A NEW PROOF OF THE GASCA - MAEZTU CONJECTURE FOR $n = 5$

G. VARDANYAN

Yerevan State University
E-mail: *gagik.vardanyan2000@gmail.com*

Abstract. An n-correct node set $\mathcal{X}$ is called GC_n set if the fundamental polynomial of each node is a product of n linear factors. In 1982 Gasca and Maeztu conjectured that for every GC_n set there is a line passing through $n + 1$ of its nodes. So far, this conjecture has been confirmed only for $n \leq 5$. The case $n = 4$, was first proved by J. R. Busch [3]. Several other proofs have been published since then. For the case $n = 5$ there is only one proof by H. Hakopian, K. Jetter and G. Zimmermann (Numer Math 127:685–713, 2014). Here we give a second proof, which largely follows the first one but is much shorter and simpler.

MSC2020 number: 41A05; 41A63.

Keywords: Bivariate polynomial interpolation; the Gasca-Maeztu conjecture; n-correct set; GC_n set; maximal line.

1. INTRODUCTION

Denote by Π_n the space of bivariate polynomials of total degree at most n :

$$\Pi_n = \left\{ \sum_{i+j \leq n} a_{ij} x^i y^j \right\}, \quad N := \dim \Pi_n = \binom{n + 2}{2}.$$

Consider a set of distinct nodes $\mathcal{X}_s = \{(x_1, y_1), (x_2, y_2), \ldots, (x_s, y_s)\}$.

The problem of finding a polynomial $p \in \Pi_n$ which satisfies the conditions

$$(1.1) \qquad p(x_i, y_i) = c_i, \qquad i = 1, 2, \ldots s,$$

is called *interpolation problem.*

Definition 1.1. A set of nodes $\mathcal{X}_s$ is called *n-poised* if for any data $\{c_1, \ldots, c_s\}$ there exists a unique polynomial $p \in \Pi_n$, satisfying the conditions (1.1).

A necessary condition of n-poisedness is: $\#\mathcal{X}_s = s = N$. If this latter equality takes place then the following holds:

Proposition 1.1. *A set of nodes $\mathcal{X}_N$ is n-poised if and only if*

$$p \in \Pi_n, \ p(x_i, y_i) = 0 \quad i = 1, \ldots, N \implies p = 0.$$

A polynomial $p \in \Pi_n$ is called an *n-fundamental polynomial* for a node $A = (x_k, y_k) \in \mathcal{X}_s$ if

$$p(x_i, y_i) = \delta_{ik}, \ i = 1, \ldots, s,$$

where δ is the Kronecker symbol. We denote the *n*-fundamental polynomial of $A \in \mathcal{X}_s$ by $p_A^\star = p_{A,\mathcal{X}}^\star$.

Definition 1.2. A set of nodes $\mathcal{X}_s$ is called *n-independent* if each node has *n*-fundamental polynomial. Otherwise, $\mathcal{X}_s$ is called *n-dependent*. A set of nodes $\mathcal{X}_s$ is called *essentially n-dependent* if none of its nodes has *n*-fundamental polynomial.

Fundamental polynomials are linearly independent. Therefore a necessary condition of *n*-indepen-dence is $\#\mathcal{X}_s = s \leq N$.

One can readily verify that a node set $\mathcal{X}_s$ is *n*-independent if and only if the interpolation problem (1.1) is solvable, meaning that for any data $\{c_1, \ldots, c_s\}$ there exists a (not necessarily unique) polynomial $p \in \Pi_n$ satisfying the conditions (1.1).

A *plane algebraic curve* is the zero set of some bivariate polynomial of degree ≥ 1. To simplify notation, we shall use the same letter, say p, to denote the polynomial p and the curve given by the equation $p(x, y) = 0$. In particular, by ℓ, we denote a linear polynomial $\ell \in \Pi_1$ and the line defined by the equation $\ell(x, y) = 0$.

Definition 1.3. Let $\mathcal{X}$ be an *n*-poised set. We say, that a node $A \in \mathcal{X}$ *uses a line* ℓ, if ℓ is a factor of the fundamental polynomial $p_A^\star$, i.e.,

$$(1.2) \qquad\qquad\qquad\qquad p_A^\star = \ell q,$$

where $q \in \Pi_{n-1}$.

