let $1 < s < \infty$, $\alpha \in \mathbb{R}$, $\beta \geq 2s-1$ and $u \in h_p(\mathbb{B})$. Then the following statements are equivalent:*

(a) $u \in \mathcal{A}^s_{\alpha,\beta}(\mathbb{B})$;

(b) There exists a positive continuous function $g \in L^s(\mathbb{B}, dv_{\alpha,\beta})$ such that

$$|u(x) - u(y)| \leq \frac{|x-y|}{[x,y]^2} \big(g(x) + g(y)\big)$$

for all $x, y \in \mathbb{B}$;

(c) There exists a positive continuous function $g \in L^s(\mathbb{B}, dv_{\alpha,\beta})$ such that

$$|u(x) - u(y)| \leq d_\lambda(x,y) \big(g(x) + g(y)\big)$$

for all $x, y \in \mathbb{B}$;

(d) There exists a positive continuous function $h \in L^s(\mathbb{B}, dv_{\alpha+2s,\beta})$ *such that*

$$|u(x) - u(y)| \le |x - y|\big(h(x) + h(y)\big)$$

for all $x, y \in \mathbb{B}$.

Remark 1.1. Theorem 1.2 is a generalization of [5, Theorem A] to the setting of p-harmonic functions.

In recent years a special class of Möbius invariant function spaces in the unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$, the so-called holomorphic $\mathbb{Q}_k$ space, has attracted much attention. See [23, 24] for a summary of recent research about $\mathbb{Q}_k$ spaces in the unit disk $\mathbb{D}$. Recall that for $0 < k < \infty$, a holomorphic function f is said to belong to the $\mathbb{Q}_k$ space if

$$\|f\|_{\mathbb{Q}_k} = \sup_{a\in\mathbb{D}} \int_{\mathbb{B}} |f'(z)|^2 (1 - |\varphi_a(z)|^2)^k dA(z) < \infty.$$

It is well-known that $\mathbb{Q}_k = \mathcal{B}$, the holomorphic Bloch space if $k > 1$ and $\mathbb{Q}_k =$ BMOA if $k = 1$.

In our final results, we focus on the borderline case $p = n$. It is known that n-harmonic functions are Möbius invariant, and thus we are able to generalize some properties of holomorphic $\mathbb{Q}_k$ spaces to the n-harmonic setting.

Definition 1.2. For $0 < k < \infty$, the Q_k space consists of all $u \in h_n(\mathbb{B})$ such that

$$\|u\|_{Q_k} = \sup_{a\in\mathbb{B}} \int_{\mathbb{B}} |\nabla u(x)|^n (1 - |\varphi_a(x)|^2)^k dv(x) < \infty,$$

where φ_a is the Möbius transformation on the real unit ball $\mathbb{B}$ that interchanges the points 0 and a (see the definition in Section 2).

In [13], Latvala characterized n-harmonic Q_k and $BMO(\mathbb{B})$ spaces by means of certain Möbius invariant weighted Dirichlet integrals. Motivated by the results in [13, 22], we show a derivative-free characterization of Q_k as follows.

Theorem 1.3. *Let* $0 < k < n$ *and* $u \in h_p(\mathbb{B})$*. Then* $u \in Q_k$ *if and only if*

$$\sup_{a\in\mathbb{B}} \int_{\mathbb{B}} \int_{\mathbb{B}} \frac{|u(x) - u(y)|^n}{[x, y]^{2n}} (1 - |\varphi_a(x)|^2)^k dv(x) dv(y) < \infty.$$

For $0 < r < 1$ and $u \in h_n(\mathbb{B})$, we define the oscillation of u at x in the pesudo-hyperbolic metric as $o_r(u)(x)$ which is given by

$$o_r(u)(x) = \sup_{y\in E(x,r)} |u(x) - u(y)|.$$

Similarly, define another oscillation of u at x as

$$\widehat{o}_r(u)(x) = \sup_{y\in E(x,r)} |\widehat{u}_r(x) - u(y)|,$$

where

$$\widehat{u}_r(x) = \frac{1}{|E(x,r)|} \int_{E(x,r)} u(y)dv(y).$$

Theorem 1.4. *Let $0 < r < 1$ and $u \in h_n(\mathbb{B})$. Then the following statements are equivalent:*

(a) $u \in Q_k$;

(b) $\sup_{a\in\mathbb{B}} \int_{\mathbb{B}} |o_r(u)(x)|^n (1-|\varphi_a(x)|^2)^k d\tau(x) < \infty,$

(c) $\sup_{a\in\mathbb{B}} \int_{\mathbb{B}} |\widehat{o}_r(u)(x)|^n (1-|\varphi_a(x)|^2)^k d\tau(x) < \infty,$

where $d\tau(x) = (1-|x|^2)^{-n}dv(x)$ is the invariant measure on $\mathbb{B}$.