Since the fundamental polynomial of a node in an *n*-poised set is unique we get

Lemma 1.1 ([9], Lemma 2.5). *Suppose $\mathcal{X}$ is a poised set and a node $A \in \mathcal{X}$ uses a line ℓ. Then ℓ passes through at least two nodes from $\mathcal{X}$, at which q from (1.2) does not vanish.*

Definition 1.4. Let $\mathcal{X}$ be a set of nodes. We say, that a line ℓ is a *k-node line* if it passes through exactly k nodes of $\mathcal{X}$: $\ell \cap \mathcal{X} = k$.

The following proposition is well-known (see e.g. [8] Proposition 1.3):

Proposition 1.2. *Suppose that a polynomial $p \in \Pi_n$ vanishes at $n + 1$ points of a line ℓ. Then we have that $p = \ell r$, where $r \in \Pi_{n-1}$.*

From here we readily get that at most $n + 1$ nodes of an n-poised set $\mathcal{X}_N$ can be collinear. In view of this an $(n + 1)$-node line ℓ is called a *maximal line* [2].

Next, let us bring the Cayley-Bacharach theorem (see e.g. [6], Th. CB4; [8], Prop. 4.1).

Theorem 1.1. *Assume that two algebraic curves of degree m and n, respectively, intersect at mn distinct points. Then the set $\mathcal{X}$ of these intersection points is essentially $(m+n-3)$-dependent.*

We are going to consider a special type of n-poised sets defined by Chung and Yao:

Definition 1.5 ([5]). An n-poised set $\mathcal{X}$ is called GC_n *set*, if the n-fundamental polynomial of each node $A \in \mathcal{X}$ is a product of n linear factors.

Now we are in a position to present the Gasca-Maeztu conjecture.

Conjecture 1.1 ([7]). *For any GC_n set $\mathcal{X}$ there is a maximal line, i.e., a line passing through its $n + 1$ nodes.*

Since now the Gasca-Maeztu conjecture was proved to be true only for $n \leq 5$. The case $n = 2$ is trivial, and the case $n = 3$ is easy to verify. The case $n = 4$ first was proved by J. R. Busch [3]. Several other proofs have been published since then (see e.g. [4], [9], [1]). For the case $n = 5$ there is only one proof by H. Hakopian, K. Jetter and G. Zimmermann [10]. Here we give a second proof, which largely follows the first one but is much shorter and simpler.

1.1. **The m-distribution sequence of a node.** In this section we bring a number of concepts, properties and results from [10].

Suppose that $\mathcal{X}$ is a GC_n set. Consider a node $A \in \mathcal{X}$ together with the set of n used lines denoted by $\mathcal{L}_A$. The $N - 1$ nodes of $\mathcal{X} \setminus \{A\}$ belong to the lines of $\mathcal{L}_A$.

Let us order the lines of $\mathcal{L}_A$ in the following way:

The line ℓ_1 is a line in $\mathcal{L}_A$ that passes through maximal number of nodes of $\mathcal{X}$, denoted by $k_1 : \mathcal{X} \cap \ell_1 = k_1$.

The line ℓ_2 is a line in $\mathcal{L}_A \setminus \{\ell_1\}$ that passes through maximal number of nodes of $\mathcal{X} \setminus \ell_1$, denoted by $k_2 : (\mathcal{X} \setminus \ell_1) \cap \ell_2 = k_2$.

In the general case the line ℓ_s, $s = 1, \ldots, n$, is a line in $\mathcal{L}_A \setminus \{\ell_1, \ldots, \ell_{s-1}\}$ that passes through maximal number of nodes of the set $\mathcal{X} \setminus \cup_{i=1}^{s-1} \ell_i$, denoted by $k_s : (\mathcal{X} \setminus \cup_{i=1}^{s-1} \ell_i) \cap \ell_s = k_s$.

63

G. VARDANYAN

A correspondingly ordered line sequence

$$\mathcal{S} = (\ell_1, \ldots, \ell_n)$$

is called a *maximal line sequence* or briefly an *m-line sequence* if the respective sequence $(k_1, \ldots, k_n)$ is the maximal in the lexicographic order [10]. Then the latter sequence is called a *maximal distribution sequence* or briefly an *m-d sequence*.

Evidently, for the m-d sequence we have that

$$(1.3) \qquad k_1 \geq k_2 \geq \cdots \geq k_n \text{ and } k_1 + \cdots + k_n = N - 1.$$

Though the m-distribution sequence for a node A is unique, it may correspond to several m-line sequences.