The rest of this paper is organized as follows. In Section 2, some necessary terminology and notation will be introduced. In Section 3, we shall prove Theorem 1.1. The proof of Theorem 1.2 will be presented in Section 4 by applying Theorem 1.1. The final Section 5 is devoted to the proofs of Theorems 1.3 and 1.4. Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. For nonnegative quantities X and Y, $X \lesssim Y$ means that X is dominated by Y times some inessential positive constant. We write $X \approx Y$ if $Y \lesssim X \lesssim Y$.

2. Preliminaries

In this section, we introduce notation and collect some preliminary results that involve Möbius transformations and p-harmonic functions.

Let $a \in \mathbb{R}^n$, we write a in polar coordinate by $a = |a|a'$. For $a, b \in \mathbb{R}^n$, let

$$[a,b] = \Big| |a|b - a' \Big|.$$

The symmetric lemma shows

$$[a,b] = [b,a].$$

For any $a \in \mathbb{B}$, denote by φ_a the Möbius transformation in $\mathbb{B}$. It's an involution of $\mathbb{B}$ such that $\varphi_a(0) = a$ and $\varphi_a(a) = 0$, which is of the form

$$\varphi_a(x) = \frac{|x-a|^2 a - (1-|a|^2)(x-a)}{[x,a]^2}, x \in \mathbb{B}.$$

An elementary computation gives

$$|\varphi_a(x)| = \frac{|x-a|}{[x,a]}.$$

In terms of φ_a, the *pseudo-hyperbolic metric* ρ is given by

$$\rho(a,b) = |\varphi_a(b)|, \quad a, b \in \mathbb{B}.$$

The *pseudo-hyperbolic ball* with center a and radius r is denoted by

$$E(a,r) = \{x \in \mathbb{B} : \rho(a,x) < r\}.$$

However, $E(a,r)$ is also a Euclidean ball with center c_a and radius r_a given by

$$c_a = \frac{(1-r^2)a}{1-|a|^2r^2} \quad \text{and} \quad r_a = \frac{r(1-|a|^2)}{1-|a|^2r^2}, \tag{2.1}$$

respectively (cf. [1, 17]).

Following [5], we define a positive value function ϱ in $\mathbb{B}$ as

$$\varrho(a,b) = \frac{|a-b|}{[a,b]^2}, \quad a, b \in \mathbb{B}.$$

The ball $B_r(a)$ associated with ϱ is given by

$$B_r(a) = \{x \in \mathbb{B} : \varrho(a,x) < r\}.$$

Obviously, one see that $\varrho(a,b) < r$ implies $\rho(a,b) < 2r$ for a small positive r.

Lemma 2.1. *Let r be a small positive number and $x \in B_r(a)$ (resp. $E(a,r)$). Then*

$$1-|x|^2 \approx 1-|a|^2 \approx [a,x], \qquad d_\lambda(a,x) \approx \varrho(a,x)$$

and

$$|B_r(a)| \approx (1-|a|^2)^{2n}, \qquad (\textit{resp. } |E(a,r)| \approx (1-|a|^2)^n)$$

where $|B_r(a)|$ and $|E(a,r)|$ denote the Euclidean volume of $B_r(a)$ and $E(a,r)$, respectively.

Proof. It is obvious from [17, Lemma 2.1].

By Lemma 2.1, the following comparable results can be easily derived.

Lemma 2.2. *For a small $r > 0$, there exist two positive constants r_1, r_2 such that*

$$\mathbb{B}(a, r_1(1-|a|^2)^2) \subseteq B_r(a) \subseteq \mathbb{B}(a, r_2(1-|a|^2)^2), \quad a \in \mathbb{B}.$$

Let $u \in h_p(\mathbb{B})$, for convenience, we denote

$$⨍_{\mathbb{B}(x,r)} u(y)dv(y) = \frac{1}{|\mathbb{B}(x,r)|}\int_{\mathbb{B}(x,r)} u(y)dv(y).$$

We end this section with some useful inequalities concerning p-harmonic functions which are crucial for our investigations (cf. [11]).