Note that, an intersection point of several lines of $\mathcal{L}_A$ is counted for the line containing it which appears in $\mathcal{S}$ first. Each node in $\mathcal{X}$ is called a *primary* node for the line it is counted for, and a *secondary* node for the other lines containing it.

According to Lemma 1.1, every used line contains at least two primary nodes, i.e.,

$$(1.4) \qquad k_i \geq 2 \quad \text{for } i = 1, \ldots, n.$$

Let $\mathcal{S} = (\ell_1, \ldots, \ell_n)$ be an m-line sequence with the associated m-d sequence $(k_1, \ldots, k_n)$.

Lemma 1.2 ([10], Lemma 2.5). *Assume that $k_i = k_{i+1} =: k$ for some i. If the intersection point of lines ℓ_i and ℓ_{i+1} belongs to $\mathcal{X}$, then it is a secondary node for both ℓ_i and ℓ_{i+1}. Moreover, interchanging ℓ_i and ℓ_{i+1} in $\mathcal{S}$ still yields an m-line sequence.*

We say that a polynomial has $(s_i, \ldots, s_j)$ *primary zeroes* in the lines $(\ell_i, \ldots, \ell_j)$ if the zeroes are primary nodes in the respective lines. From Proposition 1.2 we get

Corollary 1.1. *If a polynomial $p \in \Pi_{m-1}$ has $(m, m-1, \ldots, m-k)$ primary zeroes in the lines $(\ell_{m-k}, \ell_{m-k+1} \ldots, \ell_m)$ then we have that $p = \ell_m \ell_{m-1} \cdots \ell_{m-k} r$, where $r \in \Pi_{m-k}$.*

In some cases a particular line $\widetilde{\ell}$ used by a node is fixed and then the properties of the other factors of the fundamental polynomial are studied.

In this case in the corresponding m-line sequence, called $\widetilde{\ell}$-*m-line sequence*, one takes as the first line ℓ_1 the line $\widetilde{\ell}$, no matter through how many nodes it passes. Then the second and subsequent lines are chosen, as in the case of the m-line sequence.

Thus the line ℓ_2 is a line in $\mathcal{L}_A \setminus \{\widetilde{\ell}_1\}$ that passes through maximal number of nodes of $\mathcal{X} \setminus \widetilde{\ell}_1$, and so on.

Correspondingly the $\widetilde{\ell}$-*m-distribution sequence* is defined.

2. The Gasca-Maeztu conjecture for $n = 5$

Let us formulate the Gasca-Maeztu conjecture for $n = 5$ as:

Theorem 2.1. *For any GC_5 set $\mathfrak{X}$ of 21 nodes there is a maximal line, i.e., a 6-node line.*

To prove the theorem assume by way of contradiction the following.

Assumption 2.1. The set $\mathfrak{X}$ is a GC_5 set with no maximal line.

In view of (1.3) and (1.4) the only possible m-d sequences for any node $A \in \mathfrak{X}$ are

$$(2.1) \qquad (5,5,5,3,2); \quad (5,5,4,4,2); \quad (5,5,4,3,3); \quad (5,4,4,4,3); \quad (4,4,4,4,4).$$

The results from [10] below show how many times a line can be used, depending the number of nodes it passes through. In each statement it is assumed that $\mathfrak{X}$ is a GC_5 set with no maximal line.

Proposition 2.1 ([10], Prop. 2.11). *Suppose that $\widetilde{\ell}$ is a 2-node line. Then $\widetilde{\ell}$ can be used by at most one node of $\mathfrak{X}$.*

Proposition 2.2 ([10], Prop. 2.12). *Suppose that $\widetilde{\ell}$ is a 3-node line and is used by two nodes $A, B \in \mathfrak{X}$. Then there exists a third node C using $\widetilde{\ell}$. Furthermore, A, B, and C share three other lines, each passing through five primary nodes. For each of the three nodes, the m-d sequence is $(5,5,5,3,2)$, and the other two nodes are the primary nodes in the respective fifth line. In particular, $\widetilde{\ell}$ is used exactly three times.*

Proposition 2.3 ([10], Prop. 2.13). *Suppose that a line $\widetilde{\ell}$ is used by three nodes A, B, $C \in \mathfrak{X}$. Then $\widetilde{\ell}$ passes through at least three nodes of $\mathfrak{X}$.*

If $\widetilde{\ell}$ is a 4-node line, then A, B, and C share $\widetilde{\ell}$ and three other lines, ℓ_2 and ℓ_3 passing through five and ℓ_4 through four primary nodes. For each of the three nodes, the $\widetilde{\ell}$-m-distribution sequence with respect to $\widetilde{\ell}$ is $(4,5,5,4,2)$. $\widetilde{\ell}$ can only be used by A, B, and C, i.e., it is used exactly three times.