Lemma 2.3. *Assume that $u \in h_p(\mathbb{B})$. Then we have the following inequalities.*
(1) *For each $\delta > 1$, there is a positive constant C such that*

$$\int_{\mathbb{B}(x,r)} |\nabla u(y)|^p dv(y) \leq \frac{C}{r^p} \int_{\mathbb{B}(x,\delta r)} |u(y)|^p dv(y),$$

whenever $\mathbb{B}(x, \delta r) \subset \mathbb{B}$.
(2) *For each $\delta > 1$ and $0 < s \leq t$, there is a positive constant C such that*

$$|u(x)| \leq C\Big(⨍_{\mathbb{B}(x,r)} |u(y)|^t dv(y)\Big)^{\frac{1}{t}} \leq C\Big(⨍_{\mathbb{B}(x,\delta r)} |u(y)|^s dv(y)\Big)^{\frac{1}{s}},$$

whenever $\mathbb{B}(x, \delta r) \subset \mathbb{B}$.
(3) *For each $\delta > 1$ and $0 < s \leq t$, there is a positive constant C such that*

$$|\nabla u(x)| \leq C\Big(⨍_{\mathbb{B}(x,r)} |\nabla u(y)|^t dv(y)\Big)^{\frac{1}{t}} \leq C\Big(⨍_{\mathbb{B}(x,\delta r)} |\nabla u(y)|^s dv(y)\Big)^{\frac{1}{s}},$$

whenever $\mathbb{B}(x, \delta r) \subset \mathbb{B}$.
(4) *For each $t > 0$ and $\delta > 1$, there is a positive constant C such that*

$$osc_{x \in \mathbb{B}(y,r)} u(x) \leq C\Big(⨍_{\mathbb{B}(y,\delta r)} |\nabla u(y)|^t dv(y)\Big)^{\frac{1}{t}},$$

whenever $\mathbb{B}(y, \delta r) \subset \mathbb{B}$.

3. Proof of Theorem 1.1

Proposition 3.1. *Let $1 < s < \infty$, $\alpha \in \mathbb{R}$ and $\beta > 0$, then*

$$|u(0)|^s + \int_{\mathbb{B}} (1 - |x|)^s |\nabla u(x)|^s dv_{\alpha,\beta}(x) \lesssim \int_{\mathbb{B}} |u(x)|^s dv_{\alpha,\beta}(x) \tag{3.1}$$

for all $u \in h_p(\mathbb{B})$.

Proof. By Lemma 2.3, we have

$$|u(0)| \leq C\Big(\int_{\mathbb{B}_{\frac{1}{2}}} |u(x)|^s dv_{\alpha,\beta}(x)\Big)^{\frac{1}{s}} \lesssim \Big(\int_{\mathbb{B}} |u(x)|^s dv_{\alpha,\beta}(x)\Big)^{\frac{1}{s}}.$$

Hence it is sufficient to prove without the term $|u(0)|^s$. It follows from Lemma 2.3 again that for each fixed $x \in \mathbb{B}$,

$$\begin{aligned} |\nabla u(x)| &\leq C\Big(⨍_{\mathbb{B}(x,\frac{(1-|x|)}{4})} |\nabla u(y)|^p dv(y)\Big)^{\frac{1}{p}} \\ &\lesssim \Big((1-|x|)^{-p} ⨍_{\mathbb{B}(x,\frac{(1-|x|)}{3})} |u(y)|^p v(y)\Big)^{\frac{1}{p}} \\ &\lesssim (1-|x|)^{-1} \Big(⨍_{\mathbb{B}(x,\frac{(1-|x|)}{2})} |u(y)|^s v(y)\Big)^{\frac{1}{s}}. \end{aligned}$$

Combing this with Lemma 2.1 and Fubini's theorem, we conclude that

$$\begin{aligned}\int_{\mathbb{B}}|\nabla u(x)|^s(1-|x|)^s dv_{\alpha,\beta}(x) &\lesssim \int_{\mathbb{B}}\fint_{\mathbb{B}(x,\frac{(1-|x|)}{2})}|u(y)|^s dv(y)dv_{\alpha,\beta}(x)\\ &\lesssim \int_{\mathbb{B}}\fint_{\mathbb{B}(x,\frac{(1-|x|)}{2})}|u(y)|^s dv_{\alpha,\beta}(y)dv(x)\\ &\lesssim \int_{\mathbb{B}}|u(y)|^s\fint_{\mathbb{B}(y,\frac{(1-|y|)}{2})}dv(x)dv_{\alpha,\beta}(y)\\ &\lesssim \int_{\mathbb{B}}|u(y)|^s dv_{\alpha,\beta}(y).\end{aligned}$$

This proves the result.

Proposition 3.2. *Let $1<s<\infty$, $\alpha\in\mathbb{R}$ and $\beta\geq s-1$, then*

$$\int_{\mathbb{B}}|u(x)|^s dv_{\alpha,\beta}(x)\lesssim |u(0)|^s+\int_{\mathbb{B}}|\nabla u(x)|^s(1-|x|)^s dv_{\alpha,\beta}(x) \tag{3.2}$$

for all $u\in h_p(\mathbb{B})$.