Corollary 2.1 ([10], Cor. 2.14). *Suppose that a line $\widetilde{\ell}$ is used by four nodes in $\mathfrak{X}$. Then $\widetilde{\ell}$ is a 5-node line.*

Proposition 2.4 ([10], Prop. 2.15). *Suppose that a line $\widetilde{\ell}$ is used by five nodes in $\mathcal{X}$. Then $\widetilde{\ell}$ is a 5-node line, and it is actually used by exactly six nodes in $\mathcal{X}$. These six nodes form a GC_2 set and share two more lines with five primary nodes each, i.e., each of these six nodes has the m-d sequence $(5, 5, 5, 3, 2)$.*

At the end we bring a (part of a) table from [10] which follows from Propositions 2.1, 2.2, 2.3, 2.4 and Corollary 2.1. It shows under which conditions a k-node line $\widetilde{\ell}$, $2 \leq k \leq 5$, can be used at most how often, provided that the considered GC_5 set has no maximal line.

(2.2)

total # of nodes in $\widetilde{\ell}$	maximal # of nodes using $\widetilde{\ell}$	
	in general	no node uses $(5, 5, 5, 3, 2)$ m-d sequence
5	6	4
4	3	3
3	3	1
2	1	1

2.1. **The case** $(5, 5, 5, 3, 2)$. In this and the following sections, we will prove the following

Proposition 2.5. *Assume that $\mathcal{X}$ is a GC_5 set with no maximal line. Then for no node in $\mathcal{X}$ the m-d sequence is $(5, 5, 5, 3, 2)$.*

Assume by way of contradiction the following.

Assumption 2.2. $\mathcal{X}$ contains a node for which an m-line sequence $(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5)$ implies the m-d sequence $(5, 5, 5, 3, 2)$.

Set $\mathcal{X} = \mathcal{A} \cup \mathcal{B}$ (see Fig. 2.1), with

$$\mathcal{A} = \mathcal{X} \cap \{\ell_1 \cup \ell_2 \cup \ell_3\}, \quad \#\mathcal{A} = 15, \quad \text{and} \quad \mathcal{B} = \mathcal{X} \setminus \mathcal{A}, \quad \#\mathcal{B} = 6.$$

Denote $\mathcal{L}_3 := \{\ell_1, \ell_2, \ell_3\}$. Note that no intersection point of the three lines of $\mathcal{L}_3$ belongs to $\mathcal{X}$.

Below we bring a simple proof for

Lemma 2.1 ([10], Lemma 3.2).
    (i) *The set $\mathcal{B}$ is a GC_2 set, and each node $B \in \mathcal{B}$ uses the three lines of $\mathcal{L}_3$ and the two lines it uses within $\mathcal{B}$, i.e.,*

(2.3)
$$p^{\star}_{B,\mathcal{X}} = \ell_1\,\ell_2\,\ell_3\,p^{\star}_{B,\mathcal{B}}\,.$$
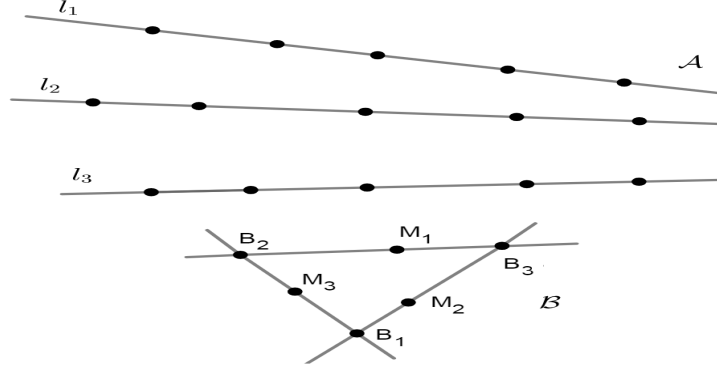
Рис. 2.1. The case $(5, 5, 5, 3, 2)$ with $\mathfrak{X} = \mathcal{A} \cup \mathcal{B}$.