Proof. Assume that $u(0)=0$. We divide the integral on the left-hand side of (3.2) into two parts:

$$\int_{\mathbb{B}}|u(x)|^s dv_{\alpha,\beta}(x)=\int_{\mathbb{B}_{\frac{1}{3}}}+\int_{\mathbb{B}\setminus\mathbb{B}_{\frac{1}{3}}}.$$

It is easy to see that the integral over $\mathbb{B}_{\frac{1}{3}}$ is dominated by

$$\begin{aligned}\int_{\mathbb{B}_{\frac{1}{3}}}|u(x)|^s dv_{\alpha,\beta}(x) &\lesssim \left(osc_{x\in\mathbb{B}_{\frac{1}{3}}}u(x)\right)^s\\ &\lesssim \int_{\mathbb{B}_{\frac{1}{2}}}|\nabla u(x)|^s(1-|x|)^s dv_{\alpha,\beta}(x)\\ &\lesssim \int_{\mathbb{B}}|\nabla u(x)|^s(1-|x|)^s dv_{\alpha,\beta}(x).\end{aligned}$$

We now estimate the integral over $\mathbb{B}\setminus\mathbb{B}_{\frac{1}{3}}$. Since u is $C^1(\mathbb{B})$, for $\zeta\in\mathbb{S}$, we have

$$|u(r\zeta)-u(\frac{1}{3}\zeta)| \lesssim C\int_{\frac{1}{3}}^{r}|\nabla u(t\zeta)|dt.$$

Thus

$$\begin{aligned}\int_{\mathbb{B}\setminus\mathbb{B}_{\frac{1}{3}}}|u(x)|^s dv_{\alpha,\beta}(x) &= \int_{\mathbb{S}}\int_{\frac{1}{3}}^{1}nr^{n-1}|u(r\zeta)|^s\omega_{\alpha,\beta}(r)drd\sigma(\zeta)\\ &\lesssim \int_{\mathbb{S}}\int_{\frac{1}{3}}^{1}r^{n-1}\Big(|u(r\zeta)-u(\frac{1}{3}\zeta)|^s+|u(\frac{1}{3}\zeta)|^s\Big)\omega_{\alpha,\beta}(r)drd\sigma(\zeta).\end{aligned}$$

Note that the integral

$$\int_{\mathbb{S}}\int_{\frac{1}{3}}^{1}r^{n-1}|u(\frac{1}{3}\zeta)|^s\omega_{\alpha,\beta}(r)drd\sigma(\zeta)\lesssim\int_{\mathbb{B}}|\nabla u(x)|^s(1-|x|)^s dv_{\alpha,\beta}(x)$$

by the same reasoning as the above integral estimate over $\mathbb{B}_{\frac{1}{3}}$. It follows from Lemma 2.3 and Hölder's inequality that

$$\begin{aligned} I &= \int_{\mathbb{S}}\int_{\frac{1}{3}}^{1} r^{n-1}|u(r\zeta)-u(\frac{1}{3}\zeta)|^s\omega_{\alpha,\beta}(r)drd\sigma(\zeta) \\ &= \int_{\mathbb{S}}\int_{\frac{1}{3}}^{1} r^{n-1}\Big(\int_{\frac{1}{3}}^{r}|\nabla u(t\zeta)|dt\Big)^s\omega_{\alpha,\beta}(r)drd\sigma(\zeta) \\ &\lesssim \int_{\mathbb{S}}\int_{\frac{1}{3}}^{1} \Big(\int_{0}^{r}t^{(n-1)/s}|\nabla u(t\zeta)|dt\Big)^s\omega_{\alpha,\beta}(r)drd\sigma(\zeta) \\ &\lesssim \int_{\mathbb{S}}\int_{0}^{1}\int_{0}^{r} t^{n-1}|\nabla u(t\zeta)|^s dt\omega_{\alpha,\beta}(r)drd\sigma(\zeta) \\ &\lesssim \int_{\mathbb{S}}\int_{0}^{1} t^{n-1}|\nabla u(t\zeta)|^s dt\int_{t}^{r}\omega_{\alpha,\beta}(r)drd\sigma(\zeta). \end{aligned}$$

Observe that

$$\int_{s}^{1}\omega_{\alpha,\beta}(r)dr \lesssim (1-s)^{\beta+1}\omega_{\alpha,\beta}(s), \quad 0<s<1$$

from [18, Example 3.2], we obtain

$$\begin{aligned} I &\lesssim \int_{\mathbb{S}}\int_{0}^{1} t^{n-1}|\nabla u(t\zeta)|^s\omega_{\alpha,\beta}(t)(1-|t|)^s dt(r)d\sigma(\zeta) \\ &\lesssim \int_{\mathbb{B}}|\nabla u(x)|^s(1-|x|)^s dv_{\alpha,\beta}(x) \end{aligned}$$

from the assumption $\beta \geq s-1$.