(ii) *No node in $\mathcal{A}$ uses any of the lines of $\mathcal{L}_3$.*

Proof. (i) Suppose by way of contradiction that the set $\mathcal{B}$ is not 2-poised, i.e., it is a subset of a conic $\mathcal{C}$. Then $\mathfrak{X}$ is a subset of the zero set of the polynomial $\ell_1 \ell_2 \ell_3 \mathcal{C}$, which contradicts Proposition 1.1. Then we readily obtain the formula (2.3).

(ii) Without loss of generality assume that $A \in \ell_1$ uses the line ℓ_2. Then $p_A^\star = \ell_2 q$, where $q \in \Pi_4$. It is easily seen that q has (5,4) primary zeros in the lines (ℓ_3, ℓ_1). Therefore, in view of Corollary 1.1, we obtain that $p_A^\star = \ell_2 \ell_3 \ell_1 r$, which is a contradiction. $\square$

Evidently, any node in a GC_2 set uses a maximal line, i.e., 3-node line. Hence we conclude readily that any GC_2 set, including also $\mathcal{B}$, possesses at least three maximal lines (see Figure 2.1).

A node $A \in \mathfrak{X}$ is called a 2_m-*node* if it is the intersection point of two maximal lines. Note that the nodes B_i, $i = 1, 2, 3$, in Fig. 2.1, are 2_m-nodes for $\mathcal{B}$.

Definition 2.1. We say, that a line ℓ is a $k_{\mathcal{A}}$-*node* line if it passes through exactly k nodes of $\mathcal{A}$.

Lemma 2.2. *(i) Assume that a line $\widetilde{\ell} \notin \mathcal{L}_3$ does not intersect a line $\ell \in \mathcal{L}_3$ at a node in $\mathfrak{X}$. Then the line $\widetilde{\ell}$ can be used at most by one node from $\mathcal{A}$. Moreover, this latter node belongs to $\ell \cap \mathcal{A}$.*

(ii) If a line ℓ is $0_{\mathcal{A}}$ or $1_{\mathcal{A}}$-node line then no node from $\mathcal{A}$ uses the line ℓ.

(iii) If a line ℓ is $2_{\mathcal{A}}$-node line then ℓ can be used by at most one node from $\mathcal{A}$.

(iv) Suppose ℓ is a maximal line in $\mathcal{B}$. Then ℓ can be used by at most one node from $\mathcal{A}$.

67

Proof. (i) Without loss of generality assume that $\ell = \ell_1$ and $A \in \ell_2$ uses $\widetilde{\ell}$:

$$p_A^\star = \widetilde{\ell}\, q, \ q \in \Pi_4.$$

It is easily seen that q has $(5, 4, 3)$ primary zeros in the lines (ℓ_1, ℓ_3, ℓ_2). Therefore, in view of Corollary 1.1, we conclude that $p_A^\star = \widetilde{\ell}\,\ell_1\,\ell_2\,\ell_3\,r, \ r \in \Pi_1$, which is a contradiction.

Now assume conversely that $A, B \in \ell_1 \cap \mathfrak{X}$ use the line $\widetilde{\ell}$. Choose a point $C \in \ell_2 \setminus (\widetilde{\ell} \cup \mathfrak{X})$. Then choose numbers α and β, with $|\alpha| + |\beta| \neq 0$, such that $p(C) = 0$, where $p := \alpha p_A^\star + \beta p_B^\star$. It is easily seen that $p = \widetilde{\ell}\, q, \ q \in \Pi_4$ and the polynomial q has $(5, 4, 3)$ primary zeros in the lines (ℓ_2, ℓ_3, ℓ_1). Therefore $p = \widetilde{\ell}\,\ell_1\,\ell_2\,\ell_3\,q$, where $q \in \Pi_1$. Thus $p(A) = p(B) = 0$, implying that $\alpha = \beta = 0$, which is a contradiction.

The items (ii) and (iii) readily follow from (i). The item (iv) readily follows from (iii). $\qquad\square$

Denote by ℓ_{AB} the line passing through the points A and B.