To remove the restriction $u(0)=0$, let $u(x)=u(0)+u_1(x)$ with $\nabla u=\nabla u_1$ and $u_1(0)=0$. Therefore,

$$\begin{aligned} \int_{\mathbb{B}}|u(x)|^s dv_{\alpha,\beta}(x) &= \int_{\mathbb{B}}|u(0)+u_1(x)|^s dv_{\alpha,\beta}(x) \\ &\lesssim |u(0)|^s+\int_{\mathbb{B}}|u_1(x)|^s dv_{\alpha,\beta}(x) \\ &\lesssim |u(0)|^s+\int_{\mathbb{B}}(1-|x|)^s|\nabla u(x)|^s dv_{\alpha,\beta}(x) \end{aligned}$$

as desired. □

Proof of Theorem 1.1. Gathering Propositions 3.1 and 3.2, the assertion (1.3) follows. By a slight modification on the proof of Proposition 3.2, we can also obtain the following corollary which can view as an extension of [5, Proposition 2.10] into p-harmonic setting.

Corollary 3.1. *Let $1<s<\infty$, $\alpha\in\mathbb{R}$ and $\beta\geq 2s-1$, then*

$$\int_{\mathbb{B}}|u(x)|^s dv_{\alpha,\beta}(x) \approx |u(0)|^s+\int_{\mathbb{B}}|\nabla u(x)|^s(1-|x|)^{2s}dv_{\alpha,\beta}(x) \tag{3.3}$$

for all $u\in h_p(\mathbb{B})$.

4. Lipschitz type characterizations for $\mathcal{A}^s_{\alpha,\beta}(\mathbb{B})$

In this section, we discuss Lipschitz type characterizations of the space $\mathcal{A}^s_{\alpha,\beta}(\mathbb{B})$ by applying Corollary 3.1.

Proof of Theorem 1.2. We first prove $(b) \Rightarrow (a)$. Assume that (b) holds. Then for each fixed x and all y sufficiently close to x

$$\Big|\frac{u(x)-u(y)}{x-y}\Big| \leq \frac{1}{[x,y]^2}\big(g(x)+g(y)\big), \quad x \neq y.$$

By letting y approach x in the direction of each real coordinate axis, we see that

$$(1-|x|)^2|\nabla u(x)| \leq Cg(x)$$

for all $x \in \mathbb{B}$. It follows from the assumption $g \in L^s(\mathbb{B}, dv_{\alpha,\beta})$ that

$$\int_{\mathbb{B}}(1-|x|)^{2s}|\nabla u(x)|^s dv_{\alpha,\beta}(x) < \infty.$$

Thus $u \in \mathcal{A}^s_{\alpha,\beta}(\mathbb{B})$ by Corollary 3.1.

For the converse, we assume $u \in \mathcal{A}^s_{\alpha,\beta}(\mathbb{B})$. Fix a small $r > 0$ and consider any two points $x, y \in \mathbb{B}$ with $\varrho(x,y) < r$. By Lemma 2.1, it is given that

$$\begin{aligned}|u(x)-u(y)| &= \Big|\int_0^1 \frac{du}{dt}(ty+(1-t)x)dt\Big| \\ &\leq C|x-y|\int_0^1 |\nabla u(ty+(1-t)x)|dt \\ &\leq C\varrho(x,y)\sup\{(1-|\zeta|)^2|\nabla u(\zeta)| : \zeta \in B_r(x)\} \\ &\leq \varrho(x,y)h(x),\end{aligned}$$

where

$$h(x) = C(r)\sup\{(1-|\zeta|)^2|\nabla u(\zeta)| : \zeta \in B_r(x)\}.$$

If $\varrho(x,y) \geq r$, the triangle inequality implies

$$\begin{aligned}|u(x)-u(y)| &\leq |u(x)|+|u(y)| \\ &\leq \varrho(x,y)\Big(\frac{|u(x)|}{r}+\frac{|u(y)|}{r}\Big).\end{aligned}$$

Letting $g(x) = h(x) + \frac{|u(x)|}{r}$, then

$$|u(x)-u(y)| \leq \varrho(x,y)\big(g(x)+g(y)\big)$$

for all $x, y \in \mathbb{B}$. Note that $g(x) = h(x) + \frac{|u(x)|}{r}$ is the desired function provided that $h \in L^s(\mathbb{B}, dv_{\alpha,\beta})$.

Since r is a small positive number, by Lemma 2.2, we see that $B_r(\zeta) \subset \mathbb{B}(x, \frac{(1-|x|)^2}{4})$ for every $\zeta \in B_r(x)$. It follows from Lemma 2.3 that

$$\begin{aligned}\sup_{\zeta \in B_r(x)} |\nabla u(\zeta)| &\leq C\Big(\fint_{\mathbb{B}(x, \frac{(1-|x|)^2}{4})} |\nabla u(y)|^p dv(y)\Big)^{\frac{1}{p}} \\ &\lesssim \Big((1-|x|)^{-2p} \fint_{\mathbb{B}(x, \frac{(1-|x|)^2}{3})} |u(y)|^p v(y)\Big)^{\frac{1}{p}} \\ &\lesssim (1-|x|)^{-2}\Big(\fint_{\mathbb{B}(x, \frac{(1-|x|)^2}{2})} |u(y)|^s v(y)\Big)^{\frac{1}{s}}.\end{aligned}$$

Hence by Fubini's theorem and Lemma 2.1,

$$\begin{aligned}\|h\|^s_{\mathcal{A}^s_{\alpha,\beta}} &\lesssim \int_{\mathbb{B}} (1-|x|)^{-2n} \omega_{\alpha,\beta}(x) \int_{\mathbb{B}(x, \frac{(1-|x|)^2}{2})} |u(y)|^s dv(y) dv(x) \\ &\lesssim \int_{\mathbb{B}} |u(y)|^s \omega_{\alpha,\beta}(y) \int_{\mathbb{B}(y, \frac{(1-|y|)^2}{2})} (1-|x|)^{-2n} dv(y) dv(x) \lesssim \|u\|^s_{\mathcal{A}^s_{\alpha,\beta}},\end{aligned}$$

which implies $h \in L^s(\mathbb{B}, dv_{\alpha,\beta})$. This proves $(a) \Leftrightarrow (b)$.

$(a) \Leftrightarrow (c)$. It follows from Lemmas 2.1, 2.2 and a discussion similar to the above, the assertion follows.

$(a) \Leftrightarrow (d)$. Assume that (d) holds. Then it can be deduced that

$$(1-|x|)^2 |\nabla u(x)| \leq C(1-|x|)^2 h(x)$$

for all $x \in \mathbb{B}$. The assumption $h \in L^s(\mathbb{B}, dv_{\alpha+2s,\beta})$ implies $(1-|x|)|^2 \nabla u(x)| \in L^s(\mathbb{B}, dv_{\alpha,\beta})$ and thus, according to Corollary 3.1, means that $u \in \mathcal{A}^s_{\alpha,\beta}(\mathbb{B})$.

Conversely, suppose that $u \in \mathcal{A}^s_{\alpha,\beta}(\mathbb{B})$. Then (b) implies that there exists a positive continuous function $g \in L^s(\mathbb{B}, dv_{\alpha,\beta})$ such that

$$|u(x) - u(y)| \leq C \frac{|x-y|}{[x,y]^2} \big(g(x) + g(y)\big)$$

for all $x, y \in \mathbb{B}$. Since for $x, y \in \mathbb{B}$,

$$[x,y] \geq 1 - |x|, \quad [x,y] \geq 1 - |y|,$$

we see that

$$\begin{aligned}|u(x) - u(y)| &\leq C|x-y| \Big(\frac{g(x)}{(1-|x|)^2} + \frac{g(y)}{(1-|y|)^2}\Big) \\ &\leq |x-y| \big(h(x) + h(y)\big), \quad x, y \in \mathbb{B},\end{aligned}$$

where

$$h(x) = \frac{Cg(x)}{(1-|x|)^2}.$$

Hence $h \in L^s(\mathbb{B}, dv_{\alpha+2s,\beta})$ from the assumption $g \in L^s(\mathbb{B}, dv_{\alpha,\beta})$. □

In the following, we consider a symmetric lifting operator L which is defined as

$$Lu(x,y)=\frac{u(x)-u(y)}{x-y},\quad x\neq y$$

where $u\in h_p(\mathbb{B})$.

As an application of Theorem 1.2, we can obtain the boundedness of operator L as follows.

Theorem 4.1. *Let $1<s<\infty$, $\alpha\in\mathbb{R},\beta\geq 2s-1$. Then $L:\mathcal{A}^s_{\alpha,\beta}(\mathbb{B})\to L^s(\mathbb{B}\times\mathbb{B},dv_{\alpha+s,\beta}\times dv_{\alpha+s,\beta})\cap h_p(\mathbb{B}\times\mathbb{B})$ is bounded.*