Proposition 2.6. *Let $\ell_{B_1 M_1}$ be 5-node line, which is used by all the six nodes of a subset $\mathcal{A}_6 \subset \mathcal{A}$. Suppose also that ℓ is a 4-node line passing through B_1. If the line ℓ is used by three nodes from $\mathcal{A}$ then all these three nodes belong to $\mathcal{A}_6$.*

Proof. The six nodes of $\mathcal{A}_6$ use the 5-node line $\ell_{B_1 M_1}$. Therefore, in view of Proposition 2.4, these six nodes share also two more lines passing through five primary nodes. It is easily seen that these latter two lines are the lines $\ell_{B_2 M_2}$ and $\ell_{B_3 M_3}$. Assume by way of contradiction that the nodes $D_1, D_2, D_3 \in \mathcal{A}$ are using the line ℓ and $D_1 \notin \mathcal{A}_6$. According to Proposition 2.3 these three nodes share also two lines passing through five primary nodes.

In view of Lemma 2.2, (iv), these latter two lines cannot be maximal lines in $\mathcal{B}$. Therefore they belong to the set $\{\ell_{B_2 M_2}, \ell_{B_3 M_3}, \ell_{M_1 M_2}, \ell_{M_2 M_3}, \ell_{M_1 M_3}\}$. One of them should be $\ell_{B_2 M_2}$ or $\ell_{B_3 M_3}$, since any two lines from $\{\ell_{M_1 M_2}, \ell_{M_2 M_3}, \ell_{M_1 M_3}\}$ share a node. Therefore one of them will be used by seven nodes, namely by D_1 and the nodes of $\mathcal{A}_6$. This contradicts Proposition 2.4. $\qquad\square$

2.2. **The proof of Proposition 2.5.** Consider all the lines passing through $B := B_1$ and at least one more node of $\mathfrak{X}$. Denote the set of these lines by $\mathcal{L}(B)$. Let $m_k(B), \ k = 1, 2, 3$, be the number of $k_{\mathcal{A}}$-node lines from $\mathcal{L}(B)$.

We have that

(2.4) $$1m_1(B) + 2m_2(B) + 3m_3(B) = \#\mathcal{A} = 15.$$

Lemma 2.3. *Suppose that a line ℓ, passing through B and different from the line ℓ_{BM_1}, is a $3_{\mathcal{A}}$-node line. Then ℓ can be used by at most three nodes from $\mathcal{A}$.*

Proof. Note that ℓ is not a maximal line for $\mathcal{B}$, since otherwise ℓ will be a maximal line for $\mathcal{X}$. Therefore ℓ is a 4-node line and Proposition 2.3 completes the proof. $\square$

Lemma 2.4. *We have that $m_3(B) \leq 4$.*

Proof. The equality (2.4) implies that $m_3(B) \leq 5$. Assume by way of contradiction that five lines pass through B and three nodes in $\mathcal{A}$. Therefore these five lines intersect the three lines ℓ_1, ℓ_2, ℓ_3, at the 15 nodes of $\mathcal{A}$. Then, by Theorem 1.1, these 15 nodes are $5 + 3 - 3 = 5$-dependent, which is a contradiction. $\square$

Proof of Proposition 2.5. In view of Proposition 2.4 we divide the proof into three cases.

Case 1. Suppose that ℓ_{BM_1} is 5-node line used by six nodes from $\mathcal{A}$.

Denote the set of these six nodes by $\mathcal{A}_6 \subset \mathcal{A}$. We have that any node from $\mathcal{A}$ uses at least one line from $\mathcal{L}(B)$. Proposition 2.6 implies that all $3_{\mathcal{A}}$-node lines from $\mathcal{L}(B)$, except ℓ_{BM_1}, can be used by at most two nodes from $\mathcal{A} \setminus \mathcal{A}_6$.

From Lemma 2.2, we have that

$$(2.5) \qquad 15 - 6 \leq 0m_1(B) + 1m_2(B) + 2(m_3(B) - 1).$$

In view of (2.4) we get

$$(2.6) \qquad m_1(B) + 2m_2(B) + 3m_3(B) - 6 \leq 1m_2(B) + 2m_3(B) - 2.$$

Therefore we conclude that $m_1(B) + m_2(B) + m_3(B) \leq 4$, or, in other words, $3m_1(B) + 3m_2(B) + 3m_3(B) \leq 12$, which contradicts (2.4).

Case 2. Suppose that ℓ_{BM_1} is 5-node line used by at most four nodes of $\mathcal{A}$.

In this case we have that

$$15 \leq 1m_2(B) + 3(m_3(B) - 1) + 4.$$

In view of (2.4) we get

$$(2.7) \qquad m_1(B) + 2m_2(B) + 3m_3(B) \leq 1m_2(B) + 3m_3(B) + 1.$$

Hence $2m_1(B) + 2m_2(B) \leq 2$. Now, by using (2.4) again, we conclude that

$$(2.8) \qquad 3m_3(B_1) \geq 13,$$

which contradicts Lemma 2.4.

Case 3. Suppose that ℓ_{BM_1} is not 5-node line.

Then, in view of the table (2.2), it can be used by at most three nodes of $\mathcal{A}$. From Lemmas 2.2 and 2.3, (ii),(iii), we have that

$$(2.9) \qquad 15 \leq 1m_2(B) + 3m_3(B).$$

In view of (2.4) we get

$$(2.10) \qquad m_1(B) + 2m_2(B) + 3m_3(B) \leq m_2(B) + 3m_3(B).$$

Hence $m_1(B) = m_2(B) = 0$ and $m_3(B) \geq 5$, which contradicts Lemma 2.4. $\qquad \square$

2.3. **The cases** $(5,5,4,4,2)$ **,** $(5,5,4,3,3)$**, and** $(5,4,4,4,3)$**.** Let us fix a node $A \in \mathcal{X}$ and consider the set of lines $\mathcal{L}(A)$. Let $n_k(A)$ be the number of $(k+1)$-node lines from $\mathcal{L}_A$. In view of Assumption 2.1 we have that

$$(2.11) \qquad 1n_1(A) + 2n_2(A) + 3n_3(A) + 4n_4(A) = \#\big(\mathcal{X} \setminus \{A\}\big) = 20.$$

Next we bring a result from [10]. We present also the proof for the convenience.

Lemma 2.5 ([10], Lemma 3.13)**.** *Assume that $\mathcal{X}$ is a GC_5 set with no maximal line. By Proposition 2.5, for no node of $\mathcal{X}$ the m-d sequence is $(5, 5, 5, 3, 2)$. Then the following hold.*

(i) *There is no 3-node line and m-node line is used exactly $m - 1$ times, where $m = 2, 4, 5$.*

(ii) *No two lines used by the same node intersect at a node in $\mathcal{X}$.*

Proof. (i) Consider all the lines in $\mathcal{L}(A)$. From the third column of the table in (2.2), it follows that for the total number $M(A)$ of uses of these lines, we have that

$$(2.12) \qquad M(A) \leq 1\, n_1(A) + 1\, n_2(A) + 3\, n_3(A) + 4\, n_4(A)\,.$$

Since each node in $\mathcal{X}\setminus\{A\}$ uses at least one line through A, we must have $M(A) \geq 20$. In view of the equality (2.11) we conclude that $M(A) = 20$ and $n_2(A) = 0$.

Moreover, we deduce that any line containing m nodes including A has to be used exactly $m-1$ times, where $m = 2, 4, 5$. Since the node A is arbitrary, this is true for all lines containing at least two nodes of $\mathcal{X}$.

(ii) Assume conversely that two lines ℓ_1, ℓ_2, used by a node $A \in \mathcal{X}$ intersect at a node $B \in \mathcal{X}$. Then each of the nodes in $\mathcal{X} \setminus \{A, B\}$ uses at least one line through B, while the node A uses at least two lines. Thus we have $M(A) \geq 21$, which is a contradiction. $\qquad \square$

Corollary 2.2. *For no node in $\mathcal{X}$ the m-d sequence is $(5, 5, 4, 3, 3)$ or $(5, 4, 4, 4, 3)$.*

Proof. Suppose, that for a node $A \in \mathfrak{X}$, the m-d sequence is $(5, 5, 4, 3, 3)$ or $(5, 4, 4, 4, 3)$. In view of Lemma 2.5, (ii), there are no secondary nodes in the used lines. Thus the presence of 3 the m-d sequence implies presence of a 3-node line in an m-line sequence, which contradicts Lemma 2.5, (i). $\qquad\square$

Proposition 2.7. *For no node in $\mathfrak{X}$ the m-d sequence is* $(5, 5, 4, 4, 2)$.