Proof. Let $u\in\mathcal{A}^s_{\alpha,\beta}(\mathbb{B})$. Then there exists a positive continuous function $g\in L^s(\mathbb{B},dv_{\alpha,\beta})$ such that

$$|Lu(x,y)|^s=\Big|\frac{u(x)-u(y)}{x-y}\Big|^s\lesssim\frac{|g(x)|^s+|g(y)|^s}{[x,y]^{2s}},\quad x\neq y,$$

by Theorem 1.2. Applying Fubini's Theorem, we obtain

$$\begin{aligned}&\int_{\mathbb{B}}\int_{\mathbb{B}}|Lu(x,y)|^s dv_{\alpha+s,\beta}(x)dv_{\alpha+s,\beta}(y)\\ \leq\;& 2C\int_{\mathbb{B}}\int_{\mathbb{B}}\frac{|g(x)|^s}{[x,y]^{2s}}dv_{\alpha+s,\beta}(x)dv_{\alpha+s,\beta}(y)\\ \lesssim\;&\int_{\mathbb{B}}\int_{\mathbb{B}}\frac{|g(x)|^s}{(1-|x|)^s(1-|y|)^s}dv_{\alpha+s,\beta}(x)dv_{\alpha+s,\beta}(y)\\ \lesssim\;&\int_{\mathbb{B}}|g(x)|^s dv_{\alpha,\beta}(x)<\infty.\end{aligned}$$

Consequently, $L:\mathcal{A}^s_{\alpha,\beta}(\mathbb{B})\to L^s(\mathbb{B}\times\mathbb{B},dv_{\alpha+s,\beta}\times dv_{\alpha+s,\beta})\cap h_p(\mathbb{B}\times\mathbb{B})$ is bounded. □

5. Characterizations of Q_k spaces

In this section, we discuss some derivative-free characterizations for Q_k spaces of n-harmonic functions on the real unit ball $\mathbb{B}$ in $\mathbb{R}^n$.

Lemma 5.1. *Let $0<k<\infty$ and $u\in h_n(\mathbb{B})$. Then there exists a constant $C>0$ such that*

$$\int_{\mathbb{B}}|\nabla u(x)|^n(1-|x|^2)^k dv(x)\leq C\int_{\mathbb{B}}\int_{\mathbb{B}}\frac{|u(x)-u(y)|^n}{[x,y]^{2n}}(1-|x|^2)^k dv(x)dv(y).$$

Proof. Write

$$K=\int_{\mathbb{B}}\int_{\mathbb{B}}\frac{|u(x)-u(y)|^n}{[x,y]^{2n}}(1-|x|^2)^k dv(x)dv(y).$$

Making the change of variables $y \mapsto \varphi_x(y)$ leads to

$$\begin{aligned} K &= \int_{\mathbb{B}}\int_{\mathbb{B}} \frac{|u(x) - u\circ\varphi_x(y)|^n}{[x, \varphi_x(y)]^{2n}}(1-|x|^2)^k J\varphi_x(y)dv(x)dv(y) \\ &= \int_{\mathbb{B}}\int_{\mathbb{B}} |u\circ\varphi_x(0) - u\circ\varphi_x(y)|^n (1-|x|^2)^{k-n} dv(x)dv(y) \\ &= \int_{\mathbb{B}} (1-|x|^2)^{k-n} dv(x) \int_{\mathbb{B}} |u\circ\varphi_x(0) - u\circ\varphi_x(y)|^n dv(y). \end{aligned}$$

Note that $u\circ\varphi_x \in h_n(\mathbb{B})$, it follows from (1.2) that

$$\int_{\mathbb{B}} |u\circ\varphi_x(0) - u\circ\varphi_x(y)|^n dv(y) \approx \int_{\mathbb{B}} |\nabla(u\circ\varphi_x)(y)|^n (1-|y|^2)^n dv(y).$$

It deduces from [13, Lemma 4.4] that

$$\begin{aligned} K &\approx \int_{\mathbb{B}} (1-|x|^2)^{k-n} dv(x) \int_{\mathbb{B}} |\nabla(u\circ\varphi_x)(y)|^n (1-|y|^2)^n dv(y) \\ &\approx \int_{\mathbb{B}} (1-|x|^2)^{k-n} dv(x) \int_{\mathbb{B}} |\nabla u(y)|^n (1-|\varphi_x(y)|^2)^n dv(y) \\ &\geq C\int_{\mathbb{B}} (1-|x|^2)^{k-n} dv(x) \int_{E(x,\frac{1}{2})} |\nabla u(y)|^n (1-|\varphi_x(y)|^2)^n dv(y) \\ &\geq C\int_{\mathbb{B}} (1-|x|^2)^{k} dv(x) ⨍_{E(x,\frac{1}{2})} |\nabla u(y)|^n dv(y) \\ &\geq C\int_{\mathbb{B}} |\nabla u(x)|^n (1-|x|^2)^k dv(x). \end{aligned}$$

□

Lemma 5.2. *Let $0 < k < n$ and $u \in h_n(\mathbb{B})$. Then there exists a constant $C > 0$ such that*

$$K = \int_{\mathbb{B}}\int_{\mathbb{B}} \frac{|u(x) - u(y)|^n}{[x, y]^{2n}}(1-|x|^2)^k dv(x)dv(y) \leq C\int_{\mathbb{B}} |\nabla u(x)|^n (1-|x|^2)^k dv(x).$$