Proof. Assume that for a node $A \in \mathfrak{X}$ some m-line sequence $(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5)$ implies the m-d sequence $(5, 5, 4, 4, 2)$. In view of Lemma 2.5, (ii), the lines $\ell_1, ..., \ell_5$, contain exactly $5, 5, 4, 4, 2$, nodes, respectively. Denote by B and C the two nodes in the line ℓ_5. Then we have

$$p_B^\star = \ell_1 \, \ell_2 \, \ell_3 \, \ell_4 \, \ell_{AC} \quad \text{and} \quad p_C^\star = \ell_1 \, \ell_2 \, \ell_3 \, \ell_4 \, \ell_{AB}.$$

In view of Lemma 2.5 the line ℓ_1 is used by exactly four nodes of $\mathfrak{X}$. Therefore, there exists a node $D \in \mathfrak{X} \setminus \{A, B, C\}$, which is using the line ℓ_1.

In view of (2.1), Proposition 2.5, and Corollary 2.2, for the node $D \in \mathfrak{X}$ some m-line sequence $(\ell_1, \ell_2', \ell_3', \ell_4', \ell_5')$ yields the m-d sequence $(5, 5, 4, 4, 2)$.

Now, as above, we have that the two nodes in the line ℓ_5' use the line ℓ_1. In view of Proposition 2.1, the line ℓ_5', used by the node D, cannot coincide with the lines ℓ_{AB}, ℓ_{AC} or ℓ_{BC}. Therefore ℓ_5' contains a node different from A, B, C, D. Hence, the line ℓ_1 is used at least five times, which is a contradiction. $\qquad\square$

2.4. **Proof of theorem 2.1.** What is left to complete the proof of Theorem 2.1 is the following

Proposition 2.8. *For no node in $\mathfrak{X}$ the m-d sequence is* $(4, 4, 4, 4, 4)$.

Proof. Let us fix a node $A \in \mathfrak{X}$. In view of (2.1), Propositions 2.5, 2.7 and Corollary 2.2, for the node A, m-d sequence is $(4, 4, 4, 4, 4)$. Thus, in view of Lemma 2.5, (ii), all used lines are 4-node lines. Therefore, in view of Lemma 2.5, (i), we conclude that $n_1(A) = n_2(A) = n_4(A) = 0$. Now, the equality (2.11) implies that $3n_3(A) = 20$, which is not possible. $\qquad\square$

Список литературы

[1] V. Bayramyan, H. Hakopian, and S. Toroyan, "A simple proof of the Gasca-Maeztu conjecture for $n = 4$", Jaen J. Approx., **7** (1), 137 -– 147 (2015).

[2] C. de Boor, "Multivariate polynomial interpolation: conjectures concerning GC sets", Numer. Algorithms, **45**, 113 – 125 (2007).

[3] J. R. Busch, "A note on Lagrange interpolation in $\mathbb{R}^2$", Rev. Un. Mat. Argentina, **36**, 33 – 38 (1990).

G. VARDANYAN

[4] J. M. Carnicer and M. Gasca, "A conjecture on multivariate polynomial interpolation", Rev. R. Acad. Cienc. Exactas Fís. Nat. (Esp.), Ser. A Mat., **95**, 145 – 153 (2001).

[5] K. C. Chung and T. H. Yao, "On lattices admitting unique Lagrange interpolations", SIAM J. Numer. Anal., **14**, 735 – 743 (1977).

[6] D. Eisenbud, M. Green, and J. Harris, "Cayley-Bacharach theorems and conjectures", Bull. Amer. Math. Soc., (N. S.) **33** (3), 295 – 324 (1996).

[7] M. Gasca and J. I. Maeztu, "On Lagrange and Hermite interpolation in $\mathbb{R}^k$", Numer. Math. **39**, 1 – 14 (1982).

[8] H. Hakopian, K. Jetter, and G. Zimmermann, "Vandermonde matrices for intersection points of curves", Jaén J. Approx., **1**, 67 – 81 (2009).

[9] H. Hakopian, K. Jetter, and G. Zimmermann, "A new proof of the Gasca-Maeztu conjecture for $n = 4$", J. Approx. Theory, **159**, 224 – 242 (2009).

[10] H. Hakopian, K. Jetter, and G. Zimmermann, "The Gasca-Maeztu conjecture for $n = 5$", Numer. Math., **127**, 685 – 713 (2014).

72