Proof. From the proof of Lemma 5.1, we see that

$$K \approx \int_{\mathbb{B}} |\nabla u(y)|^n dv(y) \int_{\mathbb{B}} (1-|\varphi_x(y)|^2)^n (1-|x|^2)^{k-n} dv(x)$$

It follows from the assumption $0 < k < n$ and [17, Lemma 2.4] that

$$\begin{aligned} \int_{\mathbb{B}} (1-|\varphi_x(y)|^2)^n (1-|x|^2)^{k-n} dv(x) &= \int_{\mathbb{B}} \frac{(1-|x|^2)^k (1-|y|^2)^n}{[x,y]^{2n}} dv(x) \\ &\lesssim (1-|y|^2)^k, \end{aligned}$$

as desired. □

Proof of Theorem 1.3. By [13, Lemmas 2.3 and 4.4], we know that $u \in Q_k$ if and only if

$$\sup_{a\in\mathbb{B}} \int_{\mathbb{B}} |\nabla(u\circ\varphi_a)(x)|^n (1-|x|^2)^k dv(x) < \infty.$$

This together with Lemmas 5.1 and 5.2, the assertion follows.

Proof of Theorem 1.4. The proof will follow by the routes $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$.

$(a) \Rightarrow (b)$. Let $u \in Q_k$. By Lemma 2.3, for $0 < r < 1$ and a fixed $x \in \mathbb{B}$,

$$|o_r(u)(x)|^n \lesssim \frac{1}{|E(x,r')|} \int_{E(x,r')} |u(x) - u(y)|^n dv(y),$$

where $r < r' < 1$. From Lemmas 2.1 and 2.3, we have

$$\begin{aligned} \frac{1}{|E(x,r')|} & \int_{E(x,r')} |u(x) - u(y)|^n dv(y) \\ & \lesssim \int_{E(x,r')} |u(x) - u(y)|^n \frac{(1-|x|^2)^n}{[x,y]^{2n}} dv(y) \\ & = \int_{\mathbb{B}(0,r')} |u \circ \varphi_x(0) - u \circ \varphi_x(y)|^n dv(y) \\ & \lesssim \int_{\mathbb{B}(0,r')} |\nabla(u \circ \varphi_x)(y)|^n (1-|y|^2)^n dv(y). \end{aligned}$$

By making the change of variables and [13, Lemma 4.3],

$$|o_r(u)(x)|^n \lesssim \int_{E(x,r')} |\nabla u(y)|^n dv(y),$$

from which we see that

$$\begin{aligned} & \int_{\mathbb{B}} |o_r(u)|^n (1 - |\varphi_a(x)|^2)^k d\tau(x) \\ \lesssim & \int_{\mathbb{B}} (1 - |\varphi_a(x)|^2)^k d\tau(x) \int_{E(x,r')} |\nabla u(y)|^n dv(y) \\ \lesssim & \int_{\mathbb{B}} |\nabla u(x)|^n (1 - |\varphi_a(x)|^2)^k dv(x), \end{aligned}$$

for each $a \in \mathbb{B}$. Hence (a) implies (b).

$(b) \Rightarrow (c)$. By Lemma 2.3, for $0 < r < 1$,

$$\begin{aligned} \sup_{y \in E(x,r)} |\widehat{u}_r(x) - u(y)| & \lesssim \sup_{y \in E(x,r)} \frac{1}{|E(x,r)|} \int_{E(x,r)} |u(y) - u(z)| dv(z) \\ & \lesssim \sup_{y \in E(x,r)} \sup_{z \in E(x,r)} |u(y) - u(z)| \\ & \lesssim \sup_{y \in E(x,r)} |u(x) - u(y)|. \end{aligned}$$

Thus

$$\widehat{o}_r(u)(x) \lesssim o_r(u)(x),$$

from which $(b) \Rightarrow (c)$ follows.

$(c) \Rightarrow (a)$. For $0 < r < 1$ and $x \in \mathbb{B}$, we have

$$\begin{aligned}(1-|x|^2)^n|\nabla u(x)|^n &\lesssim \frac{1}{|E(x,r)|}\int_{E(x,r)}|u(y)-\widehat{u}_r(x)|^n dv(y)\\ &\lesssim \Big(\sup_{y\in E(x,r)}|\widehat{u}_r(x)-u(y)|\Big)^n\end{aligned}$$

by Lemma 2.3. Consequently,

$$\sup_{a\in\mathbb{B}}\int_{\mathbb{B}}|\nabla u(x)|^n(1-|\varphi_a(x)|^2)^k dv(x) \lesssim \sup_{a\in\mathbb{B}}\int_{\mathbb{B}}|\widehat{o}_r(u)(x)|^n(1-|\varphi_a(x)|^2)^k d\tau(x).$$

The proof of this theorem is complete.

